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A strong convergence theorem for solving the split feasibility and fixed point problems in Banach spaces

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Abstract. In this paper, we introduce a new parallel iterative method for finding a common solution of the multiple-set split feasibility and fixed point problems concerning left Bregman strongly nonexpansive mappings in Banach spaces.

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1. Introduction

Let C and Q be nonempty closed and convex subsets of real p-uniformly convex and uniformly smooth Banach spaces E and F, respectively. Let $A : E \longrightarrow F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. The split feasibility problem (SFP) is formulated as follows:

$$\text{Find an element} \quad x^* \in S = C \cap A^{-1}(Q) \tag{1.1}$$

The model of SFP given above was first introduced by Censor and Elfving [11] for modeling inverse problems. We also know that it plays an important role in medical image reconstruction and signal processing (see [5,7]). In view of its applications, several iterative algorithms of solving (1.1) were presented in [5,7,12,16,18,29–31,33–36] and references therein.

There are some generalizations of the SFP, for example, the multiple-set SFP (MSSFP) (see [12,22]), the split common fixed point problem (SCFPP) (see [15,23]), the split variational inequality problem (SVIP) (see [16]), the split common null point problem (SCNPP) (see [8]) and so on.

In 2014, Wang [37] modified Schopfer's algorithm [26] and proved strong convergence for the following multiple-set split feasibility problem (MSSFP):

$$\text{Find an element} \quad x^* \in S = \left(\bigcap_{i=1}^{N} C_i\right) \bigcap \left(\bigcap_{j=N+1}^{N+M} A^{-1}(Q_j)\right), \tag{1.2}$$

where C_i and Q_j are the nonempty closed convex subsets of two p-uniformly convex and uniformly smooth Banach spaces E and F, respectively. He defined for each $n \in \mathbb{N}$

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}}(x) & 1 \le i(n) \le N, \\ J_q^*[J_p(x) - t_n A^* J_p(I - P_{Q_j})A(x)] & N+1 \le i(n) \le N+M, \end{cases}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \bmod (N+M) + 1,$$

and t_n satisfies

$$0 < t \le t_n \le \left(\frac{q}{C_q \|A\|^q} \right)^{1/(q-1)}, \tag{1.3}$$

with C_q defined as in Lemma 2.1 and proposed the following algorithm: For any initial guess $x_0 = \bar{x}$, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n(x_n) \\ D_n = \{w \in E : \Delta_p(y_n, w) \le \Delta_p(x_n, w)\} \\ E_n = \{w \in E : \langle x_n - w, J_p(\bar{x}) - J_p(x_n)\rangle \ge 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(\bar{x}), \end{cases} \tag{1.4}$$

where Δ_p is the Bregman distance with respect to $f(x) = \frac{1}{p}\|x\|^p$, Π_C denotes the Bregman projection and J_p is the duality mapping. He proved the following strong convergence theorem.

Theorem 1.1. *The sequence $\{x_n\}$, generated by (1.4), converges strongly to the Bregman projection $\Pi_S \bar{x}$ of $\bar{x}$ onto the solution set S.*

Note that the algorithm (1.4) studied in the above work is not the parallel one. Therefore, it takes a lot of time in computation when the family of sets C_i and Q_j are sufficiently large.

In 2016, Shehu et al. [27] constructed an iterative scheme for solving the following problem:

$$\text{Find an element} \quad x^* \in C \cap A^{-1}(Q) \cap F(T). \tag{1.5}$$

where T is a left Bregman strongly nonexpansive mapping of C into C. If $T = I$, the identity map, then $F(T) = C$ and in this case, the problem (1.5) reduces to SFP (1.1). They proved the following result.

Theorem 1.2. *Let E and F be two p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E and F, respectively, $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that SFP (1.1) has a nonempty solution set S. Let T be a left Bregman strongly nonexpansive mapping of C into C such that $F(T) = \hat{F}(T)$ and $F(T) \cap S \ne \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. For a fixed $u \in E_1$, let sequence $\{x_n\}$ be iteratively generated by $u_1 \in E_1$*

$$\begin{cases} x_n = \Pi_C J_q[J_p(u_n) - t_n A^* J_p(I - P_Q)A(u_n)] \\ u_{n+1} = \Pi_C J_q[\alpha_n J_p(u) + (1-\alpha_n) J_p T(x_n)], \quad n \ge 1. \end{cases} \tag{1.6}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < t \le t_n \le k < \left(\frac{q}{C_q\|A\|^q}\right)^{1/(q-1)}$.

Then $\{x_n\}$ *converges strongly to an element* $x^* \in F(T) \cap S$, *where* $x^* = \Pi_{F(T)\cap S} u$.

In this paper, we study the above works for a more generalized case

$$S = \left(\bigcap_{i=1}^{N} C_i\right) \bigcap \left(\bigcap_{j=1}^{M} A^{-1}(Q_j)\right) \bigcap \left(\bigcap_{k=1}^{K} F(T_k)\right) \neq \emptyset.$$

where C_i and Q_j are the nonempty closed convex subsets of two p-uniformly convex and uniformly smooth Banach spaces E and F, respectively, $F(T_k)$ is the set of fixed point of left Bregman strongly nonexpansive mapping $T_k : E \longrightarrow E$ such that $\hat{F}(T_k) = F(T_k)$, and $A : E \longrightarrow F$ is a bounded linear operator. We shall introduce a new strongly convergent, parallel and explicit iterative algorithm with the similar condition (1.3) on the iterative parameter.

The rest of this paper is organized as follows. In Sect. 2, we list some related facts that will be used in the proof of our result. In Sect. 3, we introduce a new parallel iterative algorithm and prove a strong convergence theorem for this algorithm. Finally, in Sect. 4, we give two numerical examples for illustrating the main result.

2. Preliminaries

In this section, we recall some definitions and results which will be used later. Let E be a real Banach space with the dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$ and we use $\langle x, f\rangle$ instead of $f(x)$ for $f \in E^*$ and $x \in E$.

