



Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces

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Abstract. The paper proposes two new iterative methods for solving pseudomonotone equilibrium problems involving fixed point problems for quasi- ϕ -nonexpansive mappings in Banach spaces. The proposed algorithms combine the extended extragradient method or the linesearch method with the Halpern iteration. The strong convergence theorems are established under standard assumptions imposed on equilibrium bifunctions and mappings. The results in this paper have generalized and enriched existing algorithms for equilibrium problems in Banach spaces.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem (shortly, EP) for f on C is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C. \quad (\text{EP})$$

Let us denote by $\text{EP}(f, C)$ the solution set of problem EP. Mathematically, problem EP is a generalization of many mathematical models including variational inequality problems, fixed point problems, optimization problems and Nash equilibrium problems [5, 34]. In recent years, problem EP has received a lot of attention by many authors and some notable methods have been proposed for solving problem EP such as: gap function methods [31], auxiliary problem principle methods [30] and proximal point methods [33, 35]. One of the most popular methods for solving problem EP is the proximal-like method [15]. This method is based on the auxiliary problem principle

introduced by Cohen [13] and extended by Mastroeni to problem EP in a real Banach space [30]. Recently, the convergence of the proximal point method has been further investigated and extended in [37] under the assumptions that the function f is pseudomonotone and satisfies a Lipschitz-type condition on C . Let us recall here that f is pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x, y \in C$$

and that f satisfies the Lipschitz-type condition on C if there exist two constants $c_1, c_2 > 0$ such that, for all $x, y \in C$,

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2. \quad (1)$$

The methods studied in [37] are also called the extragradient methods due to the results obtained by Korpelevich in [27]. In Euclidean spaces, the extragradient method [15, 37] generates two iterative sequences $\{x_n\}$ and $\{y_n\}$ defined for each integer n by

$$\begin{cases} y_n = \arg \min \{ \rho f(x_n, y) + G(x_n, y) : y \in C \}, \\ x_{n+1} = \arg \min \{ \rho f(y_n, y) + G(x_n, y) : y \in C \}, \end{cases}$$

where $x_0 \in C$, ρ is a suitable parameter and $G(x, y)$ is the Bregman distance function defined on $C \times C$. In recent years, the extragradient method has been widely and intensively investigated in Hilbert spaces by several authors, see for instance [3, 10, 18–23, 36, 43, 44, 48, 49]. In Banach spaces, approximations of solutions to problem EP are frequently based on the resolvent $J_{r,f}$ of the monotone bifunction f . More precisely, given $x_0 \in C$, a sequence $\{x_n\}$ is generated by the iteration $x_{n+1} = J_{r,f}(x_n)$ where $J_{r,f}(x)$, with $x \in E$ and $r > 0$, is the unique solution of the following regularized equilibrium problem (REP):

$$\text{Find } z \in C \text{ such that: } f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2)$$

Here, $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping on E . A disadvantage of using the resolvent $J_{r,f}$ is that the non-linear inequality (2) seems to be more difficult to solve numerically than the extragradient iteration. Another advantage of the extragradient method is that it can also be used for the more general class of pseudomonotone bifunctions.

Let $S : C \rightarrow C$ be a mapping whose fixed point set is denoted by $\text{Fix}(S)$, i.e., $\text{Fix}(S) = \{x \in C : S(x) = x\}$. The problem of finding a common solution of an equilibrium problem and a fixed point problem is a task arising in various fields of applicable mathematics, sciences, engineering and economy, see for example [14, 26]. This happens, in particular, in practical models when the constraint set of the problem is expressed as the fixed point set of a mapping, see for instance [24, 28, 29]. Several methods for solving this problem in Banach spaces can be found, for instance, in [9, 38, 41, 45, 46, 50, 53]. Most of them use a hybrid method for solving simultaneously a regularized equilibrium problem and a fixed point problem. The aim of these methods is to construct closed convex subsets of the feasible set and to compute the sequence of iterates by projecting the starting point x_0 onto intersections of these subsets. For example, in uniformly smooth and uniformly convex

Banach spaces, Takahashi and Zembayashi [45] introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{cases} x_0 \in C_0 := C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \geq 0, \end{cases} \quad (3)$$

where S is a relatively nonexpansive mapping, f is monotone, ϕ is the Lyapunov functional and $\Pi_{C_{n+1}}$ denotes the generalized projection onto the set C_{n+1} . Under suitable assumptions on the sequences $\{\alpha_n\}$ and $\{r_n\}$, the authors proved that the sequence $\{x_n\}$ generated by (3) converges strongly to the projection of x_0 onto the set $\text{EP}(f, C) \cap \text{Fix}(S)$. On the other hand, a class of more general mappings than the class of relatively nonexpansive mappings is the class of quasi- ϕ -nonexpansive (or relatively quasi-nonexpansive) mappings. This class has been widely studied in recent years, see, e.g., [28, 29, 38, 53]. Hence, we have naturally the following question:

Question: *Is it possible to construct strongly convergent algorithms in Banach spaces for solving pseudomonotone equilibrium problems and fixed point problems for quasi- ϕ -nonexpansive mappings **without** using the resolvent of the bifunction and the hybrid (outer approximation) or shrinking projection methods?*

To answer this question, motivated by the results obtained in [37, 40, 49] and the Halpern iteration in [17], we propose two new iterative methods for finding a common solution of an equilibrium problem for pseudomonotone bifunctions and a fixed point problem for quasi- ϕ -nonexpansive mappings in Banach spaces. The first algorithm combines the extended extragradient method with the Halpern iteration. The strong convergence of the iterations is proved under the ϕ -Lipschitz-type assumption of the equilibrium bifunction. To avoid this slightly strong assumption, we use the linesearch technique to design the second algorithm having the same strong convergence to the first one.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results used in the next sections. Section 3 presents the Halpern extragradient method and proves its convergence. In Sect. 4, we establish the convergence of the Halpern linesearch algorithm. Finally, in Sect. 5, we study and develop several numerical experiments to illustrate the convergence of the proposed algorithms following the choice of stepsizes and parameters.

2. Preliminaries

In this section, we recall some definitions and results which will be used later. We begin with several concepts and properties of a Banach space, see [2, 16] for more details.

Definition 2.1. A Banach space E is called

- (i) strictly convex if the unit sphere $S_1(0) = \{x \in X : \|x\| \leq 1\}$ is strictly convex, i.e., the inequality $\|x + y\| < 2$ holds for all $x, y \in S_1(0), x \neq y$;
- (ii) uniformly convex if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1, \|x - y\| = \epsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds;
- (iii) smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4}$$

- exists for all $x, y \in S_1(0)$;
- (iv) uniformly smooth if the limit (4) exists uniformly for all $x, y \in S_1(0)$.

The modulus of convexity of E is the function $\delta_E : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\}$$

for all $\epsilon \in [0, 2]$. Note that E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$ and $\delta_E(0) = 0$.

Let $p > 1$. A uniformly convex Banach space E is said to be p -uniformly convex if there exists some constant $c > 0$ such that

$$\delta_E(\epsilon) \geq c\epsilon^p.$$

It is well known that the spaces L^p, l^p and W_m^p are p -uniformly convex when $p > 2$ and 2-uniformly convex when $1 < p \leq 2$. Furthermore, any Hilbert space H is uniformly smooth and 2-uniformly convex.

Let E be a real Banach space with its dual E^* . The dual product of $f \in E^*$ and $x \in E$ is denoted by $\langle x, f \rangle$ or $\langle f, x \rangle$. For the sake of simplicity, the norms of E and E^* are denoted by the same symbol $\|\cdot\|$. The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle f, x \rangle = \|x\|^2 = \|f\|^2 \right\}.$$

We have the following properties, see for instance [12, 39]:

- (i) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \rightarrow 2^{E^*}$ is single-valued, one-to-one, and onto;
- (ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm to weak $*$ continuous;
- (iii) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E ;
- (iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

Next, we assume that E is a smooth, strictly convex, and reflexive Banach space. In the sequel, we always use $\phi : E \times E \rightarrow [0, \infty)$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \forall x, y \in E.$$

From the definition of ϕ , it is easy to show that ϕ has the following properties:

(i) For all $x, y, z \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{5}$$

(ii) For all $x, y, z \in E$,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2 \langle z - x, Jy - Jz \rangle. \tag{6}$$

Particularly, when $x = y$, we have $\langle y - z, Jy - Jz \rangle = \frac{1}{2}\phi(y, z) + \frac{1}{2}\phi(z, y)$.

(iii) For all $x, y, z \in E$ and $\lambda \in [0, 1]$,

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z). \tag{7}$$

Throughout this paper, we assume that C is a nonempty closed convex subset of E .

Let $S : C \rightarrow C$ be a mapping with the fixed point set $\text{Fix}(S)$. A point $p \in C$ is called an asymptotic fixed point [11] of S if there exists a sequence $\{x_n\} \subset C$ converging weakly to p and such that $\|x_n - S(x_n)\| \rightarrow 0$. The set of asymptotic fixed points of S is denoted by $\hat{F}(S)$. A mapping $S : C \rightarrow C$ is called:

(i) nonexpansive if $\|S(x) - S(y)\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) relatively nonexpansive [6] if $\text{Fix}(S) \neq \emptyset$, $\text{Fix}(S) = \hat{F}(S)$ and

$$\phi(x^*, S(x)) \leq \phi(x^*, x), \quad \forall x^* \in \text{Fix}(S), \quad x \in C;$$

(iii) quasi- ϕ -nonexpansive if $\text{Fix}(S) \neq \emptyset$ and

$$\phi(x^*, S(x)) \leq \phi(x^*, x), \quad \forall x^* \in \text{Fix}(S), \quad x \in C;$$

(iv) demiclosed at zero if for any sequence $\{x_n\} \subset C$ converging weakly to x such that $\{x_n - S(x_n)\}$ converges strongly to 0, one has $x \in C$ and $x \in \text{Fix}(S)$.

The fixed point set $\text{Fix}(S)$ of a relatively nonexpansive mapping S in a strictly convex and smooth Banach space is closed and convex [32, Proposition 2.4]. Since the proof in [32, Proposition 2.4] does not use the assumption $\text{Fix}(S) = \hat{F}(S)$, this conclusion is still true for the class of quasi- ϕ -nonexpansive mappings.

We have the following results.

Lemma 2.1. [51] *Let E be a uniformly convex Banach space and $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, +\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),$$

for all $\lambda \in [0, 1]$ and $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.2. [25] *Let E be an uniformly convex Banach space and $\bar{r} > 0$. Then, there exists a strictly increasing, continuous and convex function $\bar{g} : [0, 2\bar{r}] \rightarrow [0, +\infty)$ such that $\bar{g}(0) = 0$ and $\bar{g}(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_{\bar{r}} = \{z \in E : \|z\| \leq \bar{r}\}$.*

Lemma 2.3. [1] *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in an uniformly convex and uniformly smooth real Banach space E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.4. [7, 51] *Let E be a 2—uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where $\frac{1}{c}$ ($0 < c \leq 1$) is the 2—uniformly convex constant of E .

Lemma 2.5. [7, 8] *Let E be a 2—uniformly convex and smooth Banach space. Then, there exists $\tau > 0$ such that*

$$\langle x - y, Jx - Jy \rangle \geq \tau \|x - y\|^2, \quad \forall x, y \in E.$$

The generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(x, y).$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$. Thus, the generalized projection Π_C is the metric projection P_C . In what follows, we need the following properties of the functional ϕ and the generalized projection Π_C .

Lemma 2.6. [1] *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E . Then, the following conclusions hold:*

- (i) $\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \leq \phi(x, y), \forall x \in C, y \in E;$
- (ii) *if $x \in E, z \in C$, then $z = \Pi_C(x)$ iff $\langle y - z, Jx - Jz \rangle \leq 0, \forall y \in C;$*
- (iii) $\phi(x, y) = 0$ iff $x = y$.

Let E be a real Banach space. In [1], Alber studied the function $V : E \times E^* \rightarrow \Re$ defined by

$$V(x, x^*) = \|x\|^2 - 2 \langle x, x^* \rangle + \|x^*\|^2.$$

Then, from the definition of the Lyapunov functional $\phi(x, y)$, it follows that

$$V(x, x^*) = \phi(x, J^{-1}x^*).$$

Lemma 2.7. [1] *Let E be a reflexive, strictly convex and smooth Banach space with its dual E^* . Then*

$$V(x, x^*) + 2 \langle J^{-1}x - x^*, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

In the sequel, we use the following definitions:

The normal cone N_C to a set C at the point $x \in C$ is defined by

$$N_C(x) = \{x^* \in E^* : \langle x - y, x^* \rangle \geq 0, \quad \forall y \in C\}.$$

Let $g : C \rightarrow \Re$ be a function. The subdifferential of g at x is defined by

$$\partial g(x) = \{w \in E^* : g(y) - g(x) \geq \langle w, y - x \rangle, \quad \forall y \in C\}.$$

Remark 2.1. It follows easily from the definitions of the subdifferential and $\phi(x, y)$ that

$$\partial_1 \phi(x, y) := \partial \phi(\cdot, y)(x) = \{2(Jx - Jy)\}.$$

We have the following result, see [47, Section 7.2, Chapter 7] for more details.

