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# Fixed point theorems for multivalued and single-valued contractive mappings on Menger PM spaces with applications

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**Abstract.** In this paper, by introducing multivalued $(\alpha, \eta)$–$\psi$-contractive mappings, we obtain new fixed point theorems for multivalued and single-valued mappings and also coupled fixed point theorems in complete Menger PM and partially ordered Menger PM spaces. We have improved, extended and generalized probabilistic version of the very important generalization of the Banach contraction principle. Some examples and also application of our results in metric spaces and an application to existence of solution of Volterra-type integral equation are given to support the obtained results.

**Mathematics Subject Classification.** 54E40, 54E35, 54H25.

**Keywords.** Fixed point, coupled fixed point, menger probabilistic metric space, $\alpha$–$\psi$-contractive mapping, multivalued mappings, volterra integral equation.

## 1. Introduction

First probabilistic metric spaces was introduced by Menger [27] in 1942. Then, Sehgal and Bharucha-Reid [33] proved the probabilistic version of the classical Banach contraction principle for $B$-contraction mappings. After this, the fixed point theory in probabilistic metric spaces for single-valued and multivalued mappings was extensively studied by many mathematician (see [16, 21–26, 36, 37]). In 2010, Jachymski [19] improved the probabilistic version of the classical Banach contraction principle, obtained by Ćirić [7] for nonlinear contractions.

On the other hand, in 2012, Samet et al. [29] introduced the notion of $\alpha$–$\psi$-contractive mappings and gave some results on fixed point of mappings in complete metric spaces. They introduced the family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. They also supposed that $T$ be a self-mapping on a metric space $(X, d)$ and

$\alpha : X \times X \to [0, \infty)$ be a function, and said the mapping $T$, $\alpha$-admissible mapping if

$$x, y \in X \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Further, The mapping $T$ was said an $\alpha$–$\psi$-contractive mapping if there exist two function $\alpha : X \times X \to [0, \infty)$ and $\psi$ such that for $x, y \in X$,

$$\alpha(x, y)\mathrm{d}(Tx, Ty) \leq \psi(\mathrm{d}(x, y)).$$

Next, they gave the following main theorem:

**Theorem 1.1.** [29] *Let $(X, d)$ be a complete metric space and Let $T$ be $\alpha$–$\psi$-contractive mapping satisfying the following conditions:*

(i) *$T$ be $\alpha$-admissible;*
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *$T$ be continuous or if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all $n$ hold.*

*Then, $T$ has a fixed point.*

More recently, by the same idea with Theorem 1.1, new results has been developed for single-valued and multivalued mappings in metric spaces, fuzzy metric spaces and probabilistic metric spaces, see [1,13,18,28]. Hence, the following question is bound to arise:

Is it possible to obtain a generalization of multivalued and probabilistic version of Theorem 1.1 and prove fixed point theorems for mappings satisfying a more general conditions?

Our purpose of this article is to give an affirmative answer of this question in Theorems 2.12 and 2.13, and also to state coupled fixed and fixed-point theorems for single-valued mappings in partially ordered and probabilistic metric spaces in Sect. 3 that extend, generalize and improve many existing results. In fact, we shall prove our existence results for a wide class of contractive multivalued and single-valued mappings in probabilistic metric spaces. Moreover, to illustrate the usability of our results, in Sect. 4, we discuss the fixed-point theorems for multivalued and single-valued mappings on metric spaces, and also, the existence of solutions for nonlinear Volterra integral equations on a Banach space.

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\Delta^+$ be the space of all probability distribution functions $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ such that $F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+; l^- F(+\infty) = 1\}$. Here, $l^- f(x)$ denotes the left limit of the function $f$ at the point $x$, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space $D^+$ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $D^+$ in this order is the distance distribution function $\varepsilon_0$, defined by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

**Definition 1.2.** [17] A function $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a $t$-norm, if it satisfies the following conditions, for any $a,b,c,d \in [0,1]$:

(1) $\triangle(a,1) = a$;
(2) $\triangle(a,b) = \triangle(b,a)$;
(3) $\triangle(a,b) \leq \triangle(c,d)$ for $a \leq c$ and $b \leq d$;
(4) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

The four basic $t$-norms are the minimum $t$-norm: $\triangle_M(x,y) = \min\{x,y\}$, the product $t$-norm: $\triangle_p(x,y) = x.y$, the Lukasiewicz $t$-norm: $\triangle_L(x,y) = \max\{x+y-1,0\}$, and the weakest $t$-norm, the drastic product: $\triangle_D(x,y) = \min\{x,y\}$ if $\max\{x,y\} = 1$ and $\triangle_D(x,y) = 0$ otherwise.

As regards, the pointwise ordering, we have the inequalities

$$\triangle_D < \triangle_L < \triangle_p < \triangle_M.$$

It is said that the $t$-norm $\triangle$ is of Hadžić-type (H-type in short) if the family $\{\triangle^n\}_{n\geq 0}$ of its iterates, defined for each $x \in [0,1]$ by

$$\triangle^1(x) = \triangle(x,x), \quad \text{and} \quad \triangle^{n+1}(x) = \triangle((\triangle^n(x)),x) \ \ \forall n \geq 1,$$

is equicontinuous at $x = 1$, that is,

$$\forall \epsilon \in (0,1) \ \ \exists \delta \in (0,1) : a > 1-\delta \Rightarrow \ \triangle^n(a) > 1-\epsilon \ \ \forall n \geq 1.$$

$\triangle_M$ is a trivial example of $t$-norm of H-type, but there are $t$-norms of H-type weaker than $\triangle_M$, see [17].
If $\triangle$ be a $t$-norm and $\{x_i\}_{i\geq 1}$ is a sequence in $[0,1]$, $\triangle_{i=1}^{\infty} x_i$ is by definition $\lim_{n\to\infty} \triangle_{i=1}^{n} x_i$, where $\triangle_{i=1}^{n} x_i$ is defined recurrently by $x_1$ if $n = 1$ and $\triangle_{i=1}^{n} x_i = \triangle(\triangle_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$.

**Proposition 1.3.** [17] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\triangle$ is of H-type, then $\lim_{n\to\infty} \triangle_{i=n}^{\infty} x_i = \lim_{n\to\infty} \triangle_{i=1}^{\infty} x_{n+i} = 1$.*

**Definition 1.4.** [17,31] The 3-tuple $(S,\mathcal{F},\triangle)$ is called a Menger probabilistic metric space (briefly, Menger PM space) if $S$ is a nonempty set, $\triangle$ is a $t$-norm, and $\mathcal{F}$ is a mapping from $S \times S$ into $D^+$ [$F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair $(x,y)$] satisfying the following conditions:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in S$, $t > 0$;
(PM3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in S$ and $t,s \geq 0$.

Schweizer et al. [31,32] point out that if the $t$-norm $\triangle$ of a Menger PM space $(S,\mathcal{F},\triangle)$ satisfies the condition $\sup_{0<t<1} \triangle(t,t) = 1$, then $(S,\mathcal{F},\triangle)$ is a Hausdorff topological space in the topology $\tau$ induced by the family of neighborhoods

$$\{U_p(\epsilon,\lambda) : p \in S, \lambda > 0, \epsilon > 0\},$$

where

$$U_p(\epsilon,\lambda) = \{x \in S : F_{x,p}(\epsilon) > 1-\lambda\}.$$

By virtue of this topology $\tau$, a sequence $\{x_n\}$ in a Menger PM space $(S, \mathcal{F}, \triangle)$ is said to be $\tau$-convergent to $x \in S$ (we write $x_n \to x$) if for any given $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ whenever $n \geq N$; $\{x_n\}$ is called a $\tau$-Cauchy sequence in $S$ if for any given $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$, whenever $n, m \geq N$; $(S, \mathcal{F}, \triangle)$ is said to be $\tau$-complete, if each Cauchy sequence in $S$ is $\tau$-convergent to some point in $S$. It is easy to prove that $\{x_n\}$ is $\tau$-convergent to $x \in S$ if and only if

$$\lim_{n\to\infty} F_{x_n,x}(t) = 1 \quad \text{for all } t > 0.$$

**Lemma 1.5.** [11,33] *Let $(S, d)$ be a usual metric space. Define a mapping $F : S \times S \to D^+$ by*

$$F_{x,y}(t) = \varepsilon_0(t - \mathrm{d}(x, y)), \quad x, y \in S,\ t > 0.$$

*Then, $(S, \mathcal{F}, \triangle_M)$ is a Menger PM space; it is called the induced Menger PM space by $(S, d)$ and it is complete if $(S, d)$ is complete.*

In the sequel, we let $(S, \mathcal{F}, \triangle)$ be a Menger PM space and denote by $CB(S)$ the family of all nonempty $\tau$-closed subsets of $S$. Let $x \in S$ and $A, B \in CB(S)$; we define two functions $F_{x,A}$ and $F_{A,B}$ by

$$F_{x,A}(t) = \sup_{y\in A} F_{x,y}(t), \quad t \geq 0, \tag{1.1}$$

and

$$\widetilde{F}_{A,B}(t) = \sup_{s<t} \triangle(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)), \quad t \geq 0, \tag{1.2}$$

respectively. Then, $F_{x,A}$ is called the probabilistic distance between $x$ and $A$, and $\widetilde{F}_{A,B}$ is called the probabilistic distance between $A$ and $B$.
The probabilistic Hausdorff metric was defined and studied by Egbert [9] in the case of Menger spaces. He proved that if the function $H$ defined for any $A$ and $B$ in $CB(S)$ by $H(A, B) = \widetilde{F}_{A,B}$, then $(CB(S), H, \triangle)$ with continuous $t$-norm $\triangle$ is a Menger PM space. The completeness with respect to probabilistic Hausdorff metric of all nonempty closed subsets of a complete general PM spaces was proved by Cobzas [8]. He in his paper proved that if the PM space $(S, F, \triangle)$ with sup-continuous $t$-norm $\triangle$ is complete, then the space $CB(S)$ is complete with respect to probabilistic Hausdorff metric provided $\triangle \geq \triangle_L$.

**Lemma 1.6.** [5] *Let $(S, \mathcal{F}, \triangle)$ be a Menger PM space and $\triangle$ be a left-continuous $t$-norm. If $A \in CB(S)$, and $x, y$ be arbitrary points of $S$, Then, the following assertions hold:*

- (i) $F_{x,A}(t) = 1$ *for all* $t > 0$ *if and only if* $x \in A$;
- (ii) $F_{x,A}(t_1 + t_2) \geq \triangle(F_{x,y}(t_1), F_{y,A}(t_2))$, *for all* $t_1, t_2 \geq 0$;
- (iii) *For any* $A, B \in CB(S)$ *and* $x \in A$, $F_{x,B}(t) \geq \widetilde{F}_{A,B}(t)$ *for all* $t \geq 0$.