The modulus of convexity $\delta_E : [0,2] \longrightarrow [0,1]$ is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \ \|x-y\| \ge \varepsilon\right\},$$

for all $\varepsilon \in [0,1]$. The modulus of smoothness $\rho_E : [0,\infty) \longrightarrow [0,\infty)$ is defined as

$$\rho_E(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\},$$

for all $\tau \in [0,\infty)$. Recall that a Banach space E is said to be

(i) uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$ and p-uniformly convex if there exists $c_p > 0$ such that $\delta_E(\varepsilon) \ge c_p\varepsilon^p$ for all $\varepsilon \in (0,2]$.

(ii) uniformly smooth if $\lim_{\tau\to 0} \rho_E(\tau)/\tau = 0$ and q-uniformly smooth if there is $C_q > 0$ such that $\rho_E(\tau) \le C_q\tau^q$ for all $\tau > 0$.

The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$. Let $1 < q \leq 2 \leq p$ with $1/p + 1/q = 1$. It is well-known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth (see [24]).

The duality mapping $J_p : E \longrightarrow 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \ \|x^*\| = \|x\|^{p-1}\}.$$

It is also well-known that if E is p-uniformly convex and uniformly smooth, then its dual space E^* is q-uniformly smooth and uniformly convex. And in this situation, the duality mapping J_p is one-to-one, single valued and satisfies $J_p = (J_q^*)^{-1}$, where J_q^* is the duality mapping of E^* (see [1,17]).

We have the following lemma:

Lemma 2.1. [32] *Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q(x)\rangle + C_q\|y\|^q. \tag{2.1}$$

Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int dom} f \longrightarrow [0, +\infty)$, defined by

$$\Delta_f(y, x) = f(y) - f(x) - \langle y - x, \bigtriangledown f(x)\rangle,$$

is called the Bregman distance with respect to f (see [13]).

If E is a smooth and strictly Banach space and $f(x) = \dfrac{1}{p}\|x\|^p$, then $\bigtriangledown f(x) = J_p(x)$ and thus the Bregman distance with respect to f is given by

$$\begin{aligned}
\Delta_p(x, y) &= \frac{1}{p}(\|x\|^p - \|y\|^p) - \langle y - x, J_p(x)\rangle \\
&= \frac{1}{q}\|x\|^p - \langle y, J_p(x)\rangle + \frac{1}{p}\|y\|^p \\
&= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle y, J_p(x) - J_p(y)\rangle.
\end{aligned}$$

It is easy to show that, for any $x, y, z \in E$, we have

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p(x) - J_p(z)\rangle, \tag{2.2}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p(x) - J_p(y)\rangle. \tag{2.3}$$

We know that if E is p-uniformly convex, then the Bregman distance has the following property:

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_p(x) - J_p(y)\rangle, \tag{2.4}$$

for all $x, y \in E$ and for some fixed number $\tau > 0$.

Now, let C be a nonempty closed convex subset of E. The metric projection

$$P_C(x) := \arg\min_{y \in C} \|x - y\|, \quad x \in E,$$

is the unique minimum point of the norm distance, which can be characterized by the following variational inequality (see [20]):

$$\langle z - P_C x, J_p(x - P_C x)\rangle \leq 0, \quad \forall z \in C. \tag{2.5}$$

The Bregman projection

$$\Pi_C(x) := \arg\min_{y\in C} \Delta_p(x,y), \quad x \in E,$$

as the minimum point of the Bregman distance (see [6]). The Bregman projection can also be characterized by the following variational inequality:

$$\langle z - \Pi_C x, J_p(x) - J_p(\Pi_C x)\rangle \le 0, \quad \forall z \in C. \tag{2.6}$$

It follows that

$$\Delta_p(\Pi_C x, z) \le \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.7}$$

Let C be a convex subset of int dom$f$ with $f(x) = \dfrac{1}{p}\|x\|^p$, $2 \le p < \infty$ and let T be a self-mapping of C. A point p in the closure of C is said to be an asymptotic fixed point of T (see [14,25]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. The operator T is called left Bregman strongly nonexpansive (L-BSNE) with respect to a nonempty $\hat{F}(T)$ (see [21]) if

$$\Delta_p(Tx, p) \le \Delta_p(x, p), \tag{2.8}$$

for all $p \in \hat{F}(T)$ and $x \in C$, and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$, and

$$\lim_{n\to\infty} (\Delta_p(x_n, p) - \Delta_p(T(x_n), p)) = 0, \tag{2.9}$$

it follows that

$$\lim_{n\to\infty} \Delta_p(T(x_n), x_n) = 0. \tag{2.10}$$

3. Main results

We consider the problem: find an element $x^\dagger$ such that

$$x^\dagger \in S = \left(\bigcap_{i=1}^N C_i\right) \bigcap \left(\bigcap_{j=1}^M A^{-1}(Q_j)\right) \bigcap \left(\bigcap_{k=1}^K F(T_k)\right) \ne \emptyset. \tag{3.1}$$

To solve the Problem (3.1), we introduce the following algorithm:

Algorithm 3.1. For any initial guess $x_0 = x \in E$, define the sequence $\{x_n\}$ by

$y_{i,n} = \Pi_{C_i} x_n, \ i = 1, 2, \ldots, N,$
Choose i_n such that $\Delta_p(y_{i_n,n}, x_n) = \max_{i=1,\ldots,N} \Delta_p(y_{i,n}, x_n)$, let $y_n = y_{i_n,n}$,
$z_{j,n} = J_q^*[J_p(y_n) - t_n A^* J_p(I - P_{Q_j})A(y_n)], \ j = 1, 2, \ldots, M$
Choose j_n such that $\Delta_p(z_{j_n,n}, y_n) = \max_{j=1,\ldots,M} \Delta_p(z_{j,n}, y_n)$, let $z_n = z_{j_n,n}$,
$t_{k,n} = T_k(z_n), \ k = 1, 2, \ldots, K,$
Choose k_n such that $\Delta_p(t_{k_n,n}, z_n) = \max_{k=1,\ldots,K} \Delta_p(t_{k,n}, z_n)$, let $t_n = t_{k_n,n}$,

$H_n = \{z \in E : \ \Delta_p(t_n, z) \le \Delta_p(z_n, z) \le \Delta_p(y_n, z) \le \Delta_p(x_n, z)\},$
$D_n = \{z \in E : \ \langle x_n - z, J_p(x_0) - J_p(x_n)\rangle \ge 0\},$
$x_{n+1} = \Pi_{H_n \cap D_n}(x_0), \ n \ge 0,$
where, $\{t_n\}$ satisfies the condition (1.3).