Lemma 2.8. *Let C be a nonempty convex subset of a Banach space E and $g : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex, subdifferentiable and lower semicontinuous function. Furthermore, the function g satisfies the following regularity condition:*

Either $\text{int}(C) \neq \emptyset$ or g is continuous at a point in C .

Then, x^ is a solution to the following convex optimization problem $\min \{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone to C at x^* .*

We also need the following two technical lemmas to prove the strong convergence of the proposed algorithms.

Lemma 2.9. [42,52] *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n$$

for all $n \geq 0$, where the sequences $\{\gamma_n\}$ in $(0, 1)$ and $\{\delta_n\}$ in \mathbb{R} satisfy the conditions: $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.10. [28, Remark 4.4] *Let $\{\epsilon_n\}$ be a sequence of non-negative real numbers. Suppose that for any integer m , there exists an integer p such that $p \geq m$ and $\epsilon_p \leq \epsilon_{p+1}$. Let n_0 be an integer such that $\epsilon_{n_0} \leq \epsilon_{n_0+1}$ and define, for all integer $n \geq n_0$,*

$$\tau(n) = \max \{k \in \mathbb{N} : n_0 \leq k \leq n, \epsilon_k \leq \epsilon_{k+1}\}.$$

Then, $0 \leq \epsilon_n \leq \epsilon_{\tau(n)+1}$ for all $n \geq n_0$. Furthermore, the sequence $\{\tau(n)\}_{n \geq n_0}$ is non-decreasing and tends to $+\infty$ as $n \rightarrow \infty$.

3. Halpern extragradient method

In this section, we introduce an algorithm for finding a solution of an equilibrium problem which is also a solution of a fixed point problem. The algorithm combines the extragradient method with the Halpern iteration. The algorithm is designed as follows:

Algorithm 3.1. (Halpern extragradient method).

Initialization. *Choose $x_0, u \in C$. The sequences $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}$ satisfy Condition C below.*

Step 1. *Solve successively the two optimization problems*

$$y_n = \arg \min \left\{ \lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n) : y \in C \right\},$$

$$z_n = \arg \min \left\{ \lambda_n f(y_n, y) + \frac{1}{2} \phi(y, x_n) : y \in C \right\}.$$

Step 2. *Compute $x_{n+1} = \Pi_C (J^{-1}(\alpha_n J u + (1 - \alpha_n)(\beta_n J z_n + (1 - \beta_n) J S z_n)))$.*

Some explicit formulas for computing J and J^{-1} in the Banach spaces L^p , L^p and W_m^p can be found in [2, page 36]. In order to establish the convergence of Algorithm 3.1, we consider the following conditions for the bifunction f , the mapping S and the parameter sequences $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$.

Condition A:

A0. Either $\text{int}(C) \neq \emptyset$ or for each $x \in C$, the function $f(x, \cdot)$ is continuous at a point in C .

A1. f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.

A2. f is ϕ -Lipschitz-type continuous on C , i.e., there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\phi(x, y) - c_2\phi(y, z), \forall x, y \in C.$$

A3. $\limsup_{n \rightarrow \infty} f(x_n, y) \leq f(x, y)$ for each sequence $\{x_n\}$ converging weakly to $x \in C$ and for all $y \in C$.

A4. $f(x, \cdot)$ is convex and subdifferentiable on C for every $x \in C$.

Condition B:

B1. S is a quasi- ϕ -nonexpansive mapping.

B2. $I - S$ is demiclosed at zero.

Condition C:

C1. $0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \min \left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}$.

C2. $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

C3. $\{\beta_n\} \subset (0, 1)$, $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

It is easy to show that the solution set $\text{EP}(f, C)$ is closed and convex when f satisfies conditions A1, A3 and A4. Since the fixed point set $\text{Fix}(S)$ of S is closed and convex under condition B1, the solution set $F := \text{EP}(f, C) \cap \text{Fix}(S)$ is also closed and convex.

In this paper, we assume that the solution set $F := \text{EP}(f, C) \cap \text{Fix}(S)$ is nonempty, with the consequence that the generalized projection $\Pi_F(u)$ exists and is unique for all $u \in E$.

Let us mention here that when E is a Hilbert space, the ϕ -Lipschitz-type condition of f becomes the Lipschitz-type condition introduced by Mastroeni in [30].

In the following three lemmas, we assume that the nonempty closed convex set C is a subset of a uniformly smooth and uniformly convex Banach space E .

Lemma 3.1. *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm 3.1. Then, the followings hold for all $x^* \in \text{EP}(f, C)$, $y \in C$ and $n \geq 0$.*

- (i) $\lambda_n f(y_n, y) \geq \langle Jy_n - Jx_n, y_n - z_n \rangle + \langle Jz_n - Jx_n, z_n - y \rangle - c_1 \lambda_n \phi(y_n, x_n) - c_2 \lambda_n \phi(z_n, y_n)$.
- (ii) $\phi(x^*, z_n) \leq \phi(x^*, x_n) - (1 - 2\lambda_n c_1) \phi(y_n, x_n) - (1 - 2\lambda_n c_2) \phi(z_n, y_n)$.

Proof. Let n be fixed.

(i) From the definition of y_n and Lemma 2.8, we have

$$0 \in \lambda_n \partial_2 f(x_n, y_n) + \frac{1}{2} \partial_1 \phi(y_n, x_n) + N_C(y_n).$$

Thus, there exist $w \in \partial_2 f(x_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\lambda_n w + Jy_n - Jx_n + \bar{w} = 0.$$

Hence,

$$\langle Jy_n - Jx_n, y_n - y \rangle = \lambda_n \langle w, y - y_n \rangle + \langle \bar{w}, y - y_n \rangle, \forall y \in C.$$

This together with the definition of N_C implies that

$$\langle Jy_n - Jx_n, y_n - y \rangle \leq \lambda_n \langle w, y - y_n \rangle, \forall y \in C.$$

Since $w \in \partial_2 f(x_n, y_n)$, we have

$$f(x_n, y) - f(x_n, y_n) \geq \langle w, y - y_n \rangle, \forall y \in C.$$

From the last two inequalities, we obtain

$$\lambda_n (f(x_n, y) - f(x_n, y_n)) \geq \langle Jy_n - Jx_n, y_n - y \rangle, \forall y \in C. \tag{8}$$

Substituting $y = z_n$ into the last inequality, we obtain

$$\lambda_n (f(x_n, z_n) - f(x_n, y_n)) \geq \langle Jy_n - Jx_n, y_n - z_n \rangle.$$

From the ϕ -Lipschitz-type condition of f , we have

$$f(y_n, z_n) \geq f(x_n, z_n) - f(x_n, y_n) - c_1 \phi(y_n, x_n) - c_2 \phi(z_n, y_n).$$

Two last inequalities imply that

$$\lambda_n f(y_n, z_n) \geq \langle Jy_n - Jx_n, y_n - z_n \rangle - c_1 \lambda_n \phi(y_n, x_n) - c_2 \lambda_n \phi(z_n, y_n). \tag{9}$$

Similarly to inequality (8), from the definition of z_n , we obtain

$$\lambda_n (f(y_n, y) - f(y_n, z_n)) \geq \langle Jz_n - Jx_n, z_n - y \rangle, \forall y \in C. \tag{10}$$

Thus,

$$\lambda_n f(y_n, y) \geq \lambda_n f(y_n, z_n) + \langle Jz_n - Jx_n, z_n - y \rangle, \forall y \in C.$$

This together with inequality (9) comes to the desired conclusion.

(ii) Substituting $y = x^*$ into (10), we obtain

$$\lambda_n (f(y_n, x^*) - f(y_n, z_n)) \geq \langle Jz_n - Jx_n, z_n - x^* \rangle.$$

Since $x^* \in \text{EP}(f, C)$, $f(x^*, y_n) \geq 0$. It follows from the pseudomonotonicity of f that $f(y_n, x^*) \leq 0$. This together with the last inequality implies that

$$\langle Jz_n - Jx_n, x^* - z_n \rangle \geq \lambda_n f(y_n, z_n). \tag{11}$$

It follows from relations (9) and (11) that

$$\begin{aligned} 2 \langle Jz_n - Jx_n, x^* - z_n \rangle &\geq 2 \langle Jy_n - Jx_n, y_n - z_n \rangle - 2c_1 \lambda_n \phi(y_n, x_n) \\ &\quad - 2c_2 \lambda_n \phi(z_n, y_n). \end{aligned}$$

This together with relation (6) leads to the desired conclusion. Lemma 3.1 is proved. □

Next, we will prove the boundedness of all the sequences generated by Algorithm 3.1.

Lemma 3.2. *The sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{Sz_n\}$ are bounded.*

Proof. Put $t_n = J^{-1}(\beta_n Jz_n + (1 - \beta_n)JSz_n)$. Thus, $x_{n+1} = \Pi_C(J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n))$. It follows from Lemma 2.6(i) and relation (7) that, for all $x^* \in F := \text{EP}(f, C) \cap \text{Fix}(S)$,

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, \Pi_C(J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n))) \\ &\leq \phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)) \\ &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, t_n). \end{aligned} \tag{12}$$

From relation (7), the property of S and Lemma 3.1(ii),

$$\begin{aligned} \phi(x^*, t_n) &= \phi(x^*, J^{-1}(\beta_n Jz_n + (1 - \beta_n)JSz_n)) \\ &\leq \beta_n \phi(x^*, z_n) + (1 - \beta_n) \phi(x^*, Sz_n) \\ &\leq \beta_n \phi(x^*, z_n) + (1 - \beta_n) \phi(x^*, z_n) \\ &= \phi(x^*, z_n) \leq \phi(x^*, x_n). \end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, x_n) \\ &\leq \max\{\phi(x^*, u), \phi(x^*, x_n)\} \\ &\leq \dots \\ &\leq \max\{\phi(x^*, u), \phi(x^*, x_0)\}. \end{aligned}$$

This implies that the sequence $\{\phi(x^*, x_n)\}$ is bounded. Thus, from (13), we also obtain the boundedness of the sequences $\{\phi(x^*, z_n)\}$ and $\{\phi(x^*, t_n)\}$. Since $\phi(x^*, Sz_n) \leq \phi(x^*, z_n)$ for all n , the sequence $\{\phi(x^*, Sz_n)\}$ is bounded. This together with relation (5) implies that the sequences $\{x_n\}$, $\{z_n\}$, $\{Sz_n\}$ are bounded. The boundedness of the sequence $\{y_n\}$ follows from Lemma 3.1(ii), the hypothesis of λ_n and relation (5). This completes the proof of Lemma 3.2. \square

It follows from Lemma 3.2 that there exists $r > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\}, \{Sz_n\} \subset B_r = \{z \in E : \|z\| \leq r\}$ for all $n \geq 0$. From Lemma 2.1 and the definition of J , there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, +\infty)$ such that $g(0) = 0$ and for all $n \geq 0$,

$$\|\beta_n Jz_n + (1 - \beta_n)JSz_n\|^2 \leq \beta_n \|z_n\|^2 + (1 - \beta_n) \|Sz_n\|^2 - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|). \tag{14}$$

Put $w_n = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)$ and $x^\dagger = \Pi_F(u)$, where $t_n = J^{-1}(\beta_n Jz_n + (1 - \beta_n)JSz_n)$, and

$$T_n = (1 - 2\lambda_n c_1) \phi(y_n, x_n) + (1 - 2\lambda_n c_2) \phi(z_n, y_n) + \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|). \tag{15}$$

From Lemma 3.2 and the conditions on $\{\lambda_n\}$ and $\{\beta_n\}$, we deduce that the sequence $\{T_n\}$ is bounded. Thus, there exists $M > 0$ such that

$$M = \sup_{n \geq 0} \{\phi(x^\dagger, u) + T_n\}. \tag{16}$$

We have the following result which plays an important role in proving the strong convergence of Algorithm 3.1.

- Lemma 3.3.** (i) $T_n \leq \phi(x^\dagger, x_n) - \phi(x^\dagger, x_{n+1}) + \alpha_n M$ for all n .
 (ii) $\phi(x^\dagger, x_{n+1}) \leq (1 - \alpha_n) \phi(x^\dagger, x_n) + 2\alpha_n \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle$ for all n .