**Theorem 1.7.** [30] *Let* $(S, \mathcal{F}, \triangle)$ *be a Menger PM space with continuous t-norm* $\triangle$*. If* $p_n$ *and* $q_n$ *be sequences such that* $p_n \to p$ *and* $q_n \to q$*, then* $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$ *for all continuity point* $t$ *of* $F_{p,q}$*.*

**Definition 1.8.** [18] Let $T : S \to 2^S$ be a set-valued function, and let $\alpha, \eta : S \times S \times (0,\infty) \to \mathbb{R}^+$ be two functions, where $\alpha$ is bounded. We say that $T$ is an $\alpha^\star$–$\eta_\star$-admissible mapping if

$$\alpha(x, y, t) \leq \eta(x, y, t) \Rightarrow \alpha^\star(Tx, Ty, t) \leq \eta_\star(Tx, Ty, t) \quad x, y \in S,\ t > 0,$$

where $\alpha^\star(A, B, t) = \sup_{x\in A, y\in B} \alpha(x, y, t)$ and $\eta_\star(A, B, t) = \inf_{x\in A, y\in B} \eta(x, y, t)$.

Let $f : S \to S$ be a single-valued mapping and $T : S \to 2^S$ be a multivalued mapping. A point $x \in S$ is a fixed point of $f$ (resp. $T$ ) if $fx = x$(resp. $x \in Tx$).

## 2. Fixed-point results for multivalued $(\alpha, \eta)$–$\psi$-contractive mappings

We begin this section with introducing the class of functions and our new notions that help us to give our fixed-point theorems for multivalued mappings in Menger PM spaces and also partially ordered Menger PM spaces.

Let $\Psi$ denote the class of all the functions $\psi$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ satisfying the following properties:

($\psi_1$) $0 < \psi(t) < t$ for all $t > 0$;

($\psi_2$) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where $\psi^n(t)$ denotes the $n$-th iterative function of $\psi(t)$.

If $\psi$ is defined by $\psi(t) = kt$, $k \in (0, 1)$, or $\psi(t) = a(t)t$, where $a : (0,\infty) \to (0, 1)$ be a monotonically decreasing function, then $\psi \in \Psi$.

**Definition 2.1.** Let $(S, \mathcal{F}, \triangle)$ be a Menger PM space. We say that $T : S \to 2^S$ has the approximative-valued property, whenever for each $x \in S$ there exists $y \in Tx$ such that $F_{x,y}(t) = F_{x,Tx}(t)$ for all $t > 0$. Also, $T$ is called to have the w-approximative valued property whenever for each $a \in S$ and all $x \in Ta$ there exists $y \in Tx$ such that

$$F_{x,y}(t) \geq \widetilde{F}_{Ta,Tx}(t) \quad \forall t > 0,$$

i.e., the mapping $\mathcal{P}(x) = \{y \in Tx; F_{x,y}(t) \geq \widetilde{F}_{Ta,Tx}(t), \forall t > 0\}$ for each $a \in S$ and all $x \in Ta$, has nonempty values.

*Remark 2.2.* Note that, if the multivalued mapping $T$ has the approximative valued property, then it will satisfy in the $w$-approximative valued property too. It is clear that every compact-valued mapping has the approximative valued property.

**Definition 2.3.** Let $T : S \to 2^S$, and $\alpha, \eta : S \times S \times (0,\infty) \to \mathbb{R}^+$ be two functions. $T$ be called $\alpha$-admissible w.r.t. $\eta$, if

$$\forall x \in S,\ \forall y \in Tx \quad \alpha(x, y, t) \leq \eta(x, y, t) \Rightarrow \alpha(y, z, t) \leq \eta(y, z, t) \ \ \forall z \in Ty,\ t > 0.$$

*Remark 2.4.* We notice that it is easy to see that every $\alpha^\star$–$\eta_\star$-admissible mapping (Definition 1.8) is a $\alpha$-admissible w.r.t. $\eta$ on $S$.

*Example 2.5.* Let $S = \mathbb{R}^+$ and $T : S \to 2^S$ be defined by $T(x) = [0, 3x]$ for each $x \in S$. If define $\alpha : S \times S \times (0, \infty) \to \mathbb{R}^+$ as

$$\alpha(x, y, t) = \begin{cases} \frac{1}{4} & \text{if } (x, y) = (0, 0), \\ 2 & \text{if } (x, y) \neq (0, 0), \end{cases}$$

then $T$ is $\alpha$-admissible w.r.t. $\eta$. In fact, if for each $x \in S$, and $y \in Tx$ we have $\alpha(x, y, t) \leq 1$, then $x = y = 0$. This implies that $Ty = \{0\}$ and so for any $z \in Ty$ we get $\alpha(y, z, t) \leq 1$.

*Example 2.6.* Let $S = [0, 1]$ and $T : S \to 2^S$ be a multivalued map defined by $T(x) = \{\frac{1}{2}\}$ for all $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $T(x) = [\frac{1}{4}, \frac{3}{4}]$ for all $x = \frac{1}{2}$ and $\eta : S \times S \times (0, \infty) \to \mathbb{R}^+$ as

$$\eta(x, y, t) = \begin{cases} \frac{1}{3} & \text{if } y = \frac{1}{2}, \\ 2 & \text{otherwise}, \end{cases}$$

then $T$ is not $\alpha$-admissible w.r.t. $\eta$ on $S$. To see this, if for each $x \in S$, and $y \in Tx$ we have $\eta(x, y, t) \geq 1$, then $y \neq \frac{1}{2}$. This implies that $Ty = \{\frac{1}{2}\}$, thus for $z \in Ty$, $z = \frac{1}{2}$ and this means $\eta(x, y, t) = \frac{1}{3} < 1$.

**Definition 2.7.** Let $(S, \mathcal{F}, \triangle)$ be a Menger PM space and $\psi \in \Psi$. The mapping $T : S \to 2^S$ is called multivalued $(\alpha, \eta)$–$\psi$-contractive if for every $x, y \in S$ that $\alpha(x, y, t) \leq \eta(x, y, t)$ then

$$\widetilde{F}_{Tx,Ty}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}, \quad t > 0.$$

**Definition 2.8.** We say that $S$ satisfies the condition $(C_{\alpha,\eta})$, whenever for each sequence $\{x_n\}$ in $S$ with $\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x, t) \leq \eta(x_{n_k}, x, t)$ for all $k \in \mathbb{N}$ hold.

*Example 2.9.* Let $S = \mathbb{R}^+$ with the metric $\mathrm{d}(x, y) = |x - y|$. Suppose that

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = \frac{t}{t + \mathrm{d}(x, y)},$$

and $\triangle = \triangle_M$, then $(S, \mathcal{F}, \triangle)$ is a Menger PM space. Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$, $T : S \to 2^S$ be defined by $T(x) = \{2x - \frac{5}{3}\}$ for all $x > 1$, $T(x) = [0, \frac{x}{3}]$ for all $0 \leq x \leq 1$ and $\eta : S \times S \times (0, \infty) \to \mathbb{R}^+$ as

$$\eta(x, y, t) = \begin{cases} \frac{1}{3} & \text{if } x, y \notin [0, 1], \\ 2 & \text{otherwise}. \end{cases}$$

We shall show that $T$ is multivalued $(\alpha, \eta)$–$\psi$-contractive and $S$ satisfies the condition $(C_{\alpha,\eta})$. If for each $x, y \in S$ we have $\eta(x, y, t) \geq 1$, then $x, y \in [0, 1]$ and so $Tx = [0, \frac{x}{3}]$ and $Ty = [0, \frac{y}{3}]$. Using the definition of the probabilistic Hausdorff metric (1.2), we get

$$\begin{aligned}\widetilde{F}_{[0,\frac{x}{3}],[0,\frac{y}{3}]}(\frac{t}{2}) &= \frac{\frac{t}{2}}{\frac{t}{2}+\frac{1}{3}|x-y|} = \frac{t}{t+\frac{2}{3}|x-y|}\\ &\geq \frac{t}{t+|x-y|} = F_{x,y}(t)\\ &\geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\},\end{aligned}$$

for all $t > 0$. This means that $T$ is multivalued $(\alpha,\eta)$–$\psi$-contractive. In addition, if $\{x_n\}$ is a sequence in $S$ such that $\eta(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then by the definition of the $\eta$, we have $x_n \in [0,1]$ for all $n$ and so $x \in [0,1]$. This shows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (here as $\{x_n\}$) such that $\eta(x_{n_k}, x, t) \geq 1$ for all $k \in \mathbb{N}$, i.e., $S$ satisfies the condition $(C_{\alpha,\eta})$.

To obtain our main theorem, we need the following lemmas.

**Lemma 2.10.** *Let $(S,\mathcal{F},\triangle)$ be a Menger PM space. Let $\psi \in \Psi$, $A \in CB(S)$ and $x \in S$. If*

$$F_{x,A}(\psi(t)) = F_{x,A}(t) \quad \text{for all } t > 0,$$

*then $x \in A$.*

*Proof.* Let $x \in S$. By (1.1), for any $y \in A$ we have $F_{x,y}(t) \leq F_{x,A}(t)$ for all $t > 0$. On the other hand, since $\lim_{t\to\infty} F_{x,y}(t) = 1$, then for $\epsilon \in (0,1)$, there exists $t_0 > 0$ such that $F_{x,y}(t_0) > 1-\epsilon$. Hence, by choosing $n$ such that $\psi^n(t_0) < \delta$ for any $n \geq N$, and using induction on $F_{x,A}$, we get

$$F_{x,A}(\delta) \geq F_{x,A}(\psi^n(t_0)) = \cdots = F_{x,A}(\psi(t_0)) = F_{x,A}(t_0) \geq F_{x,y}(t_0) > 1-\epsilon.$$

Therefore, $F_{x,A}(t) = 1$ for all $t > 0$, and then by Lemma 1.6 (i), we conclude that $x \in A$. □

**Lemma 2.11.** [35] *For $n \in \mathbb{N}$, let $g_1, g_2, \ldots, g_n : \mathbb{R} \to [0,1]$ and $F \in D^+$. If for some $\psi \in \Psi$,*

$$F(\psi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t), F(t)\} \quad \text{for all } t > 0,$$

*then $F(\psi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t)\}$ for all $t > 0$.*

The following theorem is our main result.