First of all, we prove the following propositions.

Proposition 3.1. *In the Algorithm 3.1, we have that $S \subset H_n \cap D_n$ for all $n \ge 0$.*

Proof. First, it is easy to see that H_n and D_n are closed convex subsets of E.

Let $u \in S$, we have

$$\Delta_p(t_n, u) = \Delta_p(T_{k_n}(z_n), u) \le \Delta_p(z_n, u). \tag{3.2}$$

From the property of the Bregman projection in (2.7), we have

$$\Delta_p(y_n, u) = \Delta_p(\Pi_{C_{i_n}}(x_n), u) \le \Delta_p(x_n, u). \tag{3.3}$$

Now, we will show that $\Delta_p(z_n, u) \le \Delta_p(y_n, u)$. Let $w_n = A(y_n) - P_{Q_{j_n}} A(y_n)$. Then we have

$$z_n = J_q^*(J_p(y_n) - t_n A^* J_p(w_n)).$$

From the definition of J_p and (2.5), we have

$$\begin{aligned} \langle A(y_n) - A(u), J_p(w_n)\rangle &= \|A(y_n) - P_{Q_{j_n}} A(y_n)\|^p \\ &\quad + \langle P_{Q_{j_n}} A(y_n) - A(u), J_p(w_n)\rangle \\ &\ge \|w_n\|^p. \end{aligned} \tag{3.4}$$

Thus, from Lemma 2.1 and (3.4), we get that

$$\begin{aligned} \Delta_p(z_n, u) &= \Delta_p(J_q^*(J_p(y_n) - t_n A^* J_p(w_n)), u) \\ &= \frac{1}{q}\|J_p(y_n) - t_n A^* J_p(w_n)\|^q - \langle u, J_p(y_n)\rangle \\ &\quad + t_n\langle A(u), J_p(w_n)\rangle + \frac{1}{p}\|u\|^p \\ &\le \frac{1}{q}\|J_p(y_n)\|^q - t_n\langle Ay_n, J_p(w_n)\rangle + \frac{C_q(t_n\|A\|)^q}{q}\|J_p(w_n)\|^q \\ &\quad - \langle u, J_p(y_n)\rangle + t_n\langle Au, J_p(w_n)\rangle + \frac{1}{p}\|u\|^p \\ &= \frac{1}{q}\|y_n\|^q - \langle u, J_p(y_n)\rangle + \frac{1}{p}\|u\|^p + t_n\langle A(u) - A(y_n), J_p(w_n)\rangle \\ &\quad + \frac{C_q(t_n\|A\|)^q}{q}\|w_n\|^q \\ &= \Delta_p(y_n, u) + t_n\langle A(u) - A(y_n), J_p(w_n)\rangle + \frac{C_q(t_n\|A\|)^q}{q}\|w_n\|^q \\ &\le \Delta_p(y_n, u) - (t_n - \frac{C_q(t_n\|A\|)^q}{q})\|w_n\|^p. \end{aligned}$$

From the condition (1.3), we obtain that

$$\Delta_p(z_n, u) \leq \Delta_p(y_n, u). \tag{3.5}$$

So, from (3.2), (3.3) and (3.5), we get that $u \in H_n$. Hence, $S \subset H_n$ for all $n \geq 0$.

Finally, we show that $S \subset D_n$ for all $n \geq 0$. Indeed, $D_0 = E$, so $S \subset D_0$. Suppose that $S \subset D_n$ for some $n \geq 0$, then $S \subset H_n \cap D_n$. Thus, from $x_{n+1} = \Pi_{H_n \cap D_n}(x_0)$ and (2.6), we have

$$\langle x_{n+1} - u, J_p(x_0) - J_p(x_{n+1})\rangle \geq 0,$$

so that $u \in D_{n+1}$. By induction, we obtain that $S \subset D_n$ for all $n \geq 0$. □

Proposition 3.2. *In the Algorithm 3.1, we have that $x_{n+1}-x_n \to 0$ as $n \to \infty$.*

Proof. From the Proposition 3.1, we have that the sequence $\{x_n\}$ is well-defined.

Fixing $u \in S$. It follows form $x_{n+1} = \Pi_{H_n \cap D_n}(x_0)$ and (2.7) that

$$\Delta_p(x_{n+1}, u) \leq \Delta_p(x_0, u). \tag{3.6}$$

Hence, the sequence $\{\Delta_p(x_n, u)\}$ is bounded. Thus, from (2.4), the sequence $\{x_n\}$ also is bounded.