Proof. From the definitions of t_n , ϕ , relation (14) and the property of S , we have

$$\begin{aligned} \phi(x^\dagger, t_n) &= \|x^\dagger\|^2 - 2\langle x^\dagger, Jt_n \rangle + \|t_n\|^2. \\ &= \|x^\dagger\|^2 - 2\langle x^\dagger, \beta_n Jz_n + (1 - \beta_n)JSz_n \rangle + \|\beta_n Jz_n + (1 - \beta_n)JSz_n\|^2. \\ &\leq \|x^\dagger\|^2 - 2\langle x^\dagger, \beta_n Jz_n + (1 - \beta_n)JSz_n \rangle + \beta_n \|z_n\|^2 + (1 - \beta_n)\|Sz_n\|^2 \\ &\quad - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|) \\ &= \beta_n\phi(x^\dagger, z_n) + (1 - \beta_n)\phi(x^\dagger, Sz_n) - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|) \\ &\leq \beta_n\phi(x^\dagger, z_n) + (1 - \beta_n)\phi(x^\dagger, z_n) - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|) \\ &= \phi(x^\dagger, z_n) - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|). \end{aligned}$$

This together with relation (12) with $x^* = x^\dagger$ and Lemma 3.1 (ii) with $x^* = x^\dagger$ implies that

$$\begin{aligned} \phi(x^\dagger, x_{n+1}) &\leq \alpha_n\phi(x^\dagger, u) + (1 - \alpha_n) \left[\phi(x^\dagger, z_n) - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|) \right] \\ &\leq \alpha_n\phi(x^\dagger, u) + (1 - \alpha_n) \left[\phi(x^\dagger, x_n) - (1 - 2\lambda_n c_1)\phi(y_n, x_n) - (1 - 2\lambda_n c_2)\phi(z_n, y_n) \right. \\ &\quad \left. - \beta_n(1 - \beta_n)g(\|Jz_n - JSz_n\|) \right] \\ &= \alpha_n\phi(x^\dagger, u) + (1 - \alpha_n) \left[\phi(x^\dagger, x_n) - T_n \right] \\ &= (1 - \alpha_n)\phi(x^\dagger, x_n) + \alpha_n \left[\phi(x^\dagger, u) + T_n \right] - T_n \\ &\leq \phi(x^\dagger, x_n) + M\alpha_n - T_n. \end{aligned}$$

Hence, we obtain the desired conclusion.

(ii) Note that $x_{n+1} = \Pi_C w_n$ where $w_n = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)$. From Lemmas 2.6(i), 2.7, the definitions of ϕ and V , and relation (7), we obtain

$$\begin{aligned} \phi(x^\dagger, x_{n+1}) &= \phi(x^\dagger, \Pi_C w_n) \leq \phi(x^\dagger, w_n) \\ &= \phi(x^\dagger, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)) \\ &= V(x^\dagger, \alpha_n Ju + (1 - \alpha_n)Jt_n) \\ &\leq V(x^\dagger, \alpha_n Ju + (1 - \alpha_n)Jt_n - \alpha_n(Ju - Jx^\dagger)) \\ &\quad - 2\langle J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n) - x^\dagger, -\alpha_n(Ju - Jx^\dagger) \rangle \\ &= V(x^\dagger, \alpha_n Jx^\dagger + (1 - \alpha_n)Jt_n) + 2\alpha_n \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle \\ &= \phi(x^\dagger, J^{-1}(\alpha_n Jx^\dagger + (1 - \alpha_n)Jt_n)) + 2\alpha_n \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle \\ &\leq \alpha_n\phi(x^\dagger, x^\dagger) + (1 - \alpha_n)\phi(x^\dagger, t_n) + 2\alpha_n \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle \\ &\leq (1 - \alpha_n)\phi(x^\dagger, x_n) + 2\alpha_n \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle. \end{aligned}$$

Now, we prove the first main result. □

Theorem 3.1. *Let C be a nonempty closed convex subset of an uniformly smooth and uniformly convex Banach space E . Assume that Conditions A, B and C are satisfied and that the solution set $F := \text{EP}(f, C) \cap \text{Fix}(S)$ is nonempty. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by Algorithm 3.1 converge strongly to $\Pi_F(u)$.*

Proof. We use the two inequalities of Lemma 3.3 to prove Theorem 3.1. Recalling the first inequality in that lemma, we have, for all n , that

$$T_n \leq \phi(x^\dagger, x_n) - \phi(x^\dagger, x_{n+1}) + \alpha_n M, \tag{17}$$

where T_n and M are defined, respectively, by (15) and (16), and $x^\dagger = \Pi_F(u)$. Then, we consider two cases.

Case 1. There exists $n_0 \geq 0$ such that the sequence $\{\phi(x^\dagger, x_n)\}$ is nonincreasing.

In this case, the limit of $\{\phi(x^\dagger, x_n)\}$ exists. Thus, from (17) and $\alpha_n \rightarrow 0$, we have $\lim_{n \rightarrow \infty} T_n = 0$. From the definition of T_n and the hypotheses on λ_n and β_n , we obtain

$$\phi(y_n, x_n) \rightarrow 0, \phi(z_n, y_n) \rightarrow 0, g(\|Jz_n - JSz_n\|) \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{18}$$

Thus, from Lemmas 2.1, 2.3 and 3.2 we have, when $n \rightarrow \infty$, that

$$\|x_n - y_n\| \rightarrow 0, \|z_n - y_n\| \rightarrow 0, \|Jz_n - JSz_n\| \rightarrow 0, \tag{19}$$

and so $\|x_n - z_n\| \rightarrow 0$ by the triangle inequality. Since J and J^{-1} are uniformly continuous on each bounded subset of E and E^* (resp.), we obtain

$$\|Jx_n - Jy_n\| \rightarrow 0, \|Jx_n - Jz_n\| \rightarrow 0, \|z_n - Sz_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{20}$$

Since $\{z_n\} \subset C$ is bounded, there exists a subsequence $\{z_m\}$ of $\{z_n\}$ converging weakly to $p \in C$ such that

$$\limsup_{n \rightarrow \infty} \langle z_n - x^\dagger, Ju - Jx^\dagger \rangle = \lim_{m \rightarrow \infty} \langle z_m - x^\dagger, Ju - Jx^\dagger \rangle. \tag{21}$$

From $\|z_n - Sz_n\| \rightarrow 0$ as $n \rightarrow \infty$, and the demi-closedness at zero of S , we have that $p \in \text{Fix}(S)$. From Lemma 3.1(i), relations (18), (20), the hypothesis on λ_n and (A3) and noting that $y_n \rightarrow p$ as $n \rightarrow \infty$, we obtain that

$$0 \leq \limsup_{n \rightarrow \infty} f(y_n, y) \leq f(p, y), \forall y \in C.$$

Thus, $p \in \text{EP}(f, C)$ and $p \in F := \text{EP}(f, C) \cap \text{Fix}(S)$. From (21), $x^\dagger = \Pi_{Fu}$ and Lemma 2.6 (ii), we have

$$\limsup_{n \rightarrow \infty} \langle z_n - x^\dagger, Ju - Jx^\dagger \rangle = \lim_{m \rightarrow \infty} \langle p - x^\dagger, Ju - Jx^\dagger \rangle \leq 0. \tag{22}$$

From $w_n = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)$, $\alpha_n \rightarrow 0$ and the boundedness of $\{Ju - Jt_n\}$, we obtain

$$\|Jw_n - Jt_n\| = \alpha_n \|Ju - Jt_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since J^{-1} is uniformly continuous on each bounded subset of E^* , we deduce that $\|w_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. From $t_n = J^{-1}(\beta_n Jz_n + (1 - \beta_n)JSz_n)$ and (20), we have

$$\|Jt_n - JSz_n\| = \beta_n \|Jz_n - JSz_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, $\|t_n - Sz_n\| \rightarrow 0$ and, from (19), it follows that $\|t_n - z_n\| \rightarrow 0$. Hence, $\|w_n - z_n\| \rightarrow 0$ because $\|w_n - t_n\| \rightarrow 0$. This together with (22) implies that

$$\limsup_{n \rightarrow \infty} \langle w_n - x^\dagger, Ju - Jx^\dagger \rangle = \limsup_{n \rightarrow \infty} \langle z_n - x^\dagger, Ju - Jx^\dagger \rangle \leq 0. \tag{23}$$

Thus, from Lemmas 2.9 and 3.3 (ii), we obtain $\phi(x^\dagger, x_n) \rightarrow 0$ or $x_n \rightarrow x^\dagger$.

Case 2. There exists a subsequence $\{\phi(x^\dagger, x_{n_i})\}$ of $\{\phi(x^\dagger, x_n)\}$ such that $\phi(x^\dagger, x_{n_i}) \leq \phi(x^\dagger, x_{n_i+1})$ for all $i \geq 0$.

In that case, it follows from Lemma 2.10 that

$$\phi(x^\dagger, x_{\tau(n)}) \leq \phi(x^\dagger, x_{\tau(n)+1}), \phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_{\tau(n)+1}), \forall n \geq n_0, \quad (24)$$

where $\tau(n) = \max \{k \in N : n_0 \leq k \leq n, \phi(x^\dagger, x_k) \leq \phi(x^\dagger, x_{k+1})\}$. Furthermore, the sequence $\{\tau(n)\}_{n \geq n_0}$ is non-decreasing and $\tau(n) \rightarrow +\infty$ as $n \rightarrow \infty$. From relations (17), (24), and $\alpha_{\tau(n)} \rightarrow 0$, we have

$$T_{\tau(n)} \leq \phi(x^\dagger, x_{\tau(n)}) - \phi(x^\dagger, x_{\tau(n)+1}) + \alpha_{\tau(n)}M \leq \alpha_{\tau(n)}M \rightarrow 0 \text{ as } n \rightarrow \infty.$$

From the definition of $T_{\tau(n)}$ and the hypotheses on $\lambda_{\tau(n)}$ and $\beta_{\tau(n)}$, we obtain

$$\phi(y_{\tau(n)}, x_{\tau(n)}) \rightarrow 0, \phi(z_{\tau(n)}, y_{\tau(n)}) \rightarrow 0, g(\|Jz_{\tau(n)} - JSz_{\tau(n)}\|) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (25)$$

Thus, from Lemmas 2.1, 2.3 and 3.2 we have, when $n \rightarrow \infty$, that

$$\|x_{\tau(n)} - y_{\tau(n)}\| \rightarrow 0, \|z_{\tau(n)} - y_{\tau(n)}\| \rightarrow 0, \|Jz_{\tau(n)} - JSz_{\tau(n)}\| \rightarrow 0, \quad (26)$$

and so $\|x_{\tau(n)} - z_{\tau(n)}\| \rightarrow 0$. Since J and J^{-1} are uniformly continuous on each bounded subset of E and E^* (resp.),

$$\|Jx_{\tau(n)} - Jy_{\tau(n)}\| \rightarrow 0, \|Jx_{\tau(n)} - Jz_{\tau(n)}\| \rightarrow 0, \|z_{\tau(n)} - Sz_{\tau(n)}\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (27)$$

Since $\{z_{\tau(n)}\} \subset C$ is bounded, there exists a subsequence $\{z_{\tau(m)}\}$ of $\{z_{\tau(n)}\}$ converging weakly to $p \in C$ such that

$$\limsup_{n \rightarrow \infty} \langle z_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle = \lim_{m \rightarrow \infty} \langle z_{\tau(m)} - x^\dagger, Ju - Jx^\dagger \rangle. \quad (28)$$

By arguing as in Case 1, we have $p \in F$ and

$$\limsup_{n \rightarrow \infty} \langle w_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle = \limsup_{n \rightarrow \infty} \langle z_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle \leq 0. \quad (29)$$

From Lemma 3.3(ii), one has

$$\phi(x^\dagger, x_{\tau(n)+1}) \leq (1 - \alpha_{\tau(n)})\phi(x^\dagger, x_{\tau(n)}) + 2\alpha_{\tau(n)} \langle w_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle, \quad (30)$$

which, from relation (24), implies that

$$\begin{aligned} \alpha_{\tau(n)}\phi(x^\dagger, x_{\tau(n)}) &\leq \phi(x^\dagger, x_{\tau(n)}) - \phi(x^\dagger, x_{\tau(n)+1}) + 2\alpha_{\tau(n)} \langle w_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle \\ &\leq 2\alpha_{\tau(n)} \langle w_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle. \end{aligned}$$

Thus, it follows from $\alpha_{\tau(n)} > 0$ that $\phi(x^\dagger, x_{\tau(n)}) \leq 2 \langle w_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle$. This together with (29) implies that $\limsup_{n \rightarrow \infty} \phi(x^\dagger, x_{\tau(n)}) \leq 0$ or $\lim_{n \rightarrow \infty} \phi(x^\dagger, x_{\tau(n)}) = 0$. Thus, one obtains from (30) and $\alpha_{\tau(n)} \rightarrow 0$ that $\phi(x^\dagger, x_{\tau(n)+1}) \rightarrow 0$. Since $\phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_{\tau(n)+1})$ for all $n \geq n_0$ by relation (24), one has that $\phi(x^\dagger, x_n) \rightarrow 0$ and thus that $x_n \rightarrow x^\dagger$ as $n \rightarrow \infty$. This completes the proof of Theorem 3.1. \square

4. Halpern linesearch method

The convergence of Algorithm 3.1 is established under hypothesis A2 of ϕ -Lipschitz-type condition of the bifunction f depending on two constants c_1 and c_2 . In some cases, these constants are not known or difficult to approximate. In this section, we use the linesearch technique to avoid this assumption. We also assume that the Banach space E is uniformly smooth and 2-uniformly convex with the constant c defined as in Lemma 2.4. Without assuming hypothesis A2, however, we have to slightly strengthen hypothesis A3. More precisely, we assume that the mapping S satisfies Condition B and the bifunction f satisfies the following condition.