**Theorem 2.12.** *Let $(S,\mathcal{F},\triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type. Let $T : S \to CB(S)$ has the w-approximative valued property and be a multivalued $(\alpha,\eta)$–$\psi$-contractive mapping satisfying the following conditions:*

- (i) *$T$ is $\alpha$-admissible w.r.t. $\eta$ on $S$;*
- (ii) *For some $x_0 \in S$ there exists $x_1 \in Tx_0$ such that $\alpha(x_0,x_1,t) \leq \eta(x_0,x_1,t)$ for all $t > 0$;*
- (iii) *$T$ is continuous or $S$ satisfies the condition $(C_{\alpha,\eta})$.*

*Then, $T$ has a fixed point, that is, there exists $x \in S$ such that $x \in Tx$.*

*Proof.* If $x_1 = x_0$, then we have nothing to prove. Let $x_1 \neq x_0$, i.e., $x_0 \notin Tx_0$. Since $T$ has w-approximative value property, then there exists $x_2 \in Tx_1$ such that $F_{x_1,x_2}(t) \geq \widetilde{F}_{Tx_0,Tx_1}(t)$ for all $t > 0$. For $x_2 \in Tx_1$, using this property that $T$ is $\alpha$-admissible w.r.t. $\eta$, we have $\alpha(x_1, x_2, t) \leq \eta(x_1, x_2, t)$ for all $t > 0$. If $x_1 \in Tx_1$, then $x_1$ is a fixed point of $T$ and we have nothing to prove. Suppose that $x_2 \neq x_1$. Again, by the assumptions, there exists $x_3 \in Tx_2$ such that $F_{x_2,x_3}(t) \geq \widetilde{F}_{Tx_1,Tx_2}(t)$ and $\alpha(x_2, x_3, t) \leq \eta(x_2, x_3, t)$ for all $t > 0$. By continuing this process, we obtain a sequence $\{x_n\}$ in $S$ such that

$$x_n \in Tx_{n-1}, \quad x_n \neq x_{n-1}, \quad F_{x_n,x_{n+1}}(t) \geq \widetilde{F}_{Tx_{n-1},Tx_n}(t),$$

and

$$\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t) \quad \text{for all } t > 0,\ n = 1, 2, \ldots.$$

Now, we have

$$\begin{aligned} F_{x_n,x_{n+1}}(\psi(t)) &\geq \widetilde{F}_{Tx_{n-1},Tx_n}(\psi(t)) \\ &\geq \min\{F_{x_{n-1},x_n}(t), F_{x_{n-1},Tx_{n-1}}(t), F_{x_n,Tx_n}(t)\} \\ &\geq \min\{F_{x_{n-1},x_n}(t), F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} \\ &= \min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} \quad \text{for all } t > 0, \quad n = 1, 2, \ldots, \end{aligned}$$

and so we get from Lemma 2.11, that $F_{x_n,x_{n+1}}(\psi(t)) \geq F_{x_{n-1},x_n}(t)$, thus

$$F_{x_n,x_{n+1}}(\psi^n(t)) \geq F_{x_{n-1},x_n}(\psi^{n-1}(t)) \geq \cdots \geq F_{x_1,x_2}(\psi(t)) \geq F_{x_0,x_1}(t), \tag{2.1}$$

for all $t > 0$ and $n = 1, 2, \ldots$. In addition, $(\psi_1)$ implies that

$$F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(t) \quad \text{for all } t > 0, \quad n = 1, 2, \ldots. \tag{2.2}$$

Now fix $\delta_0 > 0$ and $\epsilon_0 \in (0, 1)$. Since $S$ is a Menger PM space, we have $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$, thus there exists $t_0 > 0$ such that $F_{x_0,x_1}(t_0) > 1-\epsilon_0$. In addition, from $(\psi_2)$ and by choosing $n$ such that $\psi^n(t_0) < \delta_0$ for all $n \geq k$, (2.1) and using monotonicity $F$, we obtain

$$F_{x_n,x_{n+1}}(\delta_0) > F_{x_n,x_{n+1}}(\psi^n(t_0)) \geq F_{x_0,x_1}(t_0) > 1 - \epsilon_0 \quad \text{for all } n \geq k.$$

Thus, we infer that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1 \text{ for all } t > 0. \tag{2.3}$$

Now we prove $\{x_n\}$ is a Cauchy sequence in $S$. This means that we need to prove that for each $\delta > 0$ and $0 < \epsilon < 1$ there exists an $n_1(\delta, \epsilon)$ such that for all $m > n \geq n_1$, $F_{x_n,x_m}(\delta) > 1-\epsilon$. In order to this, set $t_n = \frac{\delta}{2^n}$, $n = 1, 2, \ldots$, then $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t_k) = 1$ for each $k$. Thus, $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t_n) = 1$. Using Proposition 1.3, we have

$$\lim_{n\to\infty} \triangle_{i=n}^{\infty} F_{x_i,x_{i+1}}(t_i) = 1.$$

Let $n_1$ be a positive integer such that $\triangle_{i=n}^{\infty} F_{x_i,x_{i+1}}(t_i) > 1 - \epsilon$ for each $n \geq n_1$, thus, for any $m, n$ that $m > n \geq n_1$ we get

$$\begin{aligned} F_{x_n,x_m}(\delta) \geq F_{x_n,x_{n+1}}\left(\sum_{i=n}^{m-1} t_i\right) &\geq \triangle_{i=n}^{m-1} F_{x_i,x_{i+1}}(t_i) \\ &\geq \triangle_{i=n}^{\infty} F_{x_i,x_{i+1}}(t_i) > 1 - \epsilon. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is Cauchy. Since the space $(S,\mathcal{F},\triangle)$ is complete, there exists $x\in S$ such that $x_n\to x$ as $n\to\infty$. If $T$ is continuous, then $\lim_{n\to\infty}\widetilde{F}_{Tx_n,Tx}(t)=1$ for any $t>0$. Then, by Lemma 1.6, we get

$$\begin{aligned}F_{x,Tx}(t)&\geq\triangle\left(F_{x,x_{n+1}}\left(\frac{t}{2}\right),F_{x_{n+1},Tx}\left(\frac{t}{2}\right)\right)\\&\geq\triangle\left(F_{x,x_{n+1}}\left(\frac{t}{2}\right),\widetilde{F}_{Tx_n,Tx}\left(\frac{t}{2}\right)\right)\to\triangle(1,1)=1,\end{aligned}$$

and so $F_{x,Tx}(t)=1$ for any $t>0$. This implies that $x\in Tx$. But, if $S$ satisfies the condition $(C_{\alpha,\eta})$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x,t)\leq\eta(x_{n_k},x,t)$ for all $k$. Thus, again from Lemma 1.6, we obtain

$$\begin{aligned}F_{x,Tx}(\psi(t))&\geq\triangle(F_{x,x_{n_k+1}}(t-\psi(t)),F_{x_{n_k+1},Tx}(\psi(t)))\\&\geq\triangle(F_{x,x_{n_k+1}}(t-\psi(t)),\widetilde{F}_{Tx_{n_k},Tx}(\psi(t)))\\&\geq\triangle(F_{x,x_{n_k+1}}(t-\psi(t)),\min\{F_{x_{n_k},x}(t),F_{x_{n_k},x_{n_k+1}}(t),F_{x,Tx}(t)\}).\end{aligned}$$

Letting $n\to\infty$, $F_{x,Tx}(\psi(t))\geq F_{x,Tx}(t)$ and then $F_{x,Tx}(\psi(t))=F_{x,Tx}(t)$ for all $t>0$. This yields from Lemma 2.10, that $x\in Tx$. The proof is complete. □

**Theorem 2.13.** *Theorem 2.12 remain true if the condition "t-norm $\triangle$ of H-type" is replaced by the following*

*(H) There exists a sequence $\{t_n\}\subset(0,\infty)$ with $\sum_{n=1}^{\infty}t_n<\infty$ such that*

$$\lim_{n\to\infty}\triangle_{i=n}^{\infty}F_{x_0,x_1}(t_i)=1.$$

*Proof.* It is sufficient to prove that the $\{x_n\}$ is a Cauchy sequence. By a similar technique in the proof of previous theorem, for any fixed $\delta>0$, if $t_n=\frac{\delta}{2^n}$, $n=1,2,\ldots$, then we see that $\sum_{n=1}^{\infty}t_n<\infty$, thus by (H) and using induction on (2.2), we obtain for any positive integer $m>n$,

$$\begin{aligned}F_{x_n,x_m}(\delta)\geq F_{x_n,x_{n+1}}\left(\sum_{i=n}^{m-1}t_i\right)&\geq\triangle_{i=n}^{\infty}F_{x_i,x_{i+1}}(t_i)\\&\geq\triangle_{i=n}^{\infty}F_{x_0,x_1}(t_i)\to1.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is Cauchy. Following the proof of Theorem 2.12, we conclude that $T$ has a fixed point. □

*Remark 2.14.* If in Theorems 2.12 and 2.13 the condition "the w-approximative valued property" with "every compact-valued mapping", and (i) with "every $\alpha^\star$–$\eta_\star$-admissible mapping" is replaced, then result holds too.

From Theorem 2.12, we can obtain the following corollary, which is generalization of multivalued version of the theorems of Ćirić [7] and Jachymeski [19].

**Corollary 2.15.** *Let $(S,\mathcal{F},\triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type. Let $T:S\to CB(S)$ has the w-approximative-valued property satisfying the following conditions:*

(i) *For* $\psi \in \Psi$*, and each* $x, y \in S$*, that* $\alpha(x,y,t) \leq \eta(x,y,t)$ *then*

$$\widetilde{F}_{Tx,Ty}(\psi(t)) \geq F_{x,y}(t), \quad t > 0;$$

(ii) $T$ *is* $\alpha$*-admissible w.r.t.* $\eta$ *on* $S$*;*

(iii) *For some* $x_0 \in S$*, there exists* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t)$ *for all* $t > 0$*;*

(iv) $T$ *is continuous or* $S$ *satisfies the condition* $(C_{\alpha,\eta})$*.*

*Then,* $T$ *has a fixed point.*

*Proof.* Consequence follows from this fact that for $x, y \in S$ and $t > 0$, we have $F_{x,y}(t) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}$. □

**Corollary 2.16.** *Let* $\eta \equiv 1$*, then Theorems 2.12 and 2.13 remain true if "every multivalued* $(\alpha, \eta)$*–*$\psi$*-contractive mapping" is replaced by the following condition:*

(i1) *For* $\psi \in \Psi$*, and each* $x, y \in S$*,*

$$\alpha(x,y,t)\widetilde{F}_{Tx,Ty}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}, \quad t > 0;$$

Now, We will state our fixed-point results in partially ordered Menger PM spaces. The following notations subserve our purpose.

**Definition 2.17.** Let $\preceq$ be an order relation on $S$. For two subset $A, B$ of $S$, we mark $A \preceq B$ if each $a \in A$ and $b \in B$ imply that $a \preceq b$.

