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Analytical discussion for the mixed integral equations

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Abstract. This paper presents a numerical method for the solution of a Volterra–Fredholm integral equation in a Banach space. Banachs fixed point theorem is used to prove the existence and uniqueness of the solution. To find the numerical solution, the integral equation is reduced to a system of linear Fredholm integral equations, which is then solved numerically using the degenerate kernel method. Normality and continuity of the integral operator are also discussed. The numerical examples in Sect. 5 illustrate the applicability of the theoretical results.

Mathematics Subject Classification. 45A05, 46B07, 65R20.

Keywords. Banach space, Volterra–Fredholm integral equation, degenerate kernel method.

1. Introduction

Integral equation is very important in continuum mechanics, geophysics, potential theory and biology, quantum mechanics, optimal control systems, population genetics, medicine and fracture mechanics, solid mechanics, economic problems, phase transitions, electrostatics, and others [12, 19]. Many problems of mathematical physics, applied mathematics, and engineering are reduced to Volterra–Fredholm integral equations, see [1–3, 5, 15, 16, 20], for this, many different methods are used to solve this type of equations analytically [11–13, 17]. In addition, for numerical methods, see [4, 7, 8, 10, 18].

In this paper, we consider the Volterra–Fredholm integral equation of the second kind with continuous kernels with respect to time and position. We use a numerical method to transform the Volterra–Fredholm integral equation to system of linear Fredholm integral equations, where the existence and the uniqueness of the solution of the system of linear Fredholm integral equations can be discussed and proved using Picard's method and Banach's fixed point method.

Birkhäuser

Consider the following Volterra–Fredholm integral equation:

$$y(u,t) = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau) w(u,v) y(v,\tau) \mathrm{d}v \mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) y(u,\tau) \mathrm{d}\tau, \tag{1.1}$$

where γ is a constant, the function $y(u,t)$ is unknown in the Banach space $L_2[0,1] \times C[0,T]$, $0 \leq T < 1$, $[0,1]$ is the domain of integration with respect to the position and the time $t \in [0,T]$. The kernels $\Phi(t,\tau)$, $\Psi(t,\tau)$ are continuous in $C[0,T]$ and the known function $g(u,t)$ is continuous in the space $L_2[0,1] \times C[0,T]$, $0 \leq t \leq T$. In addition, the kernel of position $w(u,v)$ belongs to $C([0,1] \times [0,1])$.

2. The existence and uniqueness of solution of the Volterra–Fredholm integral equation

In this paper, we assume the following conditions:

(i) $|w(u,v)| < C$, C is a constant;
(ii) $|\Phi(t,\tau)| \leq A$, $|\Psi(t,\tau)| \leq B$, A, B are constants $\forall\, t,\tau \in [0,T]$, $0 \leq \tau \leq t \leq T$;
(iii) $\|g(u,t)\| = \max_{0<t\leq T} \int_0^t \left[\int_0^1 g^2(u,\tau)\mathrm{d}u\right]^{\frac{1}{2}} \mathrm{d}\tau = D$, D is a constant.

Theorem 2.1. *Let the conditions (i)–(iii) be satisfied. If the condition*

$$|\, \gamma \,| < \frac{2}{T^2[AC+B]} \tag{2.1}$$

is satisfied, then Eq. (1.1) has an unique solution $y(u,t)$ in the space, $L_2[0,1] \times C[0,T]$.

Proof. To prove this theorem, we use the successive approximation method (*Picard's method*). Therefore, we assume that the solution of Eq. (1.1) takes the form:

$$y(u,t) = \lim_{n\to\infty} y_n(u,t),$$

where

$$y_n(u,t) = \sum_{i=0}^{n} G_i(u,t), \quad t \in [0,T], \quad n = 1,2,\ldots \tag{2.2}$$

where the functions $G_i(u,t)$, $i = 0,1,\ldots,n$ are continuous functions of the form:

$$\left.\begin{array}{l} G_n(u,t) = y_n(u,t) - y_{n-1}(u,t), \\ G_0(u,t) = g(u,t) \end{array}\right\}. \tag{2.3}$$

□

Now, we should prove the following lemmas:

Lemma 2.2. *The series $\sum_{i=0}^{n} G_i(u,t)$ is uniformly convergent to a continuous solution function $y(u,t)$.*

Proof. We structure a sequence $y_n(u,t)$ defined by

$$y_n(u,t) = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau) w(u,v) y_{n-1}(v,\tau) \mathrm{d}v \mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) y_{n-1}(u,\tau) \mathrm{d}\tau; \quad y_0(u,t) = g(u,t). \tag{2.4}$$

Then, we get

$$y_n(u,t) - y_{n-1}(u,t) = \gamma \int_0^t \int_0^1 \Phi(t,\tau) w(u,v) [y_{n-1}(v,\tau) - y_{n-2}(v,\tau)] \mathrm{d}v \mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) [y_{n-1}(u,t) - y_{n-2}(u,\tau)] \mathrm{d}\tau. \tag{2.5}$$