Condition D: Assumptions A0, A1, A4 hold and

A3a. f is jointly weakly continuous on the product $\Delta \times \Delta$ where Δ is an open convex set containing C , in the sense that if $x, y \in \Delta$ and $\{x_n\}, \{y_n\}$ are two sequences in Δ converging weakly to x and y , respectively, then $f(x_n, y_n) \rightarrow f(x, y)$.

It is easy to see that condition A3a implies condition A3. So the solution set $EP(f, C)$ is closed and convex. Furthermore, the fixed point set $Fix(S)$ of S being closed and convex, the solution set $F = EP(f, C) \cap Fix(S)$ is also closed and convex. As Section 3, we assume that the solution set F is nonempty. Consequently, the generalized projection $\Pi_F(u)$ exists and is unique for all $u \in E$.

On the other hand, we have to use some other parameters ν, γ, α , and since f is not assumed to satisfy condition A2, we have to consider another hypothesis on the sequence $\{\lambda_n\}$. More precisely, we introduce the following condition:

Condition E: The sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy C2, C3 and

C1a. $\{\lambda_n\} \subset [\lambda, 1]$, where $\lambda \in (0, 1)$.

C4. $\nu \in (0, \frac{c^2}{2}), \gamma \in (0, 1), \alpha \in (0, 1)$.

Now, we are in a position to formulate the second algorithm:

Algorithm 4.1. (Halpern linesearch method).

Initialization. Choose $x_0, u \in C$, the parameters ν, γ, α and the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ such that Condition E above is satisfied .

Step 1. Solve the optimization problem

$$y_n = \arg \min \left\{ \lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n) : y \in C \right\}.$$

Step 2. If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 2.1. Find m the smallest nonnegative integer such that

$$\begin{cases} z_{m,n} = (1 - \gamma^m)x_n + \gamma^m y_n, \\ f(z_{m,n}, x_n) - f(z_{m,n}, y_n) \geq \frac{\alpha}{2\lambda_n} \phi(y_n, x_n). \end{cases}$$

Step 2.2. Set $\rho_n = \gamma^m$ and $z_n = z_{m,n}$.

Step 3. Choose $g_n \in \partial_2 f(z_n, x_n)$ and compute $w_n = \Pi_C J^{-1}(Jx_n - \sigma_n g_n)$, where $\sigma_n = \frac{\nu f(z_n, x_n)}{\|g_n\|^2}$ if $y_n \neq x_n$ and $\sigma_n = 0$ otherwise.

Step 4. Compute $x_{n+1} = \Pi_C J^{-1}(\alpha_n J u + (1 - \alpha_n)(\beta_n J w_n + (1 - \beta_n) J S w_n))$.

To prove the strong convergence of this algorithm, we need the following lemmas where we assume that the nonempty closed convex set C is a subset of a uniformly smooth and 2-uniformly convex Banach space E .

Lemma 4.1. *Suppose that $y_n \neq x_n$ for some $n \geq 0$. Then, for each n ,*

- (i) *The linesearch corresponding to x_n and y_n (Step 2.1) is well defined.*
- (ii) *$f(z_n, x_n) > 0$.*
- (iii) *$0 \notin \partial_2 f(z_n, x_n)$.*

Proof. Repeating the proof as in [48, Proposition 4.1] and replacing $\|x_n - y_n\|^2$ by $\phi(y_n, x_n)$, we obtain the desired conclusion. □

Lemma 4.2. *Let $f : \Delta \times \Delta$ be a bifunction satisfying conditions A3a – A4. Let $x, z \in \Delta$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences in Δ converging weakly to x, z , respectively. Then, for any $\epsilon > 0$, there exist $\eta > 0$ and $n_\epsilon > 0$ such that*

$$\partial_2 f(z_n, x_n) \subset \partial_2 f(z, x) + \frac{\epsilon}{\eta} B$$

for every $n \geq n_\epsilon$, where B denotes the closed unit ball in E .

Proof. See, e.g., [48, Proposition 4.3]. □

In the sequel, let us denote $t_n = J^{-1}(\beta_n Jw_n + (1 - \beta_n)JSw_n)$ where w_n is defined as in Step 3 of Algorithm 4.1. Then, it follows from the definition of x_{n+1} that

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n).$$

Next, we prove the boundedness of the sequences generated by Algorithm 4.1.

Lemma 4.3. (i) *The sequences $\{x_n\}, \{w_n\}, \{t_n\}, \{Sw_n\}$ generated by Algorithm 4.1 are bounded.*

- (ii) *If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to $x \in C$, then the subsequences $\{y_{n_k}\}, \{z_{n_k}\}$ and $\{g_{n_k}\}$ are bounded.*

Proof. (i) Let n be fixed. By arguing as in relations (12) and (13), we have

$$\phi(x^*, x_{n+1}) \leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, t_n). \tag{31}$$

$$\phi(x^*, t_n) \leq \phi(x^*, w_n). \tag{32}$$

From Lemmas 2.6(i), 2.7 and the definitions of ϕ and V , we obtain successively that

$$\begin{aligned} \phi(x^*, w_n) &= \phi(x^*, \Pi_C(J^{-1}(Jx_n - \sigma_n g_n))) \\ &\leq \phi(x^*, J^{-1}(Jx_n - \sigma_n g_n)) - \phi(\Pi_C(J^{-1}(Jx_n - \sigma_n g_n)), J^{-1}(Jx_n - \sigma_n g_n)) \\ &= V(x^*, Jx_n - \sigma_n g_n) - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \\ &\leq V(x^*, Jx_n - \sigma_n g_n + \sigma_n g_n) - 2 \langle J^{-1}(Jx_n - \sigma_n g_n) - x^*, \sigma_n g_n \rangle \\ &\quad - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \\ &= \phi(x^*, x_n) - 2 \langle J^{-1}(Jx_n - \sigma_n g_n) - x^*, \sigma_n g_n \rangle - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \\ &= \phi(x^*, x_n) - 2\sigma_n \langle x_n - x^*, g_n \rangle - 2 \langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, \sigma_n g_n \rangle \\ &\quad - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)). \end{aligned} \tag{33}$$

Since $x^* \in \text{EP}(f, C)$, we have that $f(x^*, z_n) \geq 0$. Then, it follows from the pseudomonotonicity of f that $f(z_n, x^*) \leq 0$. Consequently, from the definition of g_n and σ_n , we have the following inequalities:

$$\langle x_n - x^*, g_n \rangle \geq f(z_n, x_n) - f(z_n, x^*) \geq f(z_n, x_n) = \frac{\sigma_n \|g_n\|^2}{\nu}. \tag{34}$$

From Lemma 2.4, we can write

$$\begin{aligned} -2 \langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, \sigma_n g_n \rangle &= 2 \langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, -\sigma_n g_n \rangle \\ &\leq 2 \|J^{-1}(Jx_n - \sigma_n g_n) - x_n\| \|\sigma_n g_n\| \\ &\leq \frac{4}{c^2} \|JJ^{-1}(Jx_n - \sigma_n g_n) - Jx_n\| \|\sigma_n g_n\| \\ &= \frac{4\sigma_n^2}{c^2} \|g_n\|^2. \end{aligned} \tag{35}$$

Combining relations (33)–(35) and the hypothesis on ν , we obtain that

$$\phi(x^*, w_n) \leq \phi(x^*, x_n) - \left(\frac{2}{\nu} - \frac{4}{c^2}\right) \sigma_n^2 \|g_n\|^2 - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \leq \phi(x^*, x_n). \tag{36}$$

Consequently, from relations (31), (32) and (36), deduce that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, t_n) \\ &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, w_n) \\ &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n) \phi(x^*, x_n) \\ &\leq \max \{ \phi(x^*, u), \phi(x^*, x_n) \} \\ &\leq \dots \\ &\leq \max \{ \phi(x^*, u), \phi(x^*, x_0) \} \end{aligned}$$

Thus, the sequence $\{\phi(x^*, x_n)\}$ is bounded and it follows from relations (36) and (32) that the sequences $\{\phi(x^*, w_n)\}$ and $\{\phi(x^*, t_n)\}$ are also bounded. Moreover, from the property of S , we have that $\phi(x^*, Sw_n) \leq \phi(x^*, w_n)$ for all $n \geq 0$. Thus, the sequence $\{\phi(x^*, Sw_n)\}$ is bounded. This together with relation (5) gives the desired conclusion.

(ii) Firstly, we show that the sequence $\{y_{n_k}\}$ is bounded. Without loss of generality, we can suppose that $y_{n_k} \neq x_{n_k}$ for all k . Now let k be fixed and define the function T_k by

$$T_k(y) = \lambda_{n_k} f(x_{n_k}, y) + \frac{1}{2} \phi(y, x_{n_k}), \quad y \in C.$$

The subdifferential of T_k at y is given by

$$\partial T_k(y) = \lambda_{n_k} \partial_2 f(x_{n_k}, y) + \frac{1}{2} \partial_1 \phi(y, x_{n_k}).$$

Note that $\frac{1}{2} \partial_1 \phi(y, x_{n_k}) = \{Jy - Jx_{n_k}\}$. Furthermore, for all $y_1, y_2 \in C$ and $s(y_1) \in \partial T_k(y_1)$, $s(y_2) \in \partial T_k(y_2)$, it is easy to see that there exists $w_1 \in \partial_2 f(x_{n_k}, y_1)$ and $w_2 \in \partial_2 f(x_{n_k}, y_2)$ such that $s(y_1) = \lambda_{n_k} w_1 + Jy_1 - Jx_{n_k}$, $s(y_2) = \lambda_{n_k} w_2 + Jy_2 - Jx_{n_k}$ and

$$\langle s(y_2) - s(y_1), y_2 - y_1 \rangle = \lambda_{n_k} \langle w_2 - w_1, y_2 - y_1 \rangle + \langle Jy_2 - Jy_1, y_2 - y_1 \rangle. \tag{37}$$

From $w_1 \in \partial_2 f(x_{n_k}, y_1)$ and $w_2 \in \partial_2 f(x_{n_k}, y_2)$, we have

$$\begin{aligned} f(x_{n_k}, y_2) - f(x_{n_k}, y_1) &\geq \langle w_1, y_2 - y_1 \rangle. \\ f(x_{n_k}, y_1) - f(x_{n_k}, y_2) &\geq \langle w_2, y_1 - y_2 \rangle. \end{aligned}$$

Adding both sides of the last two inequalities, we obtain

$$\langle w_2 - w_1, y_2 - y_1 \rangle \geq 0. \tag{38}$$

From relations (37), (38) and Lemma 2.5, there exists $\tau > 0$ such that

$$\langle s(y_2) - s(y_1), y_2 - y_1 \rangle \geq \tau \|y_2 - y_1\|^2. \tag{39}$$

Note that $0 \in \partial T_k(y_{n_k}) + N_C(y_{n_k})$ by definition of y_{n_k} and Lemma 2.8. Thus, there exists $s(y_{n_k}) \in \partial T_k(y_{n_k})$ such that $-s(y_{n_k}) \in N_C(y_{n_k})$, i.e.,

$$\langle -s(y_{n_k}), y - y_{n_k} \rangle \leq 0, \quad \forall y \in C.$$

Substituting $y = x_{n_k}$ into the last inequality, we have $\langle s(y_{n_k}), x_{n_k} - y_{n_k} \rangle \geq 0$. Hence, it follows from (39) with $y_2 = x_{n_k}$ and $y_1 = y_{n_k}$ that

$$\langle s(x_{n_k}), x_{n_k} - y_{n_k} \rangle \geq \langle s(y_{n_k}), x_{n_k} - y_{n_k} \rangle + \tau \|x_{n_k} - y_{n_k}\|^2 \geq \tau \|x_{n_k} - y_{n_k}\|^2. \tag{40}$$

Thus

$$\|x_{n_k} - y_{n_k}\| \leq \frac{1}{\tau} \|s(x_{n_k})\| \text{ for every } s(x_{n_k}) \in \partial T_k(x_{n_k}). \tag{41}$$