**Theorem 2.18.** *Let* $(S, \mathcal{F}, \triangle)$ *be a complete Menger PM space with continuous t-norm* $\triangle$ *of H-type and* $\preceq$ *be a partial order defined on* $S$*. Let* $T : S \to CB(S)$ *has the w-approximative-valued property satisfying the following conditions:*

(i) *For* $\psi \in \Psi$*, and each* $x, y \in S$ *with* $x \preceq y$*,*

$$\widetilde{F}_{Tx,Ty}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}, \quad t > 0;$$

(ii) *If for each* $x \in S$ *and* $y \in Tx$*,* $x \preceq y$ *implies* $\{y\} \preceq Ty$*;*

(iii) *There exist* $x_0 \in S$ *and* $x_1 \in Tx_0$ *such that* $x_0 \preceq x_1$*;*

(iv) $T$ *is continuous or for each sequence* $\{x_n\}$ *in* $S$ *with* $x_n \preceq x_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $x_{n_k} \preceq x$ *for all* $k \in \mathbb{N}$ *hold.*

*Then,* $T$ *has a fixed point.*

*Proof.* Define $\alpha \equiv 1$ and the function $\eta : S \times S \times (0, \infty) \to \mathbb{R}^+$ by

$$\eta(x,y,t) = \begin{cases} 1 & \text{if } x \preceq y, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

for all $t > 0$. Using the definition of $\alpha, \eta$, the conditions (i) and (ii) of Theorem 2.12 hold. Now, let $\{x_n\}$ be a sequence in $S$ with $\eta(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$. By the definition of $\eta$, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. From (iv), this implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$, which gives us that $\eta(x_{n_k}, x, t) = 1$ for all $k \in \mathbb{N}$ and all $t > 0$. Therefore, all the hypotheses of Theorem 2.12 are satisfied and $T$ has a fixed point. □

*Example 2.19.* Let $S = \mathbb{R}^+$, $\triangle = \triangle_M$ and

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = \frac{t}{t + \mathrm{d}(x, y)},$$

for all $x, y \in S$ and for all $t > 0$. Clearly, $(S, \mathcal{F}, \triangle)$ is a Menger PM space with $\triangle$ of H-type. Define the mapping $T$ and $\alpha$ as that be defined in Example 2.5 and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Since $T$ has the compact values then it has the w-approximative-valued property. Let $x, y \in S$ such that $\alpha(x, y, t) \leq 1$ for all $t > 0$, then $x = y = 0$ and consequently $Tx = Ty = \{0\}$. Thus, it is easy to see that $T$ is multivalued $(\alpha, \eta)$–$\psi$-contractive. In addition, if put $x_0 = x_1 = 0$, then $\alpha(x_0, x_1, t) \leq 1$ for all $t > 0$. Now since $T$ is continuous, therefore, Theorem 2.12 implies that $T$ has a fixed point.

Notice that in the above example, $T$ is not a multivalued probabilistic $\psi$-contraction (see [15] or [17], Definition 4.3), i.e., $T$ does not satisfy the following inequality:

$$\forall x, y \in S, \quad u \in Tx \ \ \exists v \in Ty; \quad F_{u,v}(\psi(t)) \geq F_{x,y}(t) \quad t > 0,$$

or equivalently

$$\forall x, y \in S \ \ \widetilde{F}_{Tx,Ty}(\psi(t)) \geq F_{x,y}(t) \quad t > 0. \tag{2.4}$$

Indeed, for $\psi(t) = kt$, let $x = 1$ and $y = \frac{4}{3}$, then $Tx = [0, 3]$ and $Ty = [0, 4]$. Hence

$$\widetilde{F}_{[0,3],[0,4]}(kt) = \frac{kt}{kt + 1} = \frac{t}{t + \frac{1}{k}} \geq \frac{t}{t + \frac{1}{3}},$$

for all $t > 0$ implies that $k \geq 3$ and this is a contradiction. That is, Corollary 2.8 of [25] cannot be applied to $T$. Note that with choosing $x = 1$ and $y = 2$, one can show that there is not $\psi \in \Psi$ such that (2.4) holds, thus the corresponding theorem of [10] is also not applicable in this case.

*Example 2.20.* Let $(S, \mathcal{F}, \triangle)$, $\psi, T$, and $\eta$ be defined as in Example 2.9. It is easy to check that $T$ is $\alpha$-admissible w.r.t. $\eta$ and by putting $x_0 = 1$, $x_1 = \frac{1}{3}$ we get $\eta(x_0, x_1, t) \geq 1$. Therefore, Theorem 2.12 with $\alpha \equiv 1$ ensures the existence of a fixed point. Here, $T$ is infinitely fixed points.

Now, if in the above example, choose $x = 1$, $y = \frac{3}{2}$ then $Tx = [0, \frac{1}{3}]$, $Ty = \{\frac{4}{3}\}$ and so we have

$$\widetilde{F}_{[0,\frac{1}{3}],\{\frac{4}{3}\}}(kt) = \frac{kt}{kt + \frac{4}{3}} = \frac{t}{t + \frac{4}{3k}} \geq \frac{t}{t + \frac{1}{2}}$$

implies that $k \geq \frac{8}{3}$, and this is a contradiction, i.e., there is no $k < 1$ such that $\widetilde{F}_{Tx,Ty}(kt) \geq F_{x,y}(t)$. In addition, suppose there exists $\psi \in \Psi$ such that $\widetilde{F}_{[0,\frac{1}{3}],\{\frac{4}{3}\}}(\psi(t)) \geq \frac{t}{t+\frac{1}{2}}$, then since $\psi(t) < t$ we have $\frac{t}{t+\frac{4}{3}} \geq \frac{t}{t+\frac{1}{2}}$, which is a contradiction. This shows that the corresponding theorem of [10] cannot be applied to $T$.

## 3. Fixed point and coupled fixed-point results for single-valued mappings

In this section, we first state our main results for single-valued mappings in Menger PM spaces and also partially ordered Menger PM spaces using the following notation:

**Definition 3.1.** Let $f$ be a single-valued mapping on $S$. We say that $f$ is $\alpha$-quasi-admissible w.r.t. $\eta$, whenever

$$\forall x \in S,\ \alpha(x, fx, t) \leq \eta(x, fx, t) \Rightarrow \alpha(fx, f^2x, t) \leq \eta(fx, f^2x, t) \quad \forall t > 0.$$

**Theorem 3.2.** *Let $(S, \mathcal{F}, \triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type. Let $f : S \to S$ is a mapping satisfying the following conditions:*

(i) *For $\psi \in \Psi$, and each $x, y \in S$ that $\alpha(x, y, t) \leq \eta(x, y, t)$, then*
$$F_{fx,fy}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}, \quad t > 0;$$
(ii) *$f$ is $\alpha$-quasi-admissible w.r.t. $\eta$ on $S$;*
(iii) *There exists $x_0 \in S$ such that $\alpha(x_0, fx_0, t) \leq \eta(x_0, fx_0, t)$ for $t > 0$;*
(iv) *$f$ is continuous or $S$ satisfies the condition $(C_{\alpha,\eta})$.*

*Then, $f$ has a fixed point, that is there exists $x \in S$ such that $x = fx$.*

*Proof.* It is sufficient that we define $T : S \to 2^S$ by $Tx = \{fx\}$ for all $x \in S$, then consequence is obtained from Theorem 2.12. □

*Remark 3.3.* The reader can show easily that if $f$ satisfies the following implication:

$$\forall x, y \in S,\ \alpha(x, y, t) \leq \eta(x, y, t) \Rightarrow \alpha(fx, fy, t) \leq \eta(fx, fy, t) \quad \forall t > 0 \tag{3.1}$$

then, $f$ is $\alpha$-quasi-admissible w.r.t. $\eta$. Therefore, we can replace the condition (ii) of above theorem with implication (3.1).

The following corollaries are obtained immediately for partially ordered Menger PM spaces.

**Corollary 3.4.** *Let $(S, \mathcal{F}, \triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type and $\preceq$ be a partial order defined on $S$. Let $f : S \to S$ is a mapping satisfying the following conditions:*

(i) *For $\psi \in \Psi$, and each $x, y \in S$ that $x \preceq y$, then*
$$F_{fx,fy}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}, \quad t > 0;$$
(ii) *For each $x \in S$, $x \preceq fx$ implies $fx \preceq f^2x$;*
(iii) *There exists $x_0 \in S$ such that $x_0 \preceq fx_0$;*
(iv) *$f$ is continuous or $S$ satisfies the condition $(C_{\alpha,\eta})$.*

*Then, $f$ has a fixed point.*

**Corollary 3.5.** *Let $(S, \mathcal{F}, \triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type and $\preceq$ be a partial order defined on $S$. Let $f : S \to S$ is a non-decreasing mappings w.r.t. $\preceq$ satisfying the following conditions:*

(i) *For* $\psi \in \Psi$*, and each* $x, y \in S$ *that* $x \preceq y$*, then*

$$F_{fx,fy}(\psi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}, \quad t > 0;$$

(ii) *There exists* $x_0 \in S$ *such that* $x_0 \preceq fx_0$*;*

(iii) $f$ *is continuous or* $S$ *satisfies the condition* $(C_{\alpha,\eta})$*.*

*Then,* $f$ *has a fixed point.*

*Proof.* Let $x \in S$ such that $x \preceq fx$. Since $f$ is a non-decreasing mappings w.r.t. $\preceq$, we have $fx \preceq f^2x$, then condition (ii) of Corollary 3.4 is satisfied. Therefore, $f$ has a fixed point. □

*Remark 3.6.* Theorem 3.2, Corollaries 3.4 and 3.5 are extensions and generalizations of Theorem 2.1 of [35], Corollary 3.6 of [6], taking $\Psi(t) = kt$ $\forall k \in (0,1)$ and Theorem 1 of [19] to partially ordered Menger PM spaces.

*Example 3.7.* Let $(\mathbb{R}^+, \mathcal{F}, \triangle_M)$ with

$$\mathcal{F}(x,y)(t) = F_{x,y}(t) = \frac{t}{t + |x-y|},$$

for all $x, y \in S$ and $t > 0$. Let $f : S \to S$ be defined by $f(x) = \frac{x}{3(x+1)}$ for all $x \in [0,1]$, $f(x) = 2x^2 + 1$ for all $x > 1$ and $\eta : S \times S \times (0,\infty) \to \mathbb{R}^+$ as

$$\eta(x,y,t) = \begin{cases} 3 & \text{if } x, y \in [0,1], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

for all $t > 0$. For each $x, y \in S$, $\eta(x,y,t) \geq 1$ implies that $x, y \in [0,1]$. Thus, we get

$$\begin{aligned} F_{\frac{x}{3(x+1)}, \frac{y}{3(y+1)}}\left(\frac{t}{3}\right) &= \frac{\frac{t}{3}}{\frac{t}{3} + |\frac{y}{3(y+1)} - \frac{x}{3(x+1)}|} \\ &\geq \frac{t}{t + |x-y|} = F_{x,y}(t), \end{aligned}$$

for all $t > 0$. Clearly, this inequality implies that

$$F_{\frac{x}{3(x+1)}, \frac{x}{3(x+1)}}(\frac{t}{3}) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}.$$

Thus, (i) of Theorem 3.2 holds. In addition, for each $x \in S$ such that $\eta(x, fx, t) \geq 1$, we have $fx \in [0,1]$, thus $f^2x \in [0,1]$. This means that $\eta(fx, f^2x, t) \geq 1$ for all $t > 0$. If $x_0 = 0$ then $\eta(0, f0, t) = 3 \geq 1$ for all $t > 0$. Now, let $\{x_n\}$ be a sequence in $S$ such that $\eta(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then we have $x_n \in [0,1]$ for all $n$ and $x \in [0,1]$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (here as $\{x_n\}$) such that $\eta(x_{n_k}, x, t) \geq 1$ for all $k \in \mathbb{N}$. Then, all the hypotheses of Theorem 3.2 are satisfied and consequently, $f$ has a fixed point. Here, $x = 0$ is a fixed point of $f$.