From Eq. (2.2) and using the properties of the norm, we obtain

$$\|G_n(u,t)\| \le |\gamma| \left\| \int_0^t \int_0^1 |\Phi(t,\tau)||w(u,v)| G_{n-1}(v,\tau) \mathrm{d}v \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t |\Psi(t,\tau)| G_{n-1}(u,\tau) \mathrm{d}\tau \right\|, \tag{2.6}$$

for $n = 1$, we get from formula (2.6)

$$\|G_1(u,t)\| \le |\gamma| \left\| \int_0^t \int_0^1 |\Phi(t,\tau)||w(u,v)| G_0(v,\tau) \mathrm{d}v \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t |\Psi(t,\tau)| G_0(u,\tau) \mathrm{d}\tau \right\|. \tag{2.7}$$

Using conditions (i), (ii) and (iii), we have

$$\begin{aligned} \|G_1(u,t)\| &\le |\gamma| ACD \left\| \int_0^t \int_0^1 \mathrm{d}v \mathrm{d}\tau \right\| + |\gamma| BD \left\| \int_0^t \mathrm{d}\tau \right\|, \\ &\le |\gamma| ACD \|t\| + |\gamma| BD \|t\|, \end{aligned} \tag{2.8}$$

where

$$\|t\| = \frac{1}{2} T^2, \quad T = \max_{0 \le t \le T} |t|,$$

so that Eq. (2.8) becomes

$$\|G_1(u,t)\| \le \frac{1}{2} |\gamma| T^2 D [AC + B], \tag{2.9}$$

by induction, we get

$$\|G_n(u,t)\| \le \eta^n D, \quad \eta = \frac{1}{2} |\gamma| T^2 [AC + B] < 1, \quad n = 1, 2, \ldots \tag{2.10}$$

Since

$$|\gamma| < \frac{2}{T^2 [AC + B]}, \tag{2.11}$$

this leads us to say that the sequence $y_n(u,t)$ has a convergent solution. So that, for $n \to \infty$, we have

$$y(u,t) = \sum_{i=0}^{\infty} G_i(u,t), \tag{2.12}$$

which represents an infinite convergent series. □

Lemma 2.3. *The function $y(u,t)$ of the series (2.12) represents an unique solution of Eq. (1.1).*

Proof. To show that $y(u,t)$ is the unique solution, we assume that there exists a different continuous solution $\tilde{y}(u,t)$ of Eq. (1.1), then we obtain

$$\begin{aligned} y(u,t) - \tilde{y}(u,t) &= \gamma \int_0^t \int_0^1 \Phi(t,\tau) w(u,v)[y(v,\tau) - \tilde{y}(v,\tau)] \mathrm{d}v \mathrm{d}\tau \\ &\quad + \gamma \int_0^t \Psi(t,\tau)[y(u,\tau) - \tilde{y}(u,\tau)] \mathrm{d}\tau. \end{aligned} \tag{2.13}$$

Using conditions (i) and (ii), we get

$$\begin{aligned} \|y(u,t) - \tilde{y}(u,t)\| &\leq |\gamma| AC \int_0^t \int_0^1 \|[y(v,\tau) - \tilde{y}(v,\tau)]\| \mathrm{d}v \mathrm{d}\tau \\ &\quad + |\gamma| B \int_0^t \|[y(u,\tau) - \tilde{y}(u,\tau)]\| \mathrm{d}\tau, \end{aligned} \tag{2.14}$$

$$\|y(u,t) - \tilde{y}(u,t)\| \leq \eta \|y(u,t) - \tilde{y}(u,t)\|, \quad \eta = \frac{1}{2}|\gamma| T^2 [AC + B] < 1. \tag{2.15}$$

The formula (2.15) can be adapted as:

$$(1 - \eta)\|y(u,t) - \tilde{y}(u,t)\| \leq 0.$$

Since $\eta < 1$, so that $y(u,t) = \tilde{y}(u,t)$, that is the solution is unique. □

3. The normality and continuity of the integral operator

Equation (1.1) can be written in the following integral operator form:

$$\overline{V} y = g(u,t) + V y, \quad V y = \Phi W y + \Psi y, \tag{3.1}$$

where

$$\Phi W y = \gamma \int_0^t \int_0^1 \Phi(t,\tau) w(u,v) y(v,\tau) \mathrm{d}v \mathrm{d}\tau, \quad \Psi y = \gamma \int_0^t \Psi(t,\tau) y(u,\tau) \mathrm{d}\tau.$$

3.1. The normality of the integral operator

For the normality, we use Eq. (3.1) to get

$$\begin{aligned} \|V y\| &\leq |\gamma| \left\| \int_0^t \int_0^1 |\Phi(t,\tau)||w(u,v)| y(v,\tau) \mathrm{d}v \mathrm{d}\tau \right\| \\ &\quad + |\gamma| \left\| \int_0^t |\Psi(t,\tau)| y(u,\tau) \mathrm{d}\tau \right\|. \end{aligned}$$

Using conditions (i) and (ii), we get

$$\|Vy\| \leq \frac{1}{2}|\gamma|T^2[AC+B]\|y\|,$$

since

$$\|Vy(u,t)\| \leq \eta\|y(u,t)\|, \quad \eta = \frac{1}{2}|\gamma|T^2[AC+B] < 1,$$

where

$$|\ \gamma\ |< \frac{2}{T^2[AC+B]}.$$

Therefore, the integral operator V has a normality, which leads immediately after using the condition (iii) to the normality of the operator $\overline{V}$.