Since $\frac{1}{2} \partial_1 \phi(x_{n_k}, x_{n_k}) = 0$, $\partial T_k(x_{n_k}) = \lambda_{n_k} \partial_2 f(x_{n_k}, x_{n_k})$. Moreover, it follows from $x_{n_k} \rightarrow x$ and Lemma 4.2 that, for any $\epsilon > 0$, there exist $\eta > 0$ and $k_0 > 0$ such that

$$\partial_2 f(x_{n_k}, x_{n_k}) \subset \partial_2 f(x, x) + \frac{\epsilon}{\eta} B$$

for every $k \geq k_0$, where B denotes the closed unit ball in E . Thus, we can conclude from the boundedness of $\partial_2 f(x, x)$ and the hypothesis on λ_{n_k} and (41) that the sequence $\{\|x_{n_k} - y_{n_k}\|\}$ is bounded. Since the sequence $\{x_{n_k}\}$ is bounded, the sequence $\{y_{n_k}\}$ is also bounded. Thus, it follows from the definition of z_{n_k} that the sequence $\{z_{n_k}\}$ is also bounded. Hence, there exists a subsequence of $\{z_{n_k}\}$, again denoted by $\{z_{n_k}\}$, converging weakly to z . It also follows from $x_{n_k} \rightarrow x$, $z_{n_k} \rightarrow z$ and Lemma 4.2 that, for any $\epsilon > 0$, there exist $\eta > 0$ and $k_0 > 0$ such that

$$\partial_2 f(z_{n_k}, x_{n_k}) \subset \partial_2 f(z, x) + \frac{\epsilon}{\eta} B$$

for every $k \geq k_0$, where B denotes the closed unit ball in E . Since $g_{n_k} \in \partial_2 f(z_{n_k}, x_{n_k})$ and since B and $\partial_2 f(z, x)$ are bounded, we also obtain that the sequence $\{g_{n_k}\}$ is bounded. This completes the proof of Lemma 4.3. \square

It follows from Lemma 4.3(i) that there exists $r > 0$ such that the sequences $\{x_n\}$, $\{w_n\}$, $\{t_n\}$ and $\{S w_n\}$ are all contained in the ball $B_r = \{z \in E : \|z\| \leq r\}$ for all $n \geq 0$. From Lemma 2.1 and the definition of J , there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, +\infty)$ such that $g(0) = 0$, and the following inequality holds for all $n \geq 0$

$$\|\beta_n J w_n + (1 - \beta_n) J S w_n\|^2 \leq \beta_n \|w_n\|^2 + (1 - \beta_n) \|S w_n\|^2 - \beta_n (1 - \beta_n) g(\|J w_n - J S w_n\|). \tag{42}$$

Lemma 4.4. *Let $\{x_{n_k}\}$ be a subsequence of the sequence $\{x_n\}$ converging weakly to $\bar{x} \in C$. Assume that*

$$\sigma_{n_k} \|g_{n_k}\| \rightarrow 0, \quad \phi(w_{n_k}, J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})) \rightarrow 0, \quad g(\|Jw_{n_k} - JSw_{n_k}\|) \rightarrow 0 \tag{43}$$

as $k \rightarrow \infty$. Then, $\bar{x} \in F := \text{EP}(f, C) \cap \text{Fix}(S)$.

Proof. From the property of function g and hypothesis (43), we have that $\|Jw_{n_k} - JSw_{n_k}\| \rightarrow 0$, and since J^{-1} is uniformly continuous on each bounded set, we can write

$$\|w_{n_k} - Sw_{n_k}\| \rightarrow 0. \tag{44}$$

Then, it follows from (43), Lemma 4.3(i) that the sequences $\{\sigma_{n_k}g_{n_k}\}$, $\{x_{n_k}\}$, $\{w_{n_k}\}$ are bounded. Since J and J^{-1} are uniformly continuous on every bounded set of E and E^* (resp.), the sequence $\{J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\}$ is also bounded. Thus, there exists $\bar{r} > 0$ such that for all k ,

$$\{J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\}, \{w_{n_k}\} \subset B_{\bar{r}} = \{z \in E : \|z\| \leq \bar{r}\}.$$

By Lemma 2.2, there exists a strictly increasing, continuous and convex function $\bar{g} : [0, 2\bar{r}] \rightarrow [0, +\infty)$ such that $\bar{g}(0) = 0$ and

$$\bar{g}(\|w_{n_k} - J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\|) \leq \phi(w_{n_k}, J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})), \text{ for all } k.$$

Thus, from (43), we obtain $\bar{g}(\|w_{n_k} - J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\|) \rightarrow 0$ as $k \rightarrow \infty$. It follows from the property of \bar{g} that $\|w_{n_k} - J^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\| \rightarrow 0$. Since J is uniformly continuous on every bounded set, $\|Jw_{n_k} - JJ^{-1}(Jx_{n_k} - \sigma_{n_k}g_{n_k})\| = \|Jw_{n_k} - (Jx_{n_k} - \sigma_{n_k}g_{n_k})\| \rightarrow 0$. This together with $\sigma_{n_k} \|g_{n_k}\| \rightarrow 0$ implies that $\|Jw_{n_k} - Jx_{n_k}\| \rightarrow 0$. Thus,

$$\|w_{n_k} - x_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty \tag{45}$$

because J is uniformly continuous on every bounded set. Since $x_{n_k} \rightharpoonup \bar{x}$, we have that $w_{n_k} \rightharpoonup \bar{x} \in C$. This together with the demi-closedness at zero of S and (44) implies that $\bar{x} \in \text{Fix}(S)$. Now, we show that $\bar{x} \in \text{EP}(f, C)$. Since $x_{n_k} \rightharpoonup \bar{x} \in C$, we obtain from Lemma 4.3 that the sequences $\{y_{n_k}\}$, $\{z_{n_k}\}$ and $\{g_{n_k}\}$ are bounded. Before proving the main result, we show that $\|x_{n_k} - y_{n_k}\| \rightarrow 0$. Without loss of generality, we can assume that $x_{n_k} \neq y_{n_k}$ for all $k \geq 0$. Thus, it follows from Lemma 4.1 that $g_{n_k} \neq 0$ and $f(z_{n_k}, x_{n_k}) > 0$. From the definition of σ_{n_k} , we have

$$\nu f(z_{n_k}, x_{n_k}) = \sigma_{n_k} \|g_{n_k}\|^2 = (\sigma_{n_k} \|g_{n_k}\|) \|g_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

where we have used, for each k , hypothesis (43) and the boundedness of the sequence $\{g_{n_k}\}$. Thus, from $\nu > 0$,

$$f(z_{n_k}, x_{n_k}) \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{46}$$

Since $f(z_{n_k}, \cdot)$ is convex for each k , we can write

$$\begin{aligned} 0 &= f(z_{n_k}, z_{n_k}) = f(z_{n_k}, (1 - \rho_{n_k})x_{n_k} + \rho_{n_k}y_{n_k}) \\ &\leq (1 - \rho_{n_k})f(z_{n_k}, x_{n_k}) + \rho_{n_k}f(z_{n_k}, y_{n_k}). \end{aligned}$$

Thus, $\rho_{n_k}[f(z_{n_k}, x_{n_k}) - f(z_{n_k}, y_{n_k})] \leq f(z_{n_k}, x_{n_k})$ for each k and it follows from the linesearch step (Step 2.1) that $\frac{\rho_{n_k}\alpha}{2\lambda_{n_k}}\phi(y_{n_k}, x_{n_k}) \leq f(z_{n_k}, x_{n_k})$. Hence, from (46) and the hypotheses on α and λ_{n_k} , we obtain

$$\rho_{n_k}\phi(y_{n_k}, x_{n_k}) \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{47}$$

In the case when $\limsup_{k \rightarrow \infty} \rho_{n_k} > 0$, we obtain from (47) that $\phi(y_{n_k}, x_{n_k}) \rightarrow 0$. Thus, from Lemma 2.3, $\|y_{n_k} - x_{n_k}\| \rightarrow 0$. In the case when $\lim_{k \rightarrow \infty} \rho_{n_k} = 0$, let $\{m_k\}$ be the sequence of the smallest positive integers such that

$$f(z_{n_k}, x_{n_k}) - f(z_{n_k}, y_{n_k}) \geq \frac{\alpha}{2\lambda_{n_k}}\phi(y_{n_k}, x_{n_k}), \text{ for all } k \tag{48}$$

where $z_{n_k} = (1 - \gamma^{m_k})x_{n_k} + \gamma^{m_k}y_{n_k}$. From $\rho_{n_k} = \gamma^{m_k} \rightarrow 0$, we can conclude that $m_k > 1$. From the linesearch inequality, we have

$$f(\bar{z}_{n_k}, x_{n_k}) - f(\bar{z}_{n_k}, y_{n_k}) < \frac{\alpha}{2\lambda_{n_k}}\phi(y_{n_k}, x_{n_k}), \text{ for each } k \tag{49}$$

where $\bar{z}_{n_k} = (1 - \gamma^{m_k-1})x_{n_k} + \gamma^{m_k-1}y_{n_k}$. So, it follows from the definition of y_{n_k} that (see 8)

$$\lambda_{n_k} (f(x_{n_k}, y) - f(x_{n_k}, y_{n_k})) \geq \langle Jy_{n_k} - Jx_{n_k}, y_{n_k} - y \rangle, \forall y \in C, \forall k. \tag{50}$$

Hence, choosing $y = x_{n_k}$, we obtain that

$$\begin{aligned} -\lambda_{n_k}f(x_{n_k}, y_{n_k}) &\geq \langle Jy_{n_k} - Jx_{n_k}, y_{n_k} - x_{n_k} \rangle = \frac{1}{2}\phi(y_{n_k}, x_{n_k}) + \frac{1}{2}\phi(x_{n_k}, y_{n_k}) \\ &\geq \frac{1}{2}\phi(y_{n_k}, x_{n_k}) \end{aligned} \tag{51}$$

where the equality follows from relation (6). Then, combining relations (49) and (51), we obtain for each k

$$f(\bar{z}_{n_k}, x_{n_k}) - f(\bar{z}_{n_k}, y_{n_k}) < \frac{\alpha}{2\lambda_{n_k}}\phi(y_{n_k}, x_{n_k}) \leq -\alpha f(x_{n_k}, y_{n_k}), \tag{52}$$

Since the sequence $\{y_{n_k}\}$ is bounded, without loss of generality, we can assume that $y_{n_k} \rightarrow \bar{y}$. Then, it follows from $x_{n_k} \rightharpoonup \bar{x}$, the definition of \bar{z}_{n_k} and $\gamma^{m_k-1} \rightarrow 0$ that $\bar{z}_{n_k} \rightharpoonup \bar{x}$. Passing to the limit in (52) as $k \rightarrow \infty$ and using conditions A3a and A1, we obtain

$$-f(\bar{x}, \bar{y}) \leq -\alpha f(\bar{x}, \bar{y}), \tag{53}$$

which implies that $f(\bar{x}, \bar{y}) \geq 0$ because $\alpha \in (0, 1)$. Taking the limit in (51) and using the hypothesis on λ_{n_k} , we get

$$\lim_{k \rightarrow \infty} \phi(y_{n_k}, x_{n_k}) \leq -2 \lim_{k \rightarrow \infty} \lambda_{n_k} f(x_{n_k}, y_{n_k}) \leq -2\lambda f(\bar{x}, \bar{y}) \leq 0.$$

Hence, $\lim_{k \rightarrow \infty} \phi(y_{n_k}, x_{n_k}) = 0$ which thanks to Lemma 2.3 implies that $\|x_{n_k} - y_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. Consequently, $\bar{x} = \bar{y}$. Now, passing to the limit in (50), noting that $x_{n_k}, y_{n_k}, z_{n_k} \rightharpoonup \bar{x}$ and using condition A1 and the hypothesis on λ_{n_k} , we get $f(\bar{x}, y) \geq 0$ for all $y \in C$, i.e., $\bar{x} \in \text{EP}(f, C)$. Lemma 4.4 is proved. \square

Let n be fixed and consider $u_n = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)$ where $t_n = J^{-1}(\beta_n Jw_n + (1 - \beta_n)JSw_n)$. From the definition of x_{n+1} , we have $x_{n+1} = \Pi_C(u_n)$. Let $x^\dagger = \Pi_F(x_0)$ and

$$\bar{T}_n = \left(\frac{2}{\nu} - \frac{4}{c^2}\right)\sigma_n^2\|g_n\|^2 + \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) + \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|). \tag{54}$$

Using the same development as in Lemma 3.3, we have the following result.

Lemma 4.5. *The following inequalities hold for each n :*

- (i) $(1 - \alpha_n)\bar{T}_n \leq \phi(x^\dagger, x_n) - \phi(x^\dagger, x_{n+1}) + \alpha_n\phi(x^\dagger, u)$
- (ii) $\phi(x^\dagger, x_{n+1}) \leq (1 - \alpha_n)\phi(x^\dagger, x_n) + 2\alpha_n\langle u_n - x^\dagger, Ju - Jx^\dagger \rangle$.