Now, let $x = 0$ and $y = \frac{3}{2}$, then

$$F_{0,\frac{11}{2}}(kt) = \frac{t}{t + \frac{11}{2k}} \geq \frac{t}{t + \frac{3}{2}} = F_{0,\frac{3}{2}}(t),$$

implies that $k \geq \frac{11}{3}$, and this is contradiction, hence the inequality $F_{f(x),f(y)}(kt) \geq F_{x,y}(t)$ is not satisfied. This shows that $f$ is not a $B$-contraction, and so the corresponding theorem of Sehgal and Bharucha-Reid [33] cannot be applied to $f$. Similarly, one can show that Theorem 1 of [19] is also not applicable in this case.

In continuation, we shall use our results to obtain the coupled fixed point theorems for single-valued mappings in complete Menger PM spaces and also partially ordered Menger PM spaces.

**Definition 3.8.** [3] Let $G : S \times S \to S$ be a given mapping. We say that $(x, y) \in S \times S$ is a coupled fixed point of $G$ if

$$G(x, y) = x \quad \text{and} \quad G(y, x) = y.$$

**Lemma 3.9.** [29] *Let $G : S \times S \to S$ be a given mapping. Define the mapping $T : S \times S \to S \times S$ by*

$$T(x, y) = (G(x, y), G(y, x)), \quad (x, y) \in S \times S. \tag{3.2}$$

*Then, $(x, y)$ is a coupled fixed point of $G$ if and only if $(x, y)$ is a fixed point of $T$.*

Now, let $(x, y), (u, v) \in S \times S$. Define distribution function $F^* : S \times S \to D^+$ by

$$F^*_{(x,y),(u,v)}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}, \quad t > 0. \tag{3.3}$$

**Theorem 3.10.** *Let $(S, \mathcal{F}, \triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type and $\eta : S^2 \times S^2 \times (0, \infty) \to \mathbb{R}^+$ be a function. Suppose that $f : S \times S \to S$ be a given mapping satisfying the following conditions:*

(i) *For $\psi \in \Psi$,$(x, y), (u, v) \in S \times S$ that $\eta((x, y), (u, v), t) \geq 1$, then*

$$F_{G(x,y),G(u,v)}(\psi(t)) \geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\}, \quad t > 0;$$

(ii) *For all $(x, y) \in S \times S$, $\eta((x, y), (G(x, y), G(y, x)), t) \geq 1$ implies*

$$\eta((G(x, y), G(y, x)), (G(G(x, y), G(y, x)), G(G(y, x), G(x, y))), t) \geq 1, \quad t > 0;$$

(iii) *There exists $(x_0, y_0) \in S \times S$ such that*

$$\eta((x_0, y_0), (G(x_0, y_0), G(y_0, x_0)), t) \geq 1,\ \eta((G(y_0, x_0), G(x_0, y_0)), (y_0, x_0), t) \geq 1;$$

(iv) *$G$ is continuous or for each two sequences $\{x_n\}$ and $\{y_n\}$ in $S$ such that*

$$\eta((x_n, y_n), (x_{n+1}, y_{n+1}), t) \geq 1, \quad \text{and} \quad \eta((y_{n+1}, x_{n+1}), (y_n, x_n), t) \geq 1,$$

*for each $n \in \mathbb{N}, t > 0$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that*

$$\eta((x_{n_k}, y_{n_k}), (x, y), t) \geq 1, \quad \text{and} \quad \eta((y, x), (y_{n_k}, x_{n_k}), t) \geq 1, \quad k \in \mathbb{N}, t > 0.$$

*Then, $G$ has a coupled fixed point, that is there exists $(x^*, y^*) \in S \times S$ such that $x^* = G(x^*, y^*)$ and $y^* = G(y^*, x^*)$.*

*Proof.* Let $Y = S \times S$, and define $F^*_{(x,y),(u,v)}(t)$ as (3.3) for each $(x,y),(u,v) \in Y$. It is not hard to see that since $(S, \mathcal{F}, \triangle)$ is a complete Menger PM space, then the space $(Y, F^*, \triangle)$ is also a complete Menger PM space. The condition (i) implies that if $\eta((x,y),(u,v),t) \geq 1$, then

$$\begin{aligned} F_{G(x,y),G(u,v)}(\psi(t)) &\geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), \\ &\quad F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\} \\ &= \min\{\min\{F_{x,u}(t), F_{y,v}(t)\}, \min\{F_{x,G(x,y)}(t), F_{y,G(y,x)}(t)\}, \\ &\quad \min\{F_{u,G(u,v)}(t),, F_{v,G(v,u)}(t)\}\} \\ &= \min\{F^*_{(x,y),(u,v)}(t), F^*_{(x,y),T(x,y)}(t), F^*_{(u,v),T(u,v)}(t)\}, \end{aligned}$$

and if $\eta((v,u),(y,x),t) \geq 1$, then similarly

$$\begin{aligned} F_{G(y,x),G(v,u)}(\psi(t)) &= F_{G(v,u),G(y,x)}(\psi(t)) \\ &\geq \min\{F^*_{(x,y),(u,v)}(t), F^*_{(x,y),T(x,y)}(t), F^*_{(u,v),T(u,v)}(t)\}, \end{aligned}$$

where $T : Y \to Y$ is given by (3.2). Therefore, we obtain

$$F^*_{T(\xi),T(\gamma)}(\psi(t)) \geq \min\{F^*_{\xi,\gamma}(t), F^*_{\xi,T(\xi)}(t), F^*_{\gamma,T(\gamma)}(t)\},$$

provided $\eta_m(\xi,\gamma,t) \geq 1$ for each $\xi = (\xi_1,\xi_2)$, $\gamma = (\gamma_1,\gamma_2)$, where

$$\eta_m((\xi_1,\xi_2),(\gamma_1,\gamma_2),t) = \min\{\eta((\xi_1,\xi_2),(\gamma_1,\gamma_2),t), \eta((\gamma_2,\gamma_1),(\xi_2,\xi_1),t)\},$$

for each $t > 0$. This shows that $T$ satisfies in the condition (i) of Theorem 3.2. Moreover, if $\xi = (\xi_1,\xi_2) \in Y$ be such that $\eta_m(\xi, T\xi, t) \geq 1$, then using condition (ii), we have $\eta_m(T\xi, T^2\xi, t) \geq 1$. Next, using the definition of $\eta_m$ and condition (iii) implies that there exists $(x_0,y_0) \in Y$ such that $\eta_m((x_0,y_0),T(x_0,y_0),t) \geq 1$. Now, if $G$ is continuous then $T$ is too. Let $\{(x_n,y_n)\}$ be a sequence in $Y$ such that $\eta_m((x_n,y_n),(x_{n+1},y_{n+1}),t) \geq 1$ for each $t > 0$ and $(x_n,y_n) \to (x,y)$. Using the definition of $\eta_m$ and condition (iv), we get that there exists a subsequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ such that

$$\eta_m((x_{n_k},y_{n_k}),(x,y),t) \geq 1, \quad t > 0.$$

Hence, Theorem 3.2 ensures the existence of a fixed point of $T$, and so by Lemma 3.9, $G$ has a coupled fixed point. The proof is completed. □

*Remark 3.11.* By putting $u = G(x,y)$ and $v = G(y,x)$ in Theorem 3.10, we also can obtain a generalization of probabilistic and multivalued version of Theorem 2.5 of [29].

In the end of this section, we shall extend this results to partially ordered Menger PM spaces by a new notion of w-mixed monotone property, and the mixed monotone property which are introduced by Bhaskar and Lakshmikantham in [3].

**Definition 3.12.** [3] Let $(S, \preceq)$ be a non-empty partially ordered set. The mapping $G : S \times S \to S$ is said to have the mixed monotone property if

$$x_1, x_2 \in S, \quad x_1 \preceq x_2 \Rightarrow G(x_1,y) \preceq G(x_2,y), \quad y \in S,$$

and

$$y_1, y_2 \in S, \quad y_1 \preceq y_2 \Rightarrow G(x, y_2) \preceq G(x, y_1), \quad x \in S.$$

**Definition 3.13.** Let $(S, \preceq)$ be a non-empty partially ordered set. We say the mapping $G : S \times S \to S$ has the w-mixed monotone property if for each $x, y \in S$,

$$x_1 \in S, \quad x_1 \preceq G(x_1, y) \Rightarrow G(x_1, y) \preceq G(G(x_1, y), y),$$

and

$$y_1 \in S, \quad y_1 \preceq G(x, y_1) \Rightarrow G(x, G(x, y_1)) \preceq G(x, y_1),$$

*Remark 3.14.* It is clear that if $G$ has the mixed monotone property then it also satisfies the w-mixed monotone property.