Now, from $x_{n+1} \in D_n$ and from the definition of D_n, we have

$$\langle x_n - x_{n+1}, J_p(x_0) - J_p(x_n)\rangle \geq 0. \tag{3.7}$$

So, we obtain that

$$\langle x_n - x_0, J_p(x_0) - J_p(x_n)\rangle \geq \langle x_{n+1} - x_0, J_p(x_0) - J_p(x_n)\rangle. \tag{3.8}$$

Thus, from (2.4), we have

$$\langle x_{n+1} - x_0, J_p(x_0) - J_p(x_n)\rangle \geq \Delta_p(x_n, x_0) + \Delta_p(x_0, x_n). \tag{3.9}$$

Hence, from (2.3), we get that

$$-\Delta_p(x_n, x_{n+1}) + \Delta_p(x_n, x_0) + \Delta_p(x_0, x_{n+1}) \geq \Delta_p(x_n, x_0) + \Delta_p(x_0, x_n).$$

This is equivalent to

$$\Delta_p(x_0, x_{n+1}) \geq \Delta_p(x_0, x_n) + \Delta_p(x_n, x_{n+1}), \tag{3.10}$$

which implies that the sequence $\{\Delta_p(x_0, x_n)\}$ is increasing. Thus, from the boundedness of $\{\Delta_p(x_0, x_n)\}$, there is the finite limit

$$a = \lim_{n\to\infty} \Delta_p(x_0, x_n).$$

So, from (3.10), we obtain that $\lim_{n\to\infty} \Delta_p(x_n, x_{n+1}) = 0$. It follows from (2.4) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

□

Proposition 3.3. *In the Algorithm 3.1, we have the sequences $\{x_n - y_n\}$, $\{x_n - z_n\}$ and $\{x_n - t_n\}$ converge to zero as $n \to \infty$.*

Proof. Since $x_{n+1} \in H_n$, we have

$$\Delta_p(t_n, x_{n+1}) \le \Delta_p(z_n, x_{n+1}) \le \Delta(y_n, x_{n+1}) \le \Delta(x_n, x_{n+1}).$$

Thus, from the Proposition 3.2 ($\Delta(x_n, x_{n+1}) \to 0$), we obtain that

$$\Delta_p(t_n, x_{n+1}) \to 0,\ \Delta_p(z_n, x_{n+1}) \to 0,\ \Delta(y_n, x_{n+1}) \to 0.$$

It follows from (2.4) that

$$\|x_{n+1} - t_n\| \to 0,\ \|x_{n+1} - z_n\| \to 0,\ \|x_{n+1} - y_n\| \to 0$$

which combining with $\|x_{n+1} - x_n\| \to 0$, we get that

$$x_n - t_n \to 0,\ x_n - z_n \to 0, \text{ and } x_n - y_n \to 0.$$

□

Proposition 3.4. *In the Algorithm 3.1, we have that* $\omega_w(x_n) \subset S$.

Proof. We will prove this proposition by several steps.

Clearly, the $\omega_w(x_n) \ne \emptyset$ because the $\{x_n\}$ is bounded. Let $\bar{x} \in \omega_w(x_n)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $\bar{x}$.

Step 1. $\bar{x} \in \bigcap_{k=1}^K F(T_k)$

From the Proposition 3.3, we have $t_n - z_n \to 0$ and it follows that $\Delta_p(t_n, z_n) \to 0$. Thus, from the definition of t_n, we obtain that $\Delta_p(t_{k,n}, z_n) \to 0$, that is $\Delta_p(T_k(z_n), z_n) \to 0$ for all $k = 1, 2, \ldots, K$. Therefore, we obtain that $\bar{x} \in \hat{F}(T_k) = F(T_k)$ for all $k = 1, 2, \ldots, K$. This implies that $\bar{x} \in \bigcap_{k=1}^K F(T_k)$.

Step 2. $\bar{x} \in \bigcap_{i=1}^N C_i$

From Proposition 3.3, we have $\Delta_p(y_n, x_n) \to 0$. So, it follows from the definition of y_n that $\Delta_p(y_{i,n}, x_n) \to 0$ and hence

$$\|y_{i,n} - x_n\| \to 0, \tag{3.11}$$

for all $i = 1, 2, \ldots, N$.

We need to prove that $\Delta_p(\bar{x}, \Pi_{C_i}(\bar{x})) = 0$ for all $i = 1, 2, \ldots, N$. Indeed, from (2.3), (2.6) and (2.4), we have the following estimate

$$\begin{aligned}
\Delta_p(\bar{x}, \Pi_{C_i}(\bar{x})) &\le \langle \bar{x} - \Pi_{C_i}\bar{x}, J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&= \langle \bar{x} - x_{n_k}, J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&\quad + \langle x_{n_k} - \Pi_{C_i}(x_{n_k}), J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&\quad + \langle \Pi_{C_i}(x_{n_k}) - \Pi_{C_i}(\bar{x}), J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&\le \langle \bar{x} - x_{n_k}, J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&\quad + \langle x_{n_k} - \Pi_{C_i}(x_{n_k}), J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&= \langle \bar{x} - x_{n_k}, J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle \\
&\quad + \langle x_{n_k} - y_{i,n_k}, J_p(\bar{x}) - J_p(\Pi_{C_i}(\bar{x}))\rangle.
\end{aligned}$$

From (3.11), letting $k \to \infty$ yields $\Delta_p(\bar{x}, \Pi_{C_i}(\bar{x})) = 0$ for all $i = 1, 2, \ldots, N$, that is $\bar{x} \in C_i$ for all $i = 1, 2, \ldots, N$ or $\bar{x} \in \bigcap_{i=1}^N C_i$.

Step 3. $\bar{x} \in \bigcap_{j=1}^{M} A^{-1}Q_j$

From the Proposition 3.3, we have $\Delta_p(z_n, y_n) \to 0$. Thus, from the definition of z_n, we get that $\Delta_p(z_{j,n}, y_n) \to 0$ and hence we obtain

$$\|z_{j,n} - y_n\| \to 0, \tag{3.12}$$

for all $j = 1, 2, \ldots, M$.