Proof. Let n be fixed. From the definition of t_n , ϕ and relation (42), we have

$$\begin{aligned} \phi(x^\dagger, t_n) &= \|x^\dagger\|^2 - 2\langle x^\dagger, Jt_n \rangle + \|t_n\|^2 \\ &= \|x^\dagger\|^2 - 2\langle x^\dagger, \beta_n Jw_n + (1 - \beta_n)JSw_n \rangle + \|\beta_n Jw_n + (1 - \beta_n)JSw_n\|^2 \\ &\leq \|x^\dagger\|^2 - 2\langle x^\dagger, \beta_n Jw_n + (1 - \beta_n)JSw_n \rangle + \beta_n\|w_n\|^2 + (1 - \beta_n)\|Sw_n\|^2 \\ &\quad - \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|) \\ &= \beta_n\phi(x^\dagger, w_n) + (1 - \beta_n)\phi(x^\dagger, Sw_n) - \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|) \\ &\leq \beta_n\phi(x^\dagger, w_n) + (1 - \beta_n)\phi(x^\dagger, w_n) - \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|) \\ &= \phi(x^\dagger, w_n) - \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|). \end{aligned}$$

This together with relation (36) implies that

$$\begin{aligned} \phi(x^\dagger, t_n) &\leq \phi(x^*, x_n) - \left(\frac{2}{\nu} - \frac{4}{c^2}\right)\sigma_n^2\|g_n\|^2 - \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \\ &\quad - \beta_n(1 - \beta_n)g(\|Jw_n - JSw_n\|). \\ &= \phi(x^*, x_n) - \bar{T}_n. \end{aligned}$$

Thus, it follows from (31) and the definition of \bar{T}_n that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \alpha_n\phi(x^*, u) + (1 - \alpha_n)(\phi(x^*, x_n) - \bar{T}_n) \\ &= \alpha_n\phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x_n) - (1 - \alpha_n)\bar{T}_n \\ &\leq \alpha_n\phi(x^*, u) + \phi(x^*, x_n) - (1 - \alpha_n)\bar{T}_n. \end{aligned}$$

So conclusion (i) is true. Conclusion (ii) can be proven as in Lemma 3.3(ii). □

Now, we prove the second main result.

Theorem 4.1. *Let C be a nonempty closed convex subset of an uniformly smooth and 2-uniformly convex Banach space E . Assume that Conditions B, D and E hold and that the solution set $F := \text{EP}(f, C) \cap \text{Fix}(S)$ is nonempty. Then, the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to $\Pi_F(u)$.*

Proof. Recall the first inequality in Lemma 4.5 as:

$$(1 - \alpha_n)\bar{T}_n \leq \phi(x^\dagger, x_n) - \phi(x^\dagger, x_{n+1}) + \alpha_n\phi(x^\dagger, u), \tag{55}$$

where $x^\dagger = \Pi_F(u)$ and \bar{T}_n is defined by (54).

We consider two cases.

Case 1. There exists $n_0 \geq 0$ such that the sequence $\{\phi(x^\dagger, x_n)\}$ is nonincreasing.

In that case, the limit of $\{\phi(x^\dagger, x_n)\}$ exists. Thus, from Lemma 4.5 (i) and $\alpha_n \rightarrow 0$, we have that $\lim_{n \rightarrow \infty} \bar{T}_n = 0$. From the definition of \bar{T}_n and the hypotheses on ν and β_n , we obtain when $n \rightarrow \infty$, that

$$\sigma_n \|g_n\| \rightarrow 0, \phi(w_n, J^{-1}(Jx_n - \sigma_n g_n)) \rightarrow 0, g(\|Jw_n - JSw_n\|) \rightarrow 0. \tag{56}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to p such that

$$\limsup_{n \rightarrow \infty} \langle x_n - x^\dagger, Ju - Jx^\dagger \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - x^\dagger, Ju - Jx^\dagger \rangle. \tag{57}$$

Since C is closed and convex in the uniformly smooth and uniformly convex Banach space E , the set C is weakly closed. Thus, from $\{x_{n_k}\} \subset C$ and $x_{n_k} \rightharpoonup p$, we obtain $p \in C$. It follows from (56) with $n = n_k$ and Lemma 4.4 that $p \in EP(f, C) \cap Fix(S)$. From (56), by arguing as in Lemma 4.4 (see 45), we also obtain

$$\|w_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{58}$$

Thus, from (71), $x^\dagger = \Pi_F u$ and Lemma 2.6 (ii), we can write

$$\limsup_{n \rightarrow \infty} \langle x_n - x^\dagger, Ju - Jx^\dagger \rangle = \langle p - x^\dagger, Ju - Jx^\dagger \rangle \leq 0. \tag{59}$$

Since $u_n = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jt_n)$ for each n , we obtain that

$$\|Ju_n - Jt_n\| = \alpha_n \|Ju - Jt_n\| \rightarrow 0, \text{ as } n \rightarrow \infty$$

because $\alpha_n \rightarrow 0$ and the sequence $\{\|Ju - Jt_n\|\}$ is bounded. Since J^{-1} is uniformly continuous on each bounded subset of E^* , we deduce that

$$\|u_n - t_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{60}$$

From $t_n = J^{-1}(\beta_n Jw_n + (1 - \beta_n)JSw_n)$, we have that $Jt_n - JSw_n = \beta_n(Jw_n - JSw_n)$. Thus, it follows from relation (56) that

$$\|Jt_n - Jw_n\| \leq \|Jt_n - JSw_n\| + \|JSw_n - Jw_n\| = (1 + \beta_n)\|Jw_n - JSw_n\| \rightarrow 0$$

when $n \rightarrow \infty$. Consequently, $\|t_n - w_n\| \rightarrow 0$ and from (72) and (60), it follows that

$$\|x_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{61}$$

This together with (59) implies that

$$\limsup_{n \rightarrow \infty} \langle u_n - x^\dagger, Ju - Jx^\dagger \rangle = \limsup_{n \rightarrow \infty} \langle x_n - x^\dagger, Ju - Jx^\dagger \rangle \leq 0. \tag{62}$$

Thus, from Lemma 2.9 and Lemma 4.5(ii) we obtain $\phi(x^\dagger, x_n) \rightarrow 0$ and $x_n \rightarrow x^\dagger$ as $n \rightarrow \infty$.

Case 2. There exists a subsequence $\{\phi(x^\dagger, x_{n_i})\}$ of $\{\phi(x^\dagger, x_n)\}$ such that $\phi(x^\dagger, x_{n_i}) \leq \phi(x^\dagger, x_{n_i+1})$ for all $i \geq 0$.

In that case, it follows from Lemma 2.10 that

$$\phi(x^\dagger, x_{\tau(n)}) \leq \phi(x^\dagger, x_{\tau(n)+1}), \phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_{\tau(n)+1}), \forall n \geq n_0. \tag{63}$$

where $\tau(n) = \max \{k \in N : n_0 \leq k \leq n, \phi(x^\dagger, x_k) \leq \phi(x^\dagger, x_{k+1})\}$. Furthermore, the sequence $\{\tau(n)\}_{n \geq n_0}$ is non-decreasing and $\tau(n) \rightarrow +\infty$ as $n \rightarrow \infty$. From (55) and Lemma 4.5(i), we can write

$$(1 - \alpha_{\tau(n)})\bar{T}_{\tau(n)} \leq \phi(x^\dagger, x_{\tau(n)}) - \phi(x^\dagger, x_{\tau(n)+1}) + \alpha_{\tau(n)}\phi(x^\dagger, u) \leq \alpha_{\tau(n)}\phi(x^\dagger, u) \rightarrow 0,$$

when $n \rightarrow \infty$, which implies that $\bar{T}_{\tau(n)} \rightarrow 0$ because $\alpha_{\tau(n)} \rightarrow 0$. From the definition of $\bar{T}_{\tau(n)}$ and the hypotheses on ν and $\beta_{\tau(n)}$, we obtain, when $n \rightarrow \infty$,

$$\sigma_{\tau(n)}\|g_{\tau(n)}\| \rightarrow 0, \phi(w_{\tau(n)}, J^{-1}(Jx_{\tau(n)} - \sigma_{\tau(n)}g_{\tau(n)})) \rightarrow 0, g(\|Jw_{\tau(n)} - JSw_{\tau(n)}\|) \rightarrow 0. \tag{64}$$

From (64) and by arguing as in Lemma 4.4 (see 45), we obtain

$$\|x_{\tau(n)} - w_{\tau(n)}\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{65}$$

Since $\{x_{\tau(n)}\} \subset C$ is bounded, there exists a subsequence $\{x_{\tau(m)}\}$ of $\{z_{\tau(n)}\}$ converging weakly to $p \in C$ such that

$$\limsup_{n \rightarrow \infty} \langle x_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle = \lim_{m \rightarrow \infty} \langle x_{\tau(m)} - x^\dagger, Ju - Jx^\dagger \rangle. \tag{66}$$

From (64) with $n = m$, Lemma 4.4 and $x_{\tau(m)} \rightharpoonup p \in C$, we obtain $p \in F$. Moreover, from (66), we have

$$\limsup_{n \rightarrow \infty} \langle x_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle = \langle p - x^\dagger, Ju - Jx^\dagger \rangle \leq 0, \tag{67}$$

where the last inequality follows from $x^\dagger = \Pi_F(u)$ and Lemma 2.6 (ii). From (64), (65) and by arguing as in Case 1, we also obtain that $\|u_{\tau(n)} - x_{\tau(n)}\| \rightarrow 0$ when $n \rightarrow \infty$ (see 61). This together with (67) implies that

$$\limsup_{n \rightarrow \infty} \langle u_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle = \limsup_{n \rightarrow \infty} \langle x_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle \leq 0, \tag{68}$$

From Lemma 4.5 (ii), one has for each n that

$$\phi(x^\dagger, x_{\tau(n)+1}) \leq (1 - \alpha_{\tau(n)})\phi(x^\dagger, x_{\tau(n)}) + 2\alpha_{\tau(n)} \langle u_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle, \tag{69}$$

which, from relation (63), implies that

$$\begin{aligned} \alpha_{\tau(n)}\phi(x^\dagger, x_{\tau(n)}) &\leq \phi(x^\dagger, x_{\tau(n)}) - \phi(x^\dagger, x_{\tau(n)+1}) + 2\alpha_{\tau(n)} \langle u_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle \\ &\leq 2\alpha_{\tau(n)} \langle u_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle. \end{aligned}$$

Thus, it follows from $\alpha_{\tau(n)} > 0$ that $\phi(x^\dagger, x_{\tau(n)}) \leq 2 \langle u_{\tau(n)} - x^\dagger, Ju - Jx^\dagger \rangle$ for each n . This together with (68) implies that $\limsup_{n \rightarrow \infty} \phi(x^\dagger, x_{\tau(n)}) \leq 0$ or $\lim_{n \rightarrow \infty} \phi(x^\dagger, x_{\tau(n)}) = 0$. Thus, it follows from (69) and $\alpha_{\tau(n)} \rightarrow 0$ that $\phi(x^\dagger, x_{\tau(n)+1}) \rightarrow 0$ as $n \rightarrow \infty$. Since $\phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_{\tau(n)+1})$ for all $n \geq n_0$, we have that $\phi(x^\dagger, x_n) \rightarrow 0$ and $x_n \rightarrow x^\dagger$ as $n \rightarrow \infty$. This completes the proof of Theorem 4.1. \square

5. Computational experiments

In this section, we consider two numerical examples, one in finite dimensional space and another in infinite dimensional space, to illustrate the convergence of Algorithms 3.1 and 4.1.

Example 1. Let $E = \mathfrak{R}^m$ and C be a box defined by $C = \{x \in \mathfrak{R}^m : -1 \leq x_i \leq 1, i = 1, \dots, m\}$. Define the bifunction f on $C \times C$ by

$$f(x, y) = \langle Px + Qy + q, y - x \rangle, \text{ for all } x, y \in C,$$

where $q \in \mathfrak{R}^m$ and P, Q are two matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is negative semidefinite. Since $\phi(x, y) = \|x - y\|^2$, it follows from the properties of P and Q that the bifunction f satisfies conditions A and D with two ϕ -Lipschitz-type constants $c_1 = c_2 = \|P - Q\|/2$, see, e.g., [19, Section 5].

Let $g : \mathfrak{R}^m \rightarrow \mathfrak{R}$ be a convex function such that $\text{lev}_{\leq g} = \{x \in \mathfrak{R}^m : g(x) \leq 0\} \neq \emptyset$. Define the subgradient projection relative to g as follows:

$$S(x) = \begin{cases} x - \frac{g(x)}{\|z(x)\|^2} z(x) & \text{if } g(x) > 0 \\ x & \text{otherwise,} \end{cases}$$

where $z(x) \in \partial g(x)$. Since $\phi(x, y) = \|x - y\|^2$, the mapping S is quasi- ϕ -nonexpansive and demiclosed at zero, see, e.g., [4, Lemma 3.1]. Besides, $\text{Fix}(S) = \text{lev}_{\leq g}$. From Algorithms 3.1 and 4.1, $\phi(x, y) = \|x - y\|^2$ and the definition of f , we need to solve the following optimization problem

$$\min \left\{ \frac{1}{2} y^T H_n y + b_n^T y : -1 \leq y_i \leq 1, i = 1, \dots, m \right\}, \tag{70}$$

where $H_n = 2\lambda_n Q + I$ and $b_n = \lambda_n(Px_n - Qx_n + q) - x_n$ or $b_n = \lambda_n(Py_n - Qy_n + q) - x_n$. Problem (70) can be solved effectively by Matlab Optimization Toolbox. Since J is the identity mapping I , all the projections onto C can be rewritten equivalently as quadratic convex optimization problems over the polyedral convex set C and they are also solved similarly to problem (70). All the programs are written in Matlab 7.0 and computed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50 GHz, RAM 2.00 GB.