**Theorem 3.15.** *Let $(S, \mathcal{F}, \triangle)$ be a complete Menger PM space with continuous t-norm $\triangle$ of H-type and $\preceq$ be a partial order defined on $S$. Suppose that the mapping $G : S \times S \to S$ has the w-mixed monotone property on $S$ and there exists $\psi \in \Psi$ with*

$$F_{G(x,y),G(u,v)}(\psi(t)) \geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\},$$

*for all $x, y, u, v \in S$ such that $x \preceq u$ and $v \preceq y$ and $t > 0$. Suppose that $G$ is continuous or $S$ has the following properties:*

(a) *If $\{x_n\}$ be a non-decreasing sequence such that $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k$.*
(b) *If $\{y_n\}$ be a non-increasing sequence such that $y_n \to y$, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y \preceq y_{n_k}$ for all $k$.*

*Moreover, if there exists $(x_0, y_0) \in S \times S$ such that $x_0 \preceq G(x_0, y_0)$ and $G(y_0, x_0) \preceq y_0$, then $G$ has a coupled fixed point.*

*Proof.* Define the mapping $\eta : S^2 \times S^2 \times (0, \infty) \to \mathbb{R}^+$ by

$$\eta((x, y), (u, v), t) = \begin{cases} 1 & \text{if } x \preceq u, \ v \preceq y, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

for all $t > 0$. By hypothesis and the definition of $\eta$, for each $x, y, u, v \in S$ that $\eta((x, y), (u, v), t) \geq 1$, we have

$$F_{G(x,y),G(u,v)}(\psi(t)) \geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\},$$

for all $t > 0$. Then, $G$ satisfies in the condition (i) of Theorem 3.10. Let $(x, y) \in S \times S$ be such that $\eta((x, y), (G(x, y), G(y, x)), t) \geq 1$. Using the definition of $\eta$, we have $x \preceq G(x, y)$ and $G(y, x) \preceq y$. Since $G$ has the w-mixed monotone property, thus $G(x, y) \preceq G(G(x, y), G(y, x))$ and $G(G(y, x), G(x, y)) \preceq G(y, x)$ and so

$$\eta((G(x, y), G(y, x)), (G(G(x, y), G(y, x)), G(G(y, x), G(x, y))), t) \geq 1,$$

for all $t > 0$. In addition, there exists $(x_0, y_0) \in S \times S$ such that $x_0 \preceq G(x_0, y_0)$ and $G(y_0, x_0) \preceq y_0$, then it follows from the definition of $\eta$ that the condition

(iii) of Theorem 3.10 holds too. Now, only it remains that show the condition (iv) of Theorem 3.10 holds. If $G$ is continuous then we have nothing to prove. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $S$ such that

$$\eta((x_n, y_n), (x_{n+1}, y_{n+1}), t) \geq 1, \quad \text{and} \quad \eta((y_{n+1}, x_{n+1}), (y_n, x_n), t) \geq 1,$$

for each $n \in \mathbb{N}, t > 0$, and $x_n \to x$, $y_n \to y$ as $n \to \infty$. Then, the definition of $\eta$ implies that $\{x_n\}$ is a non-decreasing sequence and $\{y_n\}$ is a non-increasing sequence. From (a) and (b), there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $x_{n_k} \preceq x$ and $y \preceq y_{n_k}$ for all $k$. This shows that

$$\eta((x_{n_k}, y_{n_k}), (x, y), t) \geq 1, \quad \text{and} \quad \eta((y, x), (y_{n_k}, x_{n_k}), t) \geq 1, \quad k \in \mathbb{N}, t > 0.$$

Therefore, Theorem 3.10 ensures the existence of a coupled fixed point. □

**Corollary 3.16.** [35] *Let $(S, \preceq, \mathcal{F}, \triangle)$ be a partially ordered complete Menger PM space with continuous t-norm $\triangle$ of H-type. Suppose $G : S \times S \to S$ is a mapping satisfying the mixed monotone property on $S$ and, for some $\psi \in \Psi$,*

$$F_{G(x,y),G(u,v)}(\psi(t)) \geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\},$$

*for all $x, y, u, v \in S$ such that $x \preceq u$ and $v \preceq y$ and all $t > 0$. Suppose that $G$ is continuous or $S$ has the following properties:*

(i) *If non-decreasing sequence $x_n$ tends to $x$, then $x_n \preceq x$ for all $n$.*

(ii) *If non-increasing sequence $y_n$ tends to $y$, then $y \preceq y_n$ for all $n$.*

*If there exist $x_0, y_0 \in S$ such that $x_0 \preceq G(x_0, y_0)$ and $G(y_0, x_0) \preceq y_0$, then $G$ has a coupled fixed point.*

*Proof.* Since $G$ has the mixed monotone property, then it also satisfies the w-mixed monotone property. Now, let $\{x_n\}$ be a non-decreasing sequence such that $x_n \to x$ and $\{y_n\}$ a non-increasing sequence such that $y_n \to y$, then from (i) and (ii), we have $x_n \preceq x$ and $y \preceq y_n$ for all $n$. It follows that for each subsequence of $\{x_n\}$ and $\{y_n\}$, such as $\{x_{n_k}\}$ and $\{y_{n_k}\}$, we have $x_{n_k} \preceq x$ and $y \preceq y_{n_k}$ for all $k$. Therefore, (a) and (b) of Theorem 3.15 hold, consequently $G$ has a coupled fixed point. □

*Example 3.17.* Let $S = \mathbb{R}^+$ and $\mathcal{F}(x, y)(t) = F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in S$ and $t > 0$. Then $(\mathbb{R}^+, \mathcal{F}, \triangle_M)$ is a complete Menger PM space. Define $\psi \in \psi$, the continuous mapping $G : S \times S \to S$ by

$$G(x, y) = \frac{3}{2}|x - y|,$$

and $\eta : S^2 \times S^2 \times (0, \infty) \to \mathbb{R}^+$ as

$$\eta((x, y), (u, v), t) = \begin{cases} 1 & \text{if } x = y = u = v = 0, \\ \frac{1}{2} & \text{if } (x, y, u, v) \neq (0, 0, 0, 0). \end{cases}$$

For each $x, y, u, v \in S$, $\eta((x, y), (u, v), t) \geq 1$ implies that the left-hand side of inequality in (i) of Theorem 3.10 is equal to 1 and hence (i) is obviously hold. Moreover, let $x, y \in S$ such that $\eta((x, y), (G(x, y), G(y, x)), t) \geq 1$, then $x = y = G(x, y) = G(y, x) = 0$. This implies that $G(G(x, y), G(y, x)) = G(G(y, x), G(x, y)) = G(0, 0) = 0$, and so (ii) be satisfied too. With putting

$x_0 = y_0 = 0$, the condition (iii) is also satisfied. Thus, all the conditions of Theorem 3.10 hold. Therefore, $G$ has a coupled fixed point. Indeed, $(0, 0)$ is the coupled fixed point of $G$.

*Example 3.18.* Consider $S = \mathbb{R}^+$ with metric $\mathrm{d}(x, y) = |x - y|$ and suppose that "$\geq$" be the usual ordering on $S$. We define a ordering "$\preceq$" on $S$ as follows:

$$x \preceq y \quad \Leftrightarrow \quad x \geq y, \qquad x, y \in S.$$

Then, $(S, \preceq, d)$ is a complete partially ordered metric space. Let $G : S \times S \to S$ be defined by

$$G(x, y) = \frac{x}{3 + y}, \quad x, y \in S.$$

It is easy to see that $G$ has the mixed monotone property. But, put $(x, y) = (10, 0)$ and $(u, v) = (9, 1)$, then

$$d(G(10, 0), G(9, 1)) = \frac{13}{12} \leq k = \frac{k}{2}[d(10, 9) + d(0, 1)],$$

which implies that $k > 1$. This shows that $G$ does not satisfy the contractive condition of Theorem 2.1 of [3] and so it cannot applied to $G$.
Now, for each $x, y \in S$ and all $t > 0$, suppose that

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = \begin{cases} \frac{t}{t+\max\{x,y\}} & \text{if } x \neq y, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $(S, \preceq, \mathcal{F}, \triangle_M)$ is a complete partially ordered Menger PM space. Define $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Note that, if $G(x, y) = G(u, v)$ for each $(x, y), (u, v) \in S \times S$, then by the definition of $F$, we have $F_{G(x,y),G(u,v)}(t) = 1$ for all $t > 0$. Let $G(x, y) \neq G(u, v)$ for each $(x, y), (u, v) \in S \times S$ with $x \geq u$ and $v \geq y$. Then, we have

$$\begin{aligned} F_{G(x,y),G(u,v)}(\frac{t}{2}) &= \frac{\frac{t}{2}}{\frac{t}{2} + \max\left\{\frac{x}{3+y}, \frac{u}{3+v}\right\}} = \frac{t}{t + \frac{2x}{3+y}} \\ &\geq \frac{t}{t + \frac{2}{3}x} \geq \frac{t}{t + x} = \frac{t}{t + \max\left\{x, \frac{x}{3+y}\right\}} \\ &= F_{x,G(x,y)}(t) \\ &\geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,G(x,y)}(t), \\ &\quad F_{u,G(u,v)}(t), F_{y,G(y,x)}(t), F_{v,G(v,u)}(t)\}, \end{aligned}$$

for all $t > 0$. Finally, by $x_0 = y_0 = 0$, we deduce that $G$ satisfies all the conditions given in Theorem 3.15. Moreover, $(0, 0)$ is a coupled fixed point of $G$.

*Remark 3.19.* Since every fuzzy metric space $(S, M, \triangle)$ with the condition:

$$\lim_{t \to \infty} M(x, y, t) = 1, \quad x, y \in S, \tag{3.4}$$

is a Menger PM space, where $F_{x,y}(t) = M(x, y, t)$, then our results will hold in fuzzy metric spaces satisfying the condition (3.4). For further information on fuzzy metric spaces, refer to [12, 14, 20].

## 4. Applications

In this section, first we shall use our results to study new fixed-point theorems for multivalued- and single-valued mappings in complete metric spaces. The following theorem extend and generalized Theorems 2.1 and 2.2 of [29] to multivalued mappings.

**Theorem 4.1.** *Let $(S,d)$ be a complete metric space and $\alpha,\eta : S\times S\times(0,\infty)\to \mathbb{R}^+$ be two functions. Suppose $T : S \to CB(S)$ is a multivalued mapping such that for each $a\in S$ and all $x\in Ta$ there exists $y\in Tx$ such that*

$$\mathrm{d}(x,y)\le H(Ta,Tx), \tag{4.1}$$

*where $H$ denotes the Hausdorff metric on $CB(S)$. Suppose that $T$ satisfies the following conditions:*

(i) *For some $k<1$ and every $x,y\in S$ that $\alpha(x,y,t)\le\eta(x,y,t)$, we have*

$$H(Tx,Ty)\le k\max\{\mathrm{d}(x,y),\mathrm{d}(x,Tx),\mathrm{d}(y,Ty)\};$$

(ii) *$T$ is $\alpha$-admissible w.r.t. $\eta$ on $S$;*

(iii) *For some $x_0\in S$ there exists $x_1\in Tx_0$ such that $\alpha(x_0,x_1,t)\le\eta(x_0,x_1,t)$ for all $t>0$;*

(iv) *$T$ is continuous or $S$ satisfies the condition $(C_{\alpha,\eta})$.*

*Then, $T$ has a fixed point.*

*Proof.* Let $(S,\mathcal{F},\triangle_M)$ be the induced Menger PM space by $(S,d)$. We show that the conditions of Theorem 2.12 are satisfied for $\psi(t)=kt$. Since $(S,d)$ is a complete metric space then $(S,\mathcal{F},\triangle_M)$ is complete too. Furthermore, one can prove that for any $x\in S$ and $A,B\in CB(S)$, we have

$$F_{x,A}(t)=\varepsilon_0(t-\mathrm{d}(x,A)) \quad\text{and}\quad \widetilde{F}_{A,B}(t)=\varepsilon_0(t-H(A,B)).$$

Thus, for each $x,y\in S$ that $\alpha(x,y,t)\le\eta(x,y,t)$, we get

$$\begin{aligned}\widetilde{F}_{Tx,Ty}(kt)&=\varepsilon_0(kt-H(Tx,Ty))=\varepsilon_0(t-\frac{1}{k}H(Tx,Ty))\\&\ge\varepsilon_0\left(t-\max\{\mathrm{d}(x,y),\mathrm{d}(x,Tx),\mathrm{d}(y,Ty)\}\right)\\&=\min\{\varepsilon_0(t-\mathrm{d}(x,y)),\varepsilon_0(t-\mathrm{d}(x,Tx)),\varepsilon_0(t-\mathrm{d}(y,Ty))\}\\&=\min\{F_{x,y}(t),F_{x,Tx}(t),F_{y,Ty}(t)\},\end{aligned}$$

for any $t>0$. Next, from (4.1), we have

$$F_{x,y}(t)=\varepsilon_0(t-\mathrm{d}(x,y))\ge\varepsilon_0(t-H(Ta,Tx))=\widetilde{F}_{Tx,Ta}(t),$$

for any $t>0$. Therefore, the conclusion follows from Theorem 2.12. □

*Remark 4.2.* (i) The above theorem is true if (4.1) is replaced by every compact-valued mapping.