Since E is uniformly Banach space, the duality mapping J_p is uniformly continuous on bounded sets (see [17, Theorem 2.16]) and hence we have

$$t_n A^* J_p(I - P_{Q_j})A(y_n) = J_p(y_n) - J_p(z_{j,n}) \to 0.$$

Since $0 < t \leq t_n$ for all n, we obtain

$$\|A^* J_p(I - P_{Q_j})A(y_n)\| \to 0. \tag{3.13}$$

Let us now fix some $u \in S$, then $A(u) \in Q_j$ for all $j = 1, 2, \ldots, M$. It follows from (2.5) that

$$\begin{aligned}
\|(I - P_{Q_j})A(y_{n_k})\|^p &= \langle (I - P_{Q_j})A(y_{n_k}), J_p(I - P_{Q_j})A(y_{n_k})\rangle \\
&= \langle A(y_{n_k}) - A(u), J_p(I - P_{Q_j})A(y_{n_k})\rangle \\
&\quad + \langle A(u) - P_{Q_j}A(y_{n_k}), J_p(I - P_{Q_j})A(y_{n_k})\rangle \\
&\leq \langle A(y_{n_k}) - A(u), J_p(I - P_{Q_j})A(y_{n_k})\rangle \\
&\leq K_0\|(I - P_{Q_j})A(y_{n_k})\|^{p-1},
\end{aligned}$$

which combines with (3.13), we obtain that

$$\|(I - P_{Q_j})A(y_{n_k})\| \to 0 \tag{3.14}$$

for all $j = 1, 2, \ldots, M$ and $K_0 = \|A\|(\sup_k \|y_{n_k}\| + \|u\|) < \infty$.

Now, from (2.5), we have

$$\begin{aligned}
\|(I - P_{Q_j})A(\bar{x})\|^p &= \langle A(\bar{x}) - P_{Q_j}A(\bar{x}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle \\
&= \langle A(\bar{x}) - A(y_{n_k}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle \\
&\quad + \langle A(y_{n_k}) - P_{Q_j}A(\bar{x}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle \\
&\quad + \langle P_{Q_j}A(\bar{x}) - A(y_{n_k}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle \\
&\leq \langle A(\bar{x}) - A(y_{n_k}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle \\
&\quad + \langle A(y_{n_k}) - P_{Q_j}A(\bar{x}), J_p(A(\bar{x}) - P_{Q_j}A(\bar{x}))\rangle.
\end{aligned}$$

From the continuity of A, $x_n - y_n \to 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we get that $A(y_{n_k}) \rightharpoonup A(\bar{x})$. Hence, letting $k \to \infty$ and using (3.14), we obtain

$$\|A(\bar{x}) - P_{Q_j}A(\bar{x})\| = 0,$$

for all $j = 1, 2, \ldots, M$, that is $A(\bar{x}) \in \bigcap_{j=1}^{M} A^{-1}Q_j$.

Thus, from Step 1, Step 2 and Step 3, we conclude that $\bar{x} \in S$. Since $\bar{x}$ is arbitrary, $\omega_w(x_n) \subset S$. □

Now, we are in position to prove our main result.

Theorem 3.5. *In the Algorithm 3.1, we have that the sequence $\{x_n\}$ converges strongly to $x^\dagger = \Pi_S(x_0)$, as $n \to \infty$.*

Proof. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Then, from the Proposition 3.4 we have $x^* \in S$.

Since $x_{n+1} = \Pi_{H_n \cap D_n}(x_0)$, $x_{n+1} \in D_n$. Thus, from $\Pi_S(x_0) \in S \subset D_n$, we have

$$\Delta_p(x_{n+1}, x_0) \le \Delta_p(\Pi_S x_0, x_0),$$

which combines with $\Delta_p(x_{n+1}, x_0) \ge \Delta_p(x_n, x_0)$, we obtain that

$$\Delta_p(x_n, x_0) \le \Delta_p(\Pi_S x_0, x_0), \quad \forall n \ge 0. \tag{3.15}$$

Thus, from (2.2), (2.3) and (3.15), we get

$$\begin{aligned}
\Delta_p(x_{n_k}, \Pi_S(x_0)) &= \Delta_p(x_{n_k}, x_0) + \Delta_p(x_0, \Pi_S(x_0)) \\
&\quad + \langle x_{n_k} - x_0, J_p(x_0) - J_p(\Pi_S(x_0))\rangle \\
&\le \Delta_p(\Pi_S(x_0), x_0) + \Delta_p(x_0, \Pi_S(x_0)) \\
&\quad + \langle \Pi_S(x_0) - x_0, J_p(x_0) - J_p(\Pi_S(x_0))\rangle \\
&\quad + \langle x_{n_k} - \Pi_S(x_0), J_p(x_0) - J_p(\Pi_S(x_0))\rangle \\
&= \langle x_{n_k} - \Pi_S(x_0), J_p(x_0) - J_p(\Pi_S(x_0))\rangle.
\end{aligned}$$

So, we have

$$\begin{aligned}
\limsup_{k\to\infty} \Delta_p(x_{n_k}, \Pi_S(x_0)) &\le \limsup_{k\to\infty} \langle x_{n_k} - \Pi_S(x_0), J_p(x_0) - J_p(\Pi_S(x_0))\rangle \\
&\le \langle x^* - \Pi_S(x_0), J_p(x_0) - J_p(\Pi_S(x_0))\rangle \le 0,
\end{aligned}$$

which implies that $\lim_{k\to\infty} \Delta_p(x_{n_k}, \Pi_S(x_0)) = 0$ and hence $x_{n_k} \to \Pi_S(x_0)$ thanks to (2.4). By the uniqueness of Bregman projection $\Pi_S(x_0)$, we obtain that the sequence $\{x_n\}$ converges weakly to $\Pi_S(x_0)$. Now, from (2.4), there exists a $\tau > 0$ such that

$$\tau\|x_n - \Pi_S(x_0)\| \le \langle x_n - \Pi_S(x_0), J_p(x_0) - J_p(\Pi_S(x_0))\rangle.$$

Letting $n \to \infty$, we conclude that $x_n \to x^\dagger = \Pi_S(x_0)$. □

Next, from Theorem 3.5, we have two following corollaries. First, we have an algorithm for solving the MSFP in two Banach spaces.