In all the experiments below, we work in the Euclidean space \mathfrak{R}^{10} , ($m = 10$) and choose $g(x) = \max \{0, \langle c, x \rangle + d\}$, where $d \in \mathfrak{R}_-$ is a negative real number generated randomly and uniformly in $[-m, 0]$, c is a vector in \mathfrak{R}^m with its entries being generated randomly and uniformly in $[1, m]$ and $\bar{c} \in \partial g(x)$ (if $\langle c, x \rangle = -d$) was chosen randomly. The vector q is the zero vector and the two matrices P, Q are generated randomly such that Q is symmetric positive semidefinite and $Q - P$ negative semidefinite¹. The starting point is $x_0 = (1, 1, \dots, 1)^T$ and the anchor in the Halpern iteration is $u = x_0$. Note that the normalized duality mapping $J = I$ and the constant c in

¹We randomly choose $\lambda_{1k} \in [-m, 0]$, $\lambda_{2k} \in [0, m]$, $k = 1, \dots, m$. We set $\widehat{Q}_1, \widehat{Q}_2$ as two diagonal matrices with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we construct a positive semidefinite matrix Q and a negative semidefinite matrix T using random orthogonal matrices with \widehat{Q}_2 and \widehat{Q}_1 , respectively. Finally, we set $P = Q - T$

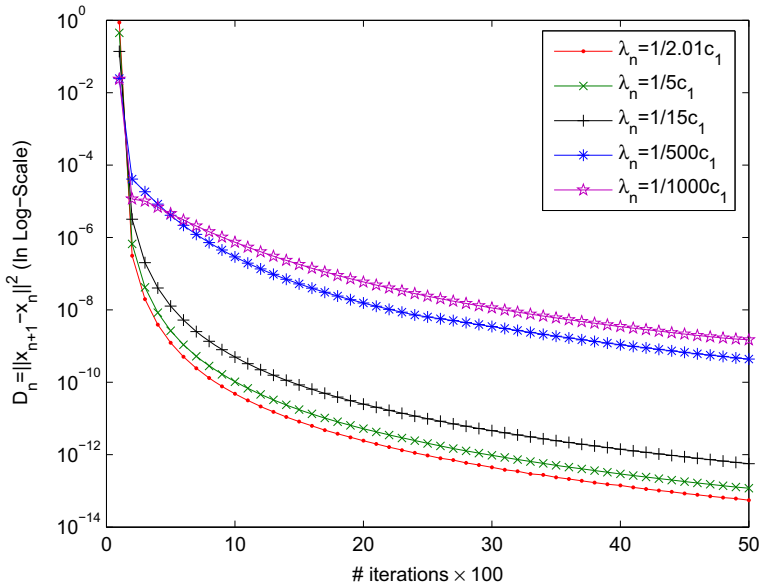


FIGURE 1. Behavior of $D_n = \|x_{n+1} - x_n\|^2$ for Algorithms 3.1 with $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$ for Example 1. Execution times for stepsizes λ_n are 207.32; 187.09; 187.40; 189.85; 191.03 (resp)

Lemma 2.4 is 1. The fixed parameters are chosen in all the experiments as $\nu = 0.25$, $\beta_n = 0.5$ and $\gamma = \alpha = 0.5$. We study numerical behaviors of the proposed algorithms on a test problem for different parameters α_n and stepsizes λ_n .

We use the sequence $D_n = \|x_{n+1} - x_n\|^2$, $n = 0, 1, 2, \dots$ to illustrate the convergence of the algorithms. Figures 1 and 2, 3 and 4 describe the results for Algorithms 3.1, 4.1, respectively. Only the 5000 first iterations have been taken into account. In these figures, the x -axes represent the number of iterations (# iterations) while the y -axes represent the values of D_n .

In Fig. 1, the performance of Algorithm 3.1 is reported following the choice of the stepsizes λ_n . From this figure, we see that the behavior of Algorithm 3.1 is the best one with $\lambda_n = \frac{1}{2.01c_1}$. It seems to be that the smaller the stepsize λ_n is, the slower the convergence of Algorithm 3.1 is.

Figure 2 shows the results for Algorithm 3.1 following the choice of the parameters α_n . It seems that it is with $\alpha_n = \frac{1}{n+1}$ that the convergence is the best one.

Figure 3 reports the results obtained by Algorithm 4.1 with different stepsizes λ_n . In this algorithm, we do not need the information on the two ϕ -Lipschitz-type constants c_1 and c_2 . The convergence of Algorithm 4.1 is the same for stepsizes $\lambda_n = 1$ and $\lambda_n = 0.1$. The sequence D_n with $\lambda_n = 0.5$ is not monotone decreasing while it is the case for the other values of λ_n .

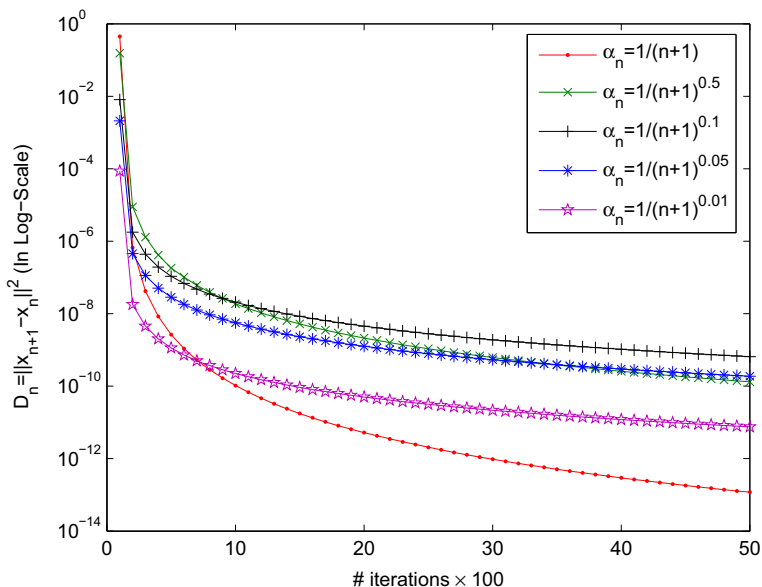


FIGURE 2. Behavior of $D_n = ||x_{n+1} - x_n||^2$ for Algorithms 3.1 with $\lambda_n = \frac{1}{5c_1}$, $\beta_n = \frac{1}{2}$ for Example 1. Execution times for parameters α_n are 182.72; 205.04; 209.77; 218.63; 229.55 (resp)

Figure 4 shows the results of Algorithm 4.1 for different values of the parameter α_n . From this figure, we see that the convergence of D_n with $\alpha_n = \frac{1}{(n+1)^{0.01}}$ is the best one when $n \rightarrow \infty$.

Example 2. Let $E = L^2[0, 1]$ with the inner product $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$ and the induced norm $||x||^2 = \int_0^1 x(t)^2 dt$ for all $x, y \in L^2[0, 1]$. Let C be the unit ball $B[0, 1]$ and the equilibrium bifunction f is of the form $f(x, y) = \langle Ax, y - x \rangle$ with the operator $A : H \rightarrow H$ defined by

$$A(x)(t) = \int_0^1 [x(t) - F_1(t, s)f_1(x(s))] ds + g_1(t), \quad x \in H, \quad t \in [0, 1], \quad (71)$$

and the mapping $S : H \rightarrow H$ is given by

$$S(x)(t) = \int_0^1 F_2(t, s)f_2(x(s))ds + g_2(t), \quad x \in H, \quad t \in [0, 1], \quad (72)$$

where

$$F_1(t, s) = \frac{2tse^{t+s}}{e\sqrt{e^2 - 1}}, \quad f_1(x) = \cos x, \quad g_1(t) = \frac{2te^t}{e\sqrt{e^2 - 1}},$$

$$F_2(t, s) = \frac{\sqrt{21}}{7}(t + s), \quad f_2(x) = \exp(-x^2), \quad g_2(t) = -\frac{\sqrt{21}}{7} \left(t + \frac{1}{2} \right).$$

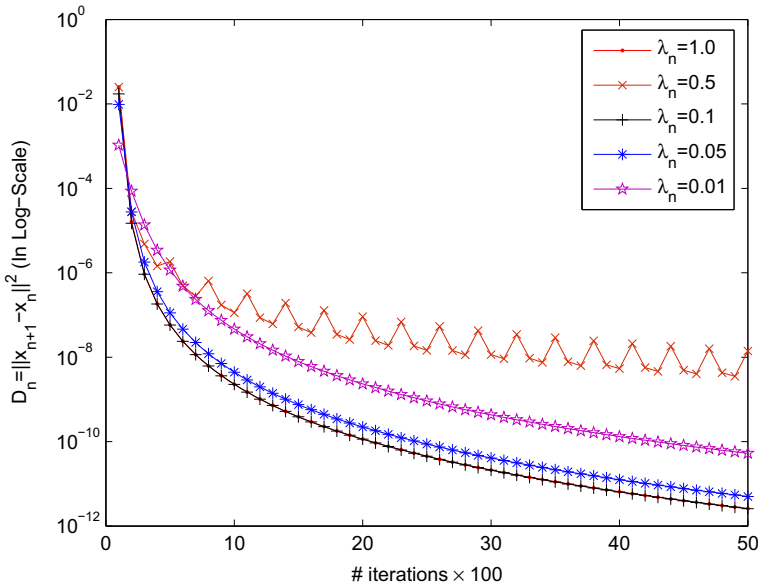


FIGURE 3. Behavior of $D_n = \|x_{n+1} - x_n\|^2$ for Algorithms 4.1 with $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$, $\nu = 0.25$, $\gamma = \alpha = 0.5$ for Example 1. Execution times for stepsizes λ_n are 264.06; 261.75; 255.67; 281.71; 290.14 (resp)

We choose here $g_1(t)$ and $g_2(t)$ such that $x^* = 0$ is the solution of the problem. We also have here that $\phi(x, y) = \|x - y\|^2$ and $J = I$. Set $S_i(x)(t) = \int_0^1 F_i(t, s)f_i(x(s))ds$, $i = 1, 2$, and we see that the mappings S_i , $i = 1, 2$ are Fréchet differentiable and $\|S'_i(x)h\| \leq \|x\|\|h\|$ for all $x, h \in H$. Thus, a straightforward computation implies that f is pseudomonotone and satisfies the ϕ -Lipschitz-type condition with $c_1 = c_2 = 1$, and U is quasi- ϕ -nonexpansive. All the optimization problems in the algorithms become the projections on C which is inherently explicit. All integrals in (71), (72) and others are computed by the trapezoidal formula with the stepsize $\tau = 0.001$. Since the solution of the problem is known, we use the sequence $F_n = \|x_n - x^*\|^2$, $n = 0, 1, 2, \dots$ to show the behavior of the algorithms. The starting point is $x_0(t) = t + 0.5 \cos t$. The numerical results for some different parameters are described in Figs. 5 and 6 for Algorithms 3.1 and 4.1, respectively.

From the above results, we see that the execution times of the Halpern linesearch algorithm are greater than those of the Halpern extragradient algorithm. This is not surprising because for each iteration in the linesearch algorithm, we have to find the smallest integer steplength, which is time consuming. Of course, the advantage of Algorithm 4.1 is not to require the ϕ -Lipschitz-type condition of cost bifunction.