(ii) Using our ideas in Theorem 2.18, it is possible to extend Theorem 4.1 to a complete metric space endowed with a partial order.

Now, we present the following result, which extend Theorems 2.1 and 2.2 of [29] for single-valued mappings in complete metric spaces.

**Theorem 4.3.** *Let $(S, d)$ be a complete metric space and $f : S \to S$ is a mapping satisfying the following conditions:*

(i) *For every $x, y \in S$ that $\alpha(x, y, t) \leq \eta(x, y, t)$, we have*

$$\mathrm{d}(fx, fy) \leq \psi(\mathrm{d}(x, y)),$$

*where $\psi \in \Psi$ is the strictly increasing function.*

(ii) *$f$ is $\alpha$-quasi-admissible with respect to $\eta$ on $S$;*

(iii) *There exists $x_0 \in S$ such that $\alpha(x_0, fx_0, t) \leq \eta(x_0, fx_0, t)$ for $t > 0$;*

(iv) *$f$ is continuous or $S$ satisfies the condition $(C_{\alpha,\eta})$.*

*Then, $f$ has a fixed point.*

*Proof.* Let $(S, \mathcal{F}, \triangle_M)$ be the induced Menger PM space by $(S, d)$. Since $(S, d)$ is a complete metric space then $(S, \mathcal{F}, \triangle_M)$ is complete. Now, let $x, y \in S$ such that $\alpha(x, y, t) \leq \eta(x, y, t)$ and $\psi \in \Psi$ is the strictly increasing function such that $\mathrm{d}(fx, fy) \leq \psi(\mathrm{d}(x, y))$, then by [19] (Theorem 2 and Remark 1), the mapping $f$ is satisfied in $F_{fx,fy}(\psi(t)) \geq F_{x,y}(t)$, on the Menger PM space $(S, \mathcal{F}, \triangle_M)$ induced by $(S, d)$. Therefore, all the conditions in Theorem 3.2 are satisfied and so $f$ has a fixed point. □

In what follows, we shall give a typical application of fixed-point theory to study the existence of the solution of nonlinear Volterra integral equations on Banach spaces. To apply the results in Sect. 3, some notations and basic definitions due to [34] are introduced here.

Let $I = [0, a]$ be a given real interval, $C(I, \mathbb{R})$ the Banach space of all real continuous functions defined on $I$ with the sup norm

$$\|x\|_\infty = \max_{t \in I} |x(t)|, \quad x \in C(I, \mathbb{R}),$$

and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ the space of all continuous functions defined on $I \times I \times C(I, \mathbb{R})$. Alternatively, the Banach space $C(I, \mathbb{R})$ can be endowed with Bielecki norm

$$\|x\|_B = \max_{t \in I}(|x(t)|e^{-Lt}), \quad x \in C(I, \mathbb{R}),\ L > 0,$$

and the induced metric $d_B(x, y) = \|x - y\|_B$ for all $x, y \in C(I, \mathbb{R})$, see [4]. Now, if the mapping is defined as $\mathcal{F} : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to D^+$ by

$$F_{x,y}(t) = \varepsilon_0(t - d_B(x, y)) \quad x, y \in C(I, \mathbb{R}),\ t > 0,$$

then the space $(C(I, \mathbb{R}), \mathcal{F}, \Delta_M)$ is the $\tau$-complete Menger PM space induced by $C(I, \mathbb{R})$, see Theorem 3 of [33]. In addition, one can prove that in the space $(C(I, \mathbb{R}), \mathcal{F}, \Delta_M)$, the convergence in norms $\|\cdot\|_\infty$ and $\|\cdot\|_B$ are equivalent to each other in $\tau$-topology.

Consider the nonlinear Volterra integral equation

$$x(t) = \int_0^t K(t, s, x(s))\mathrm{d}s + g(t), \quad t \in I, \tag{4.2}$$

and define $f : C(I, \mathbb{R}) \to C(I, \mathbb{R})$, by the formula

$$fx(t) := \int_0^t K(t, s, x(s))\mathrm{d}s + g(t), \quad g \in C(I, \mathbb{R}).$$

**Theorem 4.4.** *Consider Eq.* (4.2)*. Let* $(C(I,\mathbb{R}), \mathcal{F}, \Delta_M)$ *be Menger PM space induced by the Banach space* $C(I,\mathbb{R})$ *and suppose*

(S1) $K \in C((I \times I \times C(I,\mathbb{R})), \mathbb{R})$ *and*

$$\|K\|_\infty = \sup_{t,s\in I,\ x\in C(I,\mathbb{R})} |K(t,s,x(s))| < \infty;$$

(S2) *There exist* $\theta \in C((I\times I\times(0,\infty)), \mathbb{R})$ *and* $L > 0$ *such that if* $\theta(x,y,r) \geq 0$ *for* $x, y \in C(I,\mathbb{R})$ *and* $r > 0$*, then for every* $t, s \in I$ *we have*

$$\begin{aligned}&|K(t,s,x(s)) - K(t,s,y(s))| \\ &\quad \leq L\max\{|x(s)-y(s)|, |x(s)-fx(s)|, |y(s)-fy(s)|\};\end{aligned}$$

(S3) *There exists* $x_0 \in C(I,\mathbb{R})$ *such that* $\theta(x_0, fx_0, r) \geq 0$ *for all* $r > 0$*;*

(S4) *If* $\theta(x, fx, r) \geq 0$ *for each* $x \in C(I,\mathbb{R})$*, then* $\theta(fx, f^2x, r) \geq 0$ *for all* $r > 0$*;*

(S5) *If* $\{x_n\}$ *be a sequence in* $C(I,\mathbb{R})$ *such that* $\theta(x_n, x_{n+1}, r) \geq 0$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\theta(x_{n_k}, x, r) \geq 0$ *for all* $k \in \mathbb{N}$.

*Then, the Volterra-type integral equation* (4.2) *has a solution in* $C(I,\mathbb{R})$.

*Proof.* We shall check that hypotheses in Theorem 3.2 are satisfied. Let $S := C(I,\mathbb{R})$ be endowed with Bielecki-type norm, i.e., $\|x\|_B = \max_{t\in I}(|x(t)|e^{-Lt})$, for $x \in S$ where $L > 0$ satisfies condition $(S2)$. First, the space $(S, \mathcal{F}, \Delta_M)$ is Menger PM space, where $\Delta_M$ is a $t$-norm of H-type. In addition, let $x, y \in S$ such that $\theta(x(t), y(t), r) \geq 0$ for all $t \in I$ and $r > 0$. From $(S2)$, we have

$$\begin{aligned}d_B(fx, fy) &\leq \max_{t\in I}\int_0^t |K(t,s,x(s)) - K(t,s,y(s))|e^{L(s-t)}e^{-Ls}\mathrm{d}s \\ &\leq L\max\{d_B(x,y), d_B(x,fx), d_B(y,fy)\}\max_{t\in I}\int_0^t e^{L(s-t)}\mathrm{d}s \\ &\leq (1-e^{-aL})\max\{d_B(x,y), d_B(x,fx), d_B(y,fy)\}.\end{aligned}$$

Now, put $k = 1 - e^{-aL}$ and define the function $\eta : S \times S \times (0,\infty) \to \mathbb{R}^+$ by

$$\eta(x,y,r) = \begin{cases} 1 & \text{if } \theta(x,y,r) \geq 0, r \in I, \\ \frac{1}{2} & \text{otherwise.}\end{cases}$$

Then for any $r > 0$, $\eta(x,y,r) \geq 1$ implies

$$\begin{aligned}F_{fx,fy}(kr) &= \varepsilon_0(kr - d_B(fx,fy)) = \varepsilon_0\left(r - \frac{1}{k}d_B(fx,fy)\right) \\ &\geq \varepsilon_0(r - \max\{d_B(x,y), d_B(x,fx), d_B(y,fy)\}) \\ &= \min\{\varepsilon_0(r-d_B(x,y)), \varepsilon_0(r-d_B(x,fx)), \varepsilon_0(r-d_B(y,fy))\} \\ &= \min\{F_{x,y}(r), F_{x,fx}(r), F_{y,fy}(r)\},\end{aligned}$$

for all $x, y \in S$. This shows that the operator $f$ satisfies the condition (i) of Theorem 3.2 with respect to $\psi(r) = kr$ for $k \in (0,1)$ and $\alpha(x,y,r) = 1$ for all $x, y \in C(I,\mathbb{R})$ and $r > 0$. Finally, (ii), (iii) and (iv) of Theorem 3.2 are obviously true from $(S3)$, $(S4)$ and $(S5)$. Therefore, Theorem 3.2 ensures the existence of fixed point of $f$ that this fixed point is the solution of the integral equation (4.2). □

*Example 4.5.* Consider the continuous function $K(t,s,x) = \frac{e^{-s}}{t+1}\tilde{K}(t,x)$ where

$$\tilde{K}(t,x) = \begin{cases} -1 & \text{if } x \le 0, \\ \sqrt[3]{x} - 1 & \text{if } 0 \le x < 1, \\ \arctan(x-1) & \text{if } 1 \le x, \end{cases}$$

for all $t \in I = [0,1]$. In this case, we show that nonlinear Volterra integral equation

$$x(t) = \frac{-1}{t+1}(t + e^{-t}) + \int_0^t \frac{e^{-s}}{t+1}\tilde{K}(s, x(s))\mathrm{d}s, \ 0 \le s \le t \le 1 \tag{4.3}$$

is satisfied in conditions of Theorem 4.4. Clearly, $\|K\|_\infty = \frac{\pi}{2}$. Now, define function

$$\theta(x,y,r) = \begin{cases} 1 & \text{if } x, y \le 0, r \in I, \\ -1 & \text{otherwise.} \end{cases}$$