Corollary 3.6. *Let C_i, $i = 1, 2, \dots, N$ and Q_j, $j = 1, 2, \dots, M$ be the nonempty closed convex subsets of two p-uniformly convex and uniformly smooth Banach spaces E and F, respectively. Let $A : E \to F$ be a bounded linear operator. Suppose that $S = \left(\bigcap_{i=1}^N C_i\right) \bigcap \left(\bigcap_{j=1}^M A^{-1}(Q_j)\right) \ne \emptyset$. If the sequence $\{t_n\}$ satisfies the condition (1.3), then the sequence $\{x_n\}$ generated by $x_0 \in E$ and*

$y_{i,n} = \Pi_{C_i}(x_n),\ i = 1, 2, \dots, N,$
Choose i_n such that $\Delta_p(y_{i_n,n}, x_n) = \max_{i=1,\dots,N} \Delta_p(y_{i,n}, x_n)$, let $y_n = y_{i_n,n}$,
$z_{j,n} = J_q^*[J_p(y_n) - t_n A^* J_p(I - P_{Q_j})A(y_n)],\ j = 1, 2, \dots, M$
Choose j_n such that $\Delta_p(z_{j_n,n}, y_n) = \max_{j=1,\dots,M} \Delta_p(z_{j,n}, y_n)$, let $z_n = z_{j_n,n}$,

$H_n = \{z \in E : \ \Delta_p(z_n, z) \leq \Delta_p(y_n, z) \leq \Delta_p(x_n, z)\}$,
$D_n = \{z \in E : \ \langle x_n - z, J_p(x_0) - J_p(x_n)\rangle \geq 0\}$,
$x_{n+1} = \Pi_{H_n \cap D_n}(x_0), \ n \geq 0$,
converges strongly to $x^\dagger = \Pi_S(x_0)$*, as* $n \to \infty$*.*

Proof. Apply Theorem 3.5 with $T_k(x) = x$ for all $x \in E$ and for all $k = 1, 2, \ldots, K$, we get the proof of this corollary. □

Finally, we have the following result for the problem of finding a common fixed point of a finite family of L-BSNE operators in Banach spaces.

Corollary 3.7. *Let* E *be a* p*-uniformly convex and uniformly smooth Banach space. Let* $T_k : E \to E$*,* $k = 1, 2, \ldots, K$ *be the left Bregman strongly nonexpansive mappings such that* $\hat{F}(T_k) = F(T_k)$ *and* $S = \bigcap_{k=1}^K F(T_k) \neq \emptyset$*. Then the sequence* $\{x_n\}$ *generated by* $x_0 \in E$ *and*

$t_{k,n} = T_k(x_n), \ k = 1, 2, \ldots, K$,
Choose k_n *such that* $\Delta_p(t_{k_n,n}, x_n) = \max_{k=1,\ldots,K} \Delta_p(t_{k,n}, x_n)$*, let* $t_n = t_{k_n,n}$*,*
$H_n = \{z \in E : \ \Delta_p(t_n, z) \leq \Delta_p(x_n, z)\}$,
$D_n = \{z \in E : \ \langle x_n - z, J_p(x_0) - J_p(x_n)\rangle \geq 0\}$,
$x_{n+1} = \Pi_{H_n \cap D_n}(x_0), \ n \geq 0$,
converges strongly to $x^\dagger = \Pi_S(x_0)$*, as* $n \to \infty$*.*

Proof. Apply Theorem 3.5 with $E \equiv F$ and $C_i = Q_j = E$ for all $i = 1, 2, \ldots, N$ and for all $j = 1, 2, \ldots, M$, and $A = I$, we get the proof of this corollary. □

4. Numerical test

Example 4.1. We consider the Problem (3.1) with $C_i \subset \mathbb{R}^n$ and $Q_j \subset \mathbb{R}^m$ which are defined by

$$C_i = \{x \in \mathbb{R}^{\mathcal{N}} : \ \langle a_i^C, x\rangle \leq b_i^C\},$$
$$Q_j = \{x \in \mathbb{R}^{\mathcal{M}} : \ \langle a_j^Q, x\rangle \leq b_j^Q\},$$

where $a_i^C \in \mathbb{R}^{\mathcal{N}}, a_j^Q \in \mathbb{R}^{\mathcal{M}}$ and $b_i^C, b_j^Q \in \mathbb{R}$ for all $i = 1, 2, \ldots, N$ and for all $j = 1, 2, \ldots, M$, T_k is metric projection from $\mathbb{R}^{\mathcal{N}}$ onto S_k with

$$S_k = \{x \in \mathbb{R}^n : \ \|x - I_k\|^2 \leq R_k^2\},$$

for all $k = 1, 2, \ldots, K$, and A is bounded linear operator from $\mathbb{R}^{\mathcal{N}}$ into $\mathbb{R}^{\mathcal{M}}$ with its matrix which has the elements are randomly generated in $[2, 4]$.

Next, we take the randomly generated values of the coordinates of a_i^C, a_j^Q in $[1, 3]$ and b_i^C, b_j^Q in [2,4], the center I_k in $[-1, 1]$ and the radius R_k of S_k in $[2, 10]$, respectively.

Clearly, the $S = \left(\bigcap_{i=1}^N C_i\right) \bigcap \left(\bigcap_{j=1}^M A^{-1}(Q_j)\right) \bigcap \left(\bigcap_{k=1}^K F(T_k)\right) \neq \emptyset$, because $0 \in S$.