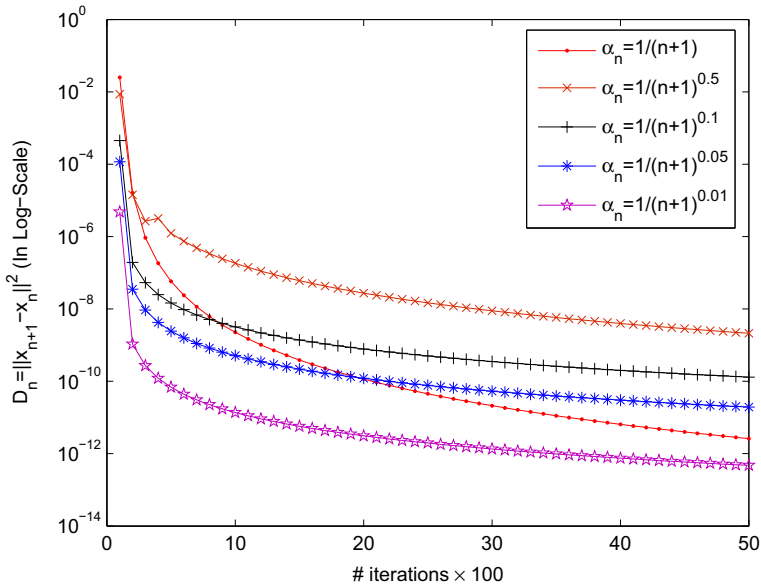


FIGURE 4. Behavior of $D_n = \|x_{n+1} - x_n\|^2$ for Algorithms 4.1 with $\lambda_n = 1$, $\beta_n = \frac{1}{2}$, $\nu = 0.25$, $\gamma = \alpha = 0.5$ for Example 1. Execution times for parameters α_n are 261.50; 285.99; 328.72; 331.84; 334.77 (resp)

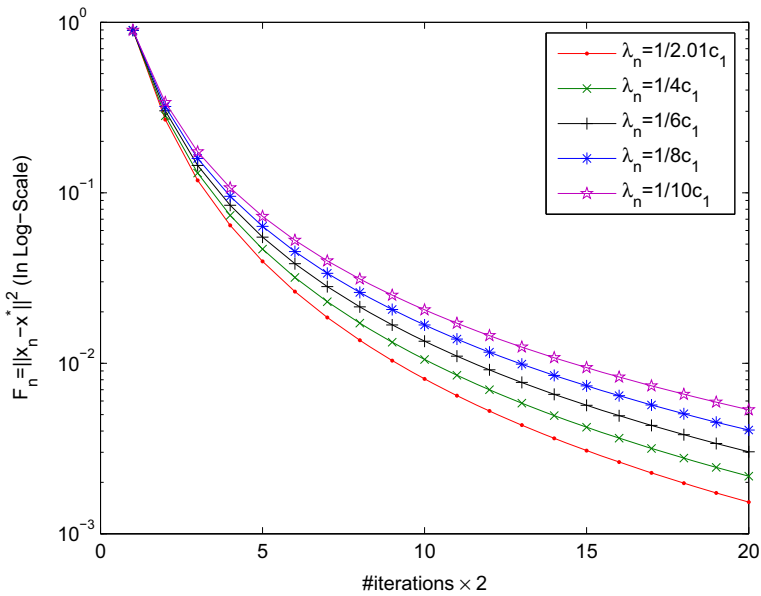


FIGURE 5. Behavior of $F_n = \|x_n - x^*\|^2$ for Algorithms 3.1 with $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$ for Example 2. Execution times for parameters α_n are 61.72, 67.58, 60.43, 68.60, 66.22 (resp)

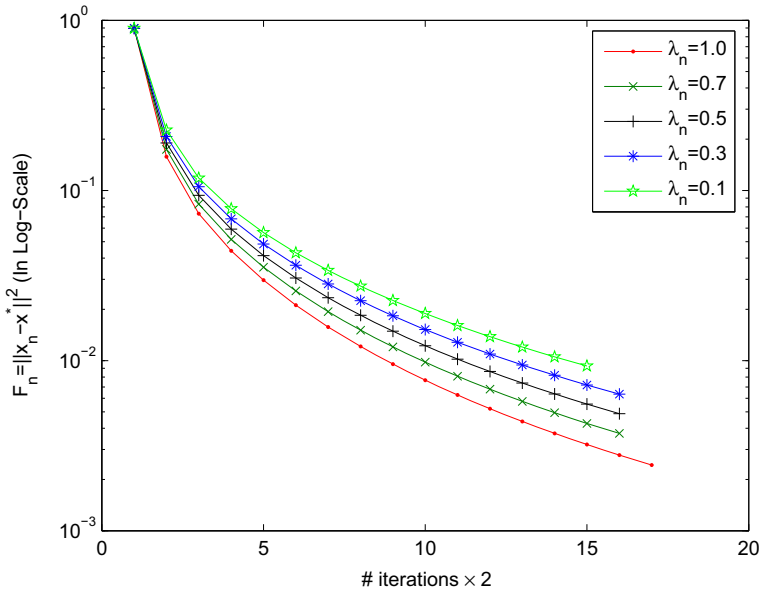


FIGURE 6. Behavior of $F_n = \|x_n - x^*\|^2$ for Algorithms 4.1 with $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$, $\nu = 0.25$, $\gamma = \alpha = 0.5$. Execution times for parameters α_n are 146.99, 145.48, 162.67, 164.62, 185.19 (resp)

6. Conclusions

In this paper, we have proposed two strong convergence algorithms for solving a pseudomonotone equilibrium problem involving a fixed point problem for a quasi- ϕ -nonexpansive mapping in Banach spaces, namely, the Halpern extragradient method and the Halpern linesearch method. The convergence theorems have been established with or without the ϕ -Lipschitz-type condition imposed on bifunctions. The paper can help us in the design and analysis of practical algorithms and also in the improvement of existing methods for solving equilibrium problems in Banach spaces. Several numerical experiments on two test problems for different stepsizes and parameters are also performed to illustrate the convergence of the proposed algorithms.

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References

- [1] Alber, Ya.: I.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartosator, A.G. (ed.) *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50. Dekker, New York (1996)
- [2] Alber, Ya I., Ryazantseva, I.: *Nonlinear Ill-Posed Problems of Monotone Type*. Springer, Dordrecht (2006)
- [3] Anh, P.N., Hieu, D.V.: Multi-step algorithms for solving EPs. *Math. Model. Anal.* **23**, 453–472 (2018)
- [4] Bauschke, H.H., Chen, J.: A projection method for approximating fixed points of quasi—nonexpansive mappings without the usual demiclosedness condition. *J. Nonlinear Convex Anal.* **15**, 129–135 (2014)
- [5] Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. *Math. Program.* **63**, 123–145 (1994)
- [6] Butnariu, D., Reich, S., Zaslavski, A.J.: Asymptotic behavior of relatively nonexpansive operators in Banach spaces. *J. Appl. Anal.* **7**, 151–174 (2001)
- [7] Bruck, R.E., Reich, S.: Nonexpansive projections and resolvents of accretive operators in Banach spaces. *Houston J. Math.* **3**, 459–470 (1977)
- [8] Chidume, C.: *Geometric Properties of Banach Spaces and Nonlinear Iterations*. Springer, London (2009)
- [9] Ceng, L.C., Yao, J.C.: A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.* **214**, 186–201 (2008)
- [10] Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **148**, 318–335 (2011)
- [11] Censor, Y., Reich, S.: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. *Optimization* **37**, 323–339 (1996)
- [12] Cioranescu, I.: *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Mathematics and Its Applications*, vol. 62. Kluwer Academic Publishers, Dordrecht (1990)
- [13] Cohen, G.: Auxiliary problem principle and decomposition of optimization problems. *J. Optim. Theory Appl.* **32**, 277–305 (1980)
- [14] Facchinei, F., Pang, J.S.: *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer, Berlin (2002)
- [15] Flam, S.D., Antipin, A.S.: Equilibrium programming and proximal-like algorithms. *Math. Program.* **78**, 29–41 (1997)
- [16] Goebel, K., Reich, S.: *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*. Marcel Dekker, New York (1984)
- [17] Halpern, B.: Fixed points of nonexpanding maps. *Bull. Am. Math. Soc.* **73**(6), 957–961 (1967)
- [18] Hieu, D.V.: An inertial-like proximal algorithm for equilibrium problems. *Math. Method Oper. Res.* (2018). <https://doi.org/10.1007/s00186-018-0640-6>
- [19] Hieu, D.V.: Parallel extragradient-proximal methods for split equilibrium problems. *Math. Model. Anal.* **21**(4), 478–501 (2016)

- [20] Hieu, D.V., Muu, L.D., Anh, P.K.: Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings. *Numer. Algorithms* **73**, 197–217 (2016)
- [21] Hieu, D.V., Anh, P.K., Muu, L.D.: Modified hybrid projection methods for finding common solutions to variational inequality problems. *Comput. Optim. Appl.* **66**, 75–96 (2017)
- [22] Hieu, D.V.: An extension of hybrid method without extrapolation step to equilibrium problems. *J. Ind. Manag. Optim.* **13**, 1723–1741 (2017)
- [23] Hieu, D.V.: Hybrid projection methods for equilibrium problems with non-Lipschitz type bifunctions. *Math. Methods Appl. Sci.* **40**, 4065–4079 (2017)
- [24] Iiduka, H., Yamada, I.: A subgradient-type method for the equilibrium problem over the fixed point set and its applications. *Optimization* **58**, 251–261 (2009)
- [25] Kamimura, S., Takahashi, W.: Strong convergence of proximal-type algorithm in Banach space. *SIAM J. Optim.* **13**, 938–945 (2002)
- [26] Konnov, I.V.: *Equilibrium Models and Variational Inequalities*. Elsevier, Amsterdam (2007)
- [27] Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. *Ekonom. Mat. Metody* **12**, 747–756 (1976)
- [28] Maingé, P.E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. *SIAM J. Control Optim.* **47**, 1499–1515 (2008)
- [29] Maingé, P.E.: Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints. *Eur. J. Oper. Res.* **205**, 501–506 (2010)
- [30] Mastroeni, G.: On Auxiliary Principle for Equilibrium Problems. Vol. 68 of the Series Nonconvex Optimization and Its Applications, pp. 289–298 (2003)
- [31] Mastroeni, G.: Gap function for equilibrium problems. *J. Glob. Optim.* **27**(4), 411–426 (2003)
- [32] Matsushita, S.-Y., Takahashi, W.: A strong convergence theorem for relatively nonexpansive mappings in a Banach space. *J. Approx. Theory* **134**(2), 257–266 (2005)
- [33] Moudafi, A.: Proximal point algorithm extended to equilibrium problem. *J. Nat. Geometry* **15**, 91–100 (1999)
- [34] Muu, L.D., Oettli, W.: Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal. TMA* **18**(12), 1159–1166 (1992)
- [35] Muu, L.D., Quoc, T.D.: Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash–Cournot equilibrium model. *J. Optim. Theory Appl.* **142**(1), 185–204 (2009)
- [36] Nguyen, T.T.V., Strodiot, J.J., Nguyen, V.H.: Hybrid methods for solving simultaneously an equilibrium problem and countably many fixed point problems in a Hilbert space. *J. Optim. Theory Appl.* **160**, 809–831 (2014)
- [37] Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. *Optimization* **57**, 749–776 (2008)
- [38] Qin, X., Cho, Y.J., Kang, S.M.: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. *J. Comput. Appl. Math.* **225**, 20–30 (2009)

- [39] Reich, S.: The book review of Geometry of Banach spaces, duality mappings and nonlinear problems by Ioana Cioranescu. *Bull. Am. Math. Soc.* **26**, 367–370 (1992)
- [40] Reich, S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces. *J. Math. Anal. Appl. Math.* **75**, 287–292 (1980)
- [41] Reich, S., Sabach, S.: Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces. *Contemp. Math.* **568**, 225–240 (2012)
- [42] Reich, S.: Constructive techniques for accretive and monotone operators. In: *Applied Nonlinear Analysis*, pp. 335–345. Academic Press, New York (1979)
- [43] Strodiot, J.J., Nguyen, T.T.V., Nguyen, V.H.: A new class of hybrid extragradient algorithms for solving quasi-equilibrium problems. *J. Glob. Optim.* **56**(2), 373–397 (2013)
- [44] Strodiot, J.J., Vuong, P.T., Nguyen, T.T.V.: A class of shrinking projection extragradient methods for solving non-monotone equilibrium problems in Hilbert spaces. *J. Glob. Optim.* **64**, 159–178 (2016)
- [45] Takahashi, W., Zembayashi, K.: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. *Nonlinear Anal.* **70**, 45–57 (2009)
- [46] Takahashi, W., Zembayashi, K.: Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings. *Fixed Point Theory Appl.* **2008**, 11 (2008). <https://doi.org/10.1155/2008/528476>. (Article ID 528476)
- [47] Tiel, J.V.: *Convex Analysis: An Introductory Text*. Wiley, New York (1984)
- [48] Vuong, P.T., Strodiot, J.J., Nguyen, V.H.: Extragradient methods and line-search algorithms for solving Ky Fan inequalities and fixed point problems. *J. Optim. Theory Appl.* **155**, 605–627 (2012)
- [49] Vuong, P.T., Strodiot, J.J., Nguyen, V.H.: On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space. *Optimization* **64**, 429–451 (2015)
- [50] Wang, Y.Q., Zeng, L.C.: Hybrid projection method for generalized mixed equilibrium problems, variational inequality problems, and fixed point problems in Banach spaces. *Appl. Math. Mech. Engl. Ed.* **32**(2), 251–264 (2011)
- [51] Xu, H.K.: Inequalities in Banach spaces with applications. *Nonlinear Anal.* **16**, 1127–1138 (1991)
- [52] Xu, H.K.: Another control condition in an iterative method for nonexpansive mappings. *Bull. Aust. Math. Soc.* **65**, 109–113 (2002)
- [53] Zegeye, H., Shahzad, N.: A hybrid scheme for finite families of equilibrium, variational inequality and fixed point problems. *Nonlinear Anal.* **74**, 263–272 (2011)

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