If $\theta(x,y,r) \ge 0$ then $x(t), y(t) \le 0$, which implies $\tilde{K}(t,x) = \tilde{K}(t,y) = -1$. That is,

$$|K(t,s,x) - K(t,s,y)| = 0 \le L\max\{|x-y|, |x-fx|, |y-fy|\},$$

for all $t,s \in I$. Moreover, $\theta(-1, f(-1), r) \ge 0$. Assume $\theta(x, fx, r) \ge 0$, then $x(t) \le 0$ and $fx(t) \le 0$, and so $\tilde{k}(t, fx) = -1$, thus

$$f^2x(t) = \frac{-1}{t+1}(t + e^{-t}) + \int_0^t \frac{e^{-s}}{t+1}\tilde{K}(s, fx(s))\mathrm{d}s = -1,$$

for all $fx(t) \le 0$. That is, $\theta(fx, f^2x, r) \ge 0$. Therefore, $\theta(x, fx, r) \ge 0$ implies $\theta(fx, f^2x, r) \ge 0$ for all $r > 0$. Further, let us assume that $x_n$ is a sequence in $C(I,\mathbb{R})$ such that $\theta(x_n, x_{n+1}, r) \ge 0$ for all $n \in \mathbb{N}$ and $x_n \to x$. Then $x_{n_k} = x_n \le 0$. That is, $\theta(x_{n_k}, x, r) \ge 0$ for all $k \in \mathbb{N}$. Thus, all conditions of Theorem 4.4 are satisfied. Hence, equation (4.3) has a solution in $C(I,\mathbb{R})$. Here, $x(t) = -1$ is a solution. Observe that, the inequality $|K(t,s,x) - K(t,s,y)| \le L\max\{|x-y|, |x-fx|, |y-fy|\}$ is not globally satisfied, i.e., there exist $t,s \in I$ and $x,y \in \mathbb{R}$ such that $|K(t,s,x) - K(t,s,y)| > L\max\{|x-y|, |x-fx|, |y-fy|\}$. To show this, let $L > 0$. For all $t,s \in I$ and each $0 < x < y < \frac{1}{\sqrt{(6eL)^3}} < 1$ we obtain

$$\begin{aligned} |K(t,s,x) - K(t,s,y)| &= \frac{e^{-s}}{t+1}|\sqrt[3]{x} - \sqrt[3]{y}| \\ &= \frac{e^{-s}}{t+1}\frac{|x-y|}{\sqrt[3]{x^2} + \sqrt[3]{y^2} + \sqrt[3]{x}\sqrt[3]{y}} > L|x-y|, \end{aligned}$$

and hence, $|\frac{K(t,s,x)-K(t,s,y)}{x-y}| \to \infty$ as $x,y \to 0$. Thus, we see the impact of the function $\theta$ in Theorem 4.4.

**Corollary 4.6.** *Consider Eq.* (4.2). *Suppose*

(i) $K : I \times I \times \mathbb{R} \to \mathbb{R}$ *is continuous with* $\|K\|_\infty < \infty$ *and* $K(t,s,\cdot)$ *is non-decreasing for each* $t,s \in I$*;*

(ii) *For each* $t,s \in I$ *and each* $x,y \in \mathbb{R}$ *with* $x \le y$*, we have*

$$|K(t,s,x) - K(t,s,y)| \le L\max\{|x-y|, |x - fx|, |y - fy|\};$$

(iii) *There exists* $x_0 \in C(I,\mathbb{R})$ *such that* $x_0(t) \le \int_0^t K(t,s,x_0(s))\mathrm{d}s + g(t)$*, for all* $t \in I$*.*

*Then, the integral equation* (4.2) *has a solution in* $C(I,\mathbb{R})$*.*

*Proof.* First, define the mapping $\theta(x,y,r) = y - x$ for $x,y \in \mathbb{R}$. From (iii), we have $x_0 \le fx_0$, then $\theta(x_0, fx_0, r) \ge 0$ and so the condition $(S3)$ of Theorem 4.4 holds. In addition, since $K$ is non-decreasing for each $t,s \in I$, thus $f$ is non-decreasing. This shows $(S4)$ and $(S5)$ of the same theorem are satisfied. Now, the existence follows from Theorem 4.4. □

**Corollary 4.7.** *Let* $(C(I,\mathbb{R}), \mathcal{F}, \Delta_M)$ *be Menger PM space induced by the Banach space* $C(I,\mathbb{R})$ *and suppose* $K \in C((I \times I \times C(I,\mathbb{R})),\mathbb{R})$ *satisfying the following conditions*

(i) $\|K\|_\infty = \sup_{t,s\in I,\ x\in C(I,\mathbb{R})} |K(t,s,x(s))| < \infty$*;*

(ii) *There exists* $L > 0$ *such that for all* $t,s \in I$ *and each* $x,y \in C(I,\mathbb{R})$*, we have*

$$\begin{aligned}&|K(t,s,x(s)) - K(t,s,y(s))|\\ &\quad\le L\max\{|x(s)-y(s)|, |x(s) - fx(s)|, |y(s) - fy(s)|\}.\end{aligned}$$

*Then, the integral equation* (4.2) *has a solution in* $C(I,\mathbb{R})$*.*

*Proof.* By Theorem 4.4 with $\theta(x,y,r) = 1$ for all $x,y \in C(I,\mathbb{R})$ and $r > 0$, we deduce the existence of the solution. □

*Example 4.8.* Consider the Volterra integral equation (4.2) where $K(t,s,x) = s\arctan(x)$ and $g(t) = \frac{1}{2}(3t - (1+t^2)\arctan t)$. Thus, we have

$$x(t) = \frac{1}{2}(3t - (1+t^2)\arctan t) + \int_0^t s\arctan(x(s))\mathrm{d}s,\ 0 \le s \le t \le 1. \tag{4.4}$$

It is obviously that $\|K\|_\infty = \frac{\pi}{2}$. Further, for arbitrarily fixed $x,y \in C(I,\mathbb{R})$ and for $t,s \in I$, we obtain

$$\begin{aligned}|K(t,s,x) - K(t,s,y)| &\le |\arctan x - \arctan y|\\ &\le |x-y| \le \max\{|x-y|, |x-fx|, |y-fy|\},\end{aligned}$$

thus, the continuous function $K$ satisfies assumption (ii) of Corollary 4.7. Hence, Eq. (4.4) has a solution in $C(I,\mathbb{R})$. Here, $x(t) = t$ is the solution of this equation.

*Example 4.9.* Consider the following nonlinear Volterra integral equation

$$x(t) = \frac{1}{3}t\cos(t^3) + t^3 - \frac{t}{3} + \int_0^t ts^2\sin(x(s))\mathrm{d}s, \quad t \in [0,1]. \tag{4.5}$$

Since, for continuous function $K$, $\|K\|_\infty = 1$ and also, for each $x, y \in C(I,\mathbb{R})$ and for $t, s \in I$, we have

$$\begin{aligned}|K(t,s,x) - K(t,s,y)| &= ts^2|\cos x - \cos y|\\ &\leq |x-y| \leq \max\{|x-y|, |x-fx|, |y-fy|\}.\end{aligned}$$

Therefore, Eq. (4.5) has a solution in $C(I,\mathbb{R})$ whose exact solution is $x(t) = t^3$.

*Example 4.10.* The equations of Hammerstein type $x(t) = \int_0^t \overline{K}(t-s)N(x(s))\,\mathrm{d}s + g(t)$ arise in nonlinear physical phenomenons such as electro-magnetic fluid dynamics, reformulation of boundary value problems with a nonlinear boundary condition (see [2]) and chemical absorption kinetics [22]. Here, we consider the following Hammerstein integral equations of nonlinear Volterra type

$$x(t) = e^t(\ln(e + e^{1-t}) - \ln 2) + \int_0^t \frac{e^{t-s}x(s)}{1+|x(s)|}\mathrm{d}s, \tag{4.6}$$

for $t \in [0,1]$, where it is a special case of equation (4.2) with $K(t,s,x) = \frac{e^{t-s}x(s)}{1+|x(s)|}$ and $g(t) = e^t(\ln(e+e^{1-t}) - \ln 2)$. It is derived that $K(t,s,x)$ is continuous and $|K(t,s,x)| \leq |e^{t-s}| \leq e$. Moreover,

$$\begin{aligned}|K(t,s,x) - K(t,s,y)| &\leq e \cdot |\frac{x}{1+|x|} - \frac{y}{1+|y|}|\\ &\leq e\cdot \mid |y| - |x| \mid \leq e|x-y|\\ &\leq e \cdot \max\{|x-y|, |x-fx|, |y-fy|\},\end{aligned}$$

for $t, s \in [0,1]$ and $x, y \in C(I,\mathbb{R})$, thus $L = e$. Then, Corollary 4.7 ensures the existence of a solution of (4.6). Here, the solution of this equation is $x(t) = e^t$.

*Example 4.11.* As the final example, consider the following Volterra integral equation

$$x(t) = \ln\left(1 + \frac{t(t+2)}{t^2+2t+2}\right) + \int_0^t \frac{1}{1+s}\tanh(x(s))\mathrm{d}s, \tag{4.7}$$

for $t \in [0,1]$. It is easily seen that $K(t,s,x) = \frac{1}{1+s}\tanh(x)$ satisfies in $\|K\|_\infty = 1$ and there exists $c \in \mathbb{R}$ such that

$$\begin{aligned}\left|\frac{\tanh x}{1+s} - \frac{\tanh y}{1+s}\right| &\leq |\tanh x - \tanh y| = (1 - \tanh^2 c)|x-y|\\ &\leq \max\{|x-y|, |x-fx|, |y-fy|\},\end{aligned}$$

for $t, s \in [0,1]$ and $x, y \in C(I,\mathbb{R})$. Hence, all the required conditions of Corollary 4.7 are satisfied, and (4.7) has a solution. Here, $x(t) = \ln(t+1)$ is the solution of this equation.

## 5. Conclusions

The new notions of contractions for multivalued and single-valued mappings in complete Menger PM spaces have been introduced. We proved the new fixed-point theorems for these new types of multivalued and single-valued mappings in complete Menger PM and partially ordered Menger PM spaces. Moreover, we extended our results to the case of coupled fixed points. These results extended, generalized and improved many existing results. In the final part of the paper, to illustrate the usability of our results, the fixed-point theorems for multivalued and single-valued mappings on metric spaces and so, the existence of solutions for nonlinear Volterra integral equations have been proved.

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