Table 1. Table of numerical results for Example 4.1

Stop condition: $\mathrm{TOL}_n < 10^{-5}$			Stop condition: $\mathrm{TOL}_n < 10^{-6}$		
No.	TOL_n	n	No.	TOL_n	n
1	9.731910437302011e−006	525	1	9.822579488456307e−007	2692
2	9.723805875618304e−006	382	2	9.883943056654116e−007	1084
3	9.740930951261035e−006	594	3	9.991787341354400e−007	1878
4	9.817880940634140e−006	793	4	9.821636271493920e−007	1922
5	9.773956068159787e−006	250	5	9.984864719310806e−007	1644

Now, we will test the Algorithm 3.1, with the initial $x_0 \in \mathbb{R}^{\mathcal{N}}$ where its coordinates are also randomly generated in $[-5, 5]$, $\mathcal{N} = 20$, $\mathcal{M} = 40$, $N = 50$, $M = 100$, $K = 200$ and $t_n = \dfrac{1}{2\|A\|^2}$. After five attempts with randomized data, we obtain the following table of results (see Table 1).

Remark 4.2. In the above example, the function TOL is defined by

$$\mathrm{TOL}_n = \frac{1}{N}\sum_{i=1}^{N}\|x_n - P_{C_i}x_n\|^2 + \frac{1}{M}\sum_{j=1}^{M}\|Ax_n - P_{Q_j}Ax_n\|^2 + \frac{1}{K}\sum_{k=1}^{K}\|x_n - T_kx_n\|^2,$$

for all $n \geq 1$. Note that, if at the nth step, $\mathrm{TOL}_n = 0$ then $x_n \in S$ that is, x_n is a solution of this problem.

Example 4.3. We take $E = F = L_2([0, 1])$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$$

and the norm

$$\|f\| = \left(\int_0^1 f^2(t)\mathrm{d}t\right)^{1/2},$$

for all $f, g \in L_2([0, 1])$.

Now, let

$$C_i = \{x \in L_2([0, 1]) :\ \langle a_i, x\rangle = b_i\},$$

where $a_i(t) = t^{i-1}$, $b_i = \dfrac{1}{i+1}$ for all $i = 1, 2, \ldots, N$ and $t \in [0, 1]$,

$$Q_j = \{x \in L_2([0, 1]) :\ \langle c_j, x\rangle \geq d_j\},$$

in which $c_j(t) = t + j$, $d_j = \dfrac{1}{4}$ for all $j = 1, 2, \ldots, M$ and $t \in [0, 1]$, and

$$T_k = P_{S_k},$$

in here $S_k = \{x \in L_2([0, 1]) :\ \|x - I_k\| \leq k + 1\}$, with $I_k(t) = t + k$ for all $k = 1, 2, \ldots, K$ and $t \in [0, 1]$.

Table 2. Table of numerical results for Example 4.3

Stop condition: $\|x_{n+1} - x_n\| < \text{err}$					
$t_n = 1,\ x_0(t) = t^2$			$t_n = 1,\ x_0(t) = \exp(t)$		
err	$\|x_{n+1} - x_n\|$	n	err	$\|x_{n+1} - x_n\|$	n
10^{-2}	9.923267213e−003	128	10^{-2}	9.069246165e−003	125
10^{-3}	9.909406522e−004	2159	10^{-3}	9.953384192e−004	1091
10^{-4}	9.983270545e−005	47,840	10^{-4}	9.979431075e−005	11,352

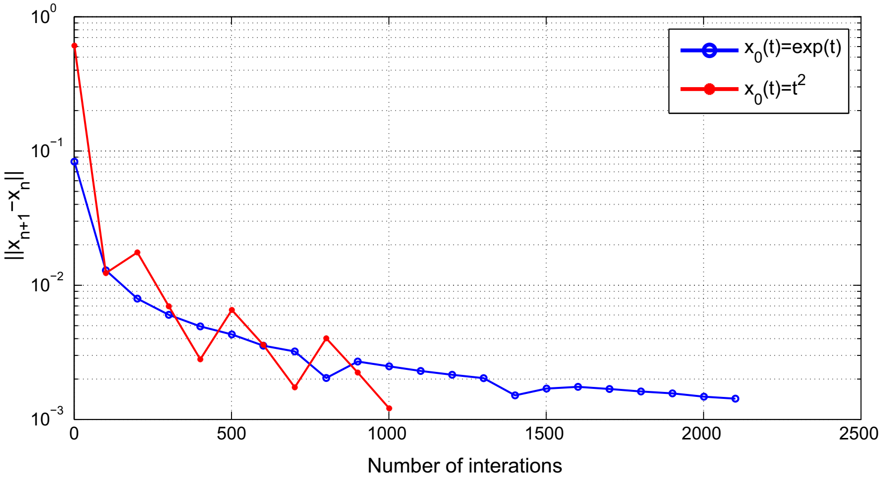

Figure 1. The behavior of $\|x_{n+1} - x_n\|$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-3}$

Let us assume that

$$A:\ L_2([0,1]) \longrightarrow L_2([0,1]),\ (Ax)(t) = \frac{x(t)}{2}.$$

We consider the problem of finding an element $x^\dagger$ such that

$$x^\dagger \in S = \left(\bigcap_{i=1}^{N} C_i\right) \bigcap \left(\bigcap_{j=1}^{M} A^{-1}(Q_j)\right) \bigcap \left(\bigcap_{k=1}^{K} F(T_k)\right). \tag{4.1}$$

It is easy to see that $S \neq \emptyset$, because $x(t) = t \in S$.

We have

$$\Pi_{C_i}(x) = P_{C_i}(x) = \frac{b_i - \langle a_i, x\rangle}{\|a_i\|^2} a_i + x,$$

$$P_{Q_j}(x) = \max\left\{0, \frac{d_j - \langle c_j, x\rangle}{\|c_j\|^2}\right\} c_j + x,$$

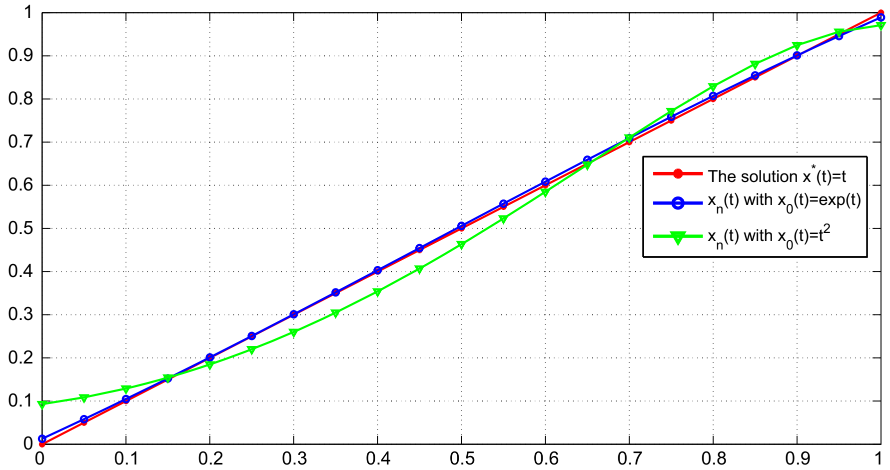

FIGURE 2. The behavior of $x_n(t)$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-2}$

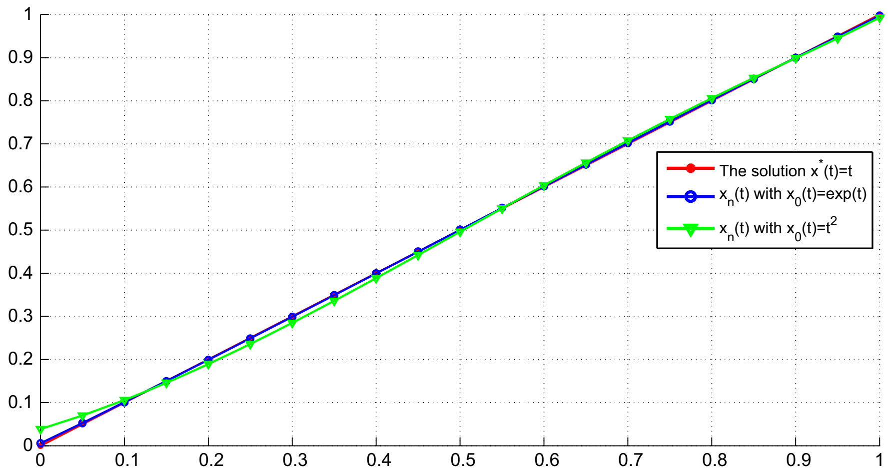

FIGURE 3. The behavior of $x_n(t)$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-3}$

and

$$T_k(x) = \begin{cases} x, & \text{if } \|x - I_k\| \le k+1, \\ I_k + \dfrac{k+1}{\|x - I_k\|}(x - I_k), & \text{otherwise.} \end{cases}$$

Using Algorithm 3.1 with $N = 10$, $M = 20$ and $K = 40$, we obtain the following table of numerical results.

The behavior of $\|x_{n+1} - x_n\|$ in Table 2 is described in the Fig. 1.

The behaviors of the approximation solution $x_n(t)$ in both of the cases $\|x_{n+1} - x_n\| < 10^{-2}$ and $\|x_{n+1} - x_n\| < 10^{-3}$ are presented in Figs. 2 and 3.

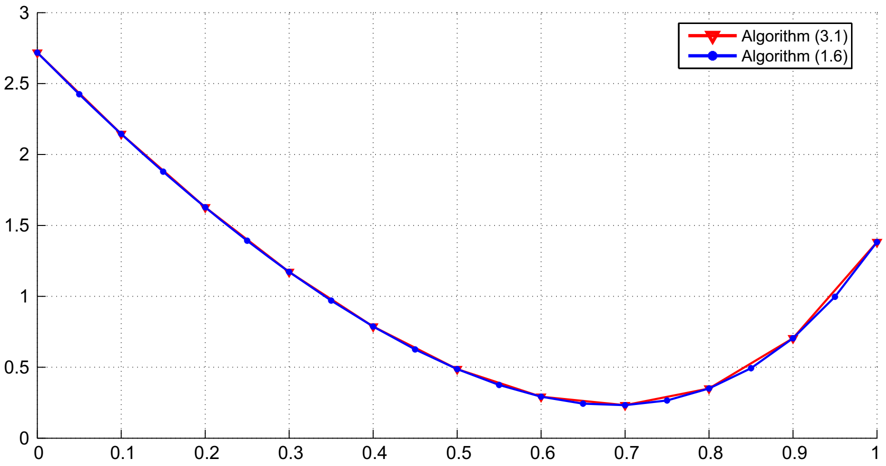

FIGURE 4. The behavior of $x_n(t)$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-7}$

TABLE 3. Table of numerical results for Problem (4.2)

Stop condition: $\|x_{n+1} - x_n\| <$ err					
Algorithm (1.6)			Algorithm (3.1)		
err	$\|x_{n+1} - x_n\|$	n	err	$\|x_{n+1} - x_n\|$	n
10^{-6}	9.814293000e−007	18	10^{-6}	7.160416379e−007	23
10^{-7}	9.750563778e−008	56	10^{-7}	9.075352447e−008	26
10^{-8}	9.976658166e−009	174	10^{-8}	3.552713678e−015	30

Now, we consider a special case of problem (4.1) as follows:

$$\text{Find an element } x^{\dagger} \in C \cap A^{-1}(Q) \cap F(T), \tag{4.2}$$

where $C = C_2$, $Q = Q_2$ and $T = T_2$.

Applying algorithms (1.6) and (3.1) with $t_n = 1$ and $\alpha_n = \dfrac{1}{n}$ for all $n \geq 1$, and $u(t) = x_0(t) = \exp(t^2 + 1)$ for all $t \in [0, 1]$, we get the following table of numerical results.

Figure 4 show the behaviors of the approximation solutions $x_n(t)$ for the case $\|x_{n+1} - x_n\| < 10^{-7}$ in Table 3.

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