3.2. The continuity of the integral operator

For the continuity, we suppose the two potential functions $y_1(u,t)$ and $y_2(u,t)$ in the space $L_2[0,1] \times C[0,T]$ are satisfied Eq. (3.1), then

$$\overline{V}y_1 = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau)w(u,v)y_1(v,\tau)\mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau)y_1(u,\tau)\mathrm{d}\tau, \tag{3.2}$$

$$\overline{V}y_2 = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau)w(u,v)y_2(v,\tau)\mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau)y_2(u,\tau)\mathrm{d}\tau. \tag{3.3}$$

Using Eqs. (3.2) and (3.3), we get

$$\overline{V}[y_1 - y_2] = \gamma \int_0^t \int_0^1 \Phi(t,\tau)w(u,v)[y_1(v,\tau) - y_2(v,\tau)]\mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau)[y_1(u,\tau) - y_2(u,\tau)]\mathrm{d}\tau.$$

Using conditions (i) and (ii), we get

$$\|\overline{V}[y_1 - y_2]\| \leq |\gamma|AC \int_0^t \int_0^1 \|[y_1(v,\tau) - y_2(v,\tau)]\|\mathrm{d}v\mathrm{d}\tau + |\gamma|B \int_0^t \|[y_1(u,\tau) - y_2(u,\tau)]\|\mathrm{d}\tau,$$

where

$$\|t\| = \frac{1}{2}T^2, \quad T = \max_{0\leq t\leq T} |t|,$$

so that last inequality becomes

$$\|\overline{V}[y_1 - y_2]\| \leq \eta\|y_1 - y_2\|, \quad \eta = \frac{1}{2}|\gamma|T^2[AC+B] < 1, \tag{3.4}$$

with

$$\mid \gamma \mid < \frac{2}{T^2[AC+B]}.$$

Inequality (3.4) leads us to the continuity of the integral operator $\overline{V}$. So that, $\overline{V}$ is a contraction operator. Therefore by Banach's fixed point theorem, there is an unique fixed point $y(u,t)$, which is the solution of the linear mixed integral Eq. (1.1).

4. The reduced system of Fredholm integral equations and its solution

4.1. Quadratic numerical method [6]

We tend to use the quadratic numerical method to reduce the solution of the Eq. (1) to the system of linear Fredholm integral equations of the second kind. We divide the interval $[0,T]$, $0 \le t \le T$, as $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T$, where $t = t_i$, $i = 0, 1, \ldots, N$; to get

$$\begin{aligned} y(u,t_i) &= g(u,t_i) + \gamma \int_0^{t_i} \int_0^1 \Phi(t_i,\tau) w(u,v) y(v,\tau) \mathrm{d}v \mathrm{d}\tau \\ &\quad + \gamma \int_0^{t_i} \Psi(t_i,\tau) y(u,\tau) \mathrm{d}\tau. \end{aligned} \tag{4.1}$$

The Volterra integral terms can be written as follows:

$$\begin{aligned} &\int_0^{t_i} \int_0^1 \Phi(t_i,\tau) w(u,v) y(v,\tau) \mathrm{d}v \mathrm{d}\tau \\ &\quad = \sum_{j=0}^{i} \mu_j \Phi(t_i,t_j) \int_0^1 w(u,v) y(v,t_j) \mathrm{d}v + O(\hbar_i^{\wp_1+1}), \end{aligned} \tag{4.2}$$

$$\int_0^{t_i} \Psi(t_i,\tau) y(u,\tau) \mathrm{d}\tau = \sum_{j=0}^{i} \nu_j \Psi(t_i,t_j) y(u,t_j) + O(\hbar_i^{\wp_2+1}), \tag{4.3}$$

where

$$(\hbar_i^{\wp_1+1} \longrightarrow 0,\ \hbar_i^{\wp_2+1} \longrightarrow 0, \quad \wp_1 > 0,\ \wp_2 > 0),$$
$$\hbar_i = \max_{0 \le j \le i} \rho_j \quad \text{and} \quad \rho_j = t_{j+1} - t_j.$$

The values of the weight formula μ_j, ν_j and the constants $\wp_1, \wp_2$ depend on the number of derivatives of $\Phi(t,\tau)$ and $\Psi(t,\tau)$, $\forall \tau \in [0,T]$, with respect to t.

Using Eqs. (4.2), (4.3) in Eq. (4.1), we get

$$\begin{aligned} y(u,t_i) &= g(u,t_i) + \gamma \sum_{j=0}^{i} \mu_j \Phi(t_i,t_j) \int_0^1 w(u,v) y(v,t_j) \mathrm{d}v \\ &\quad + \gamma \sum_{j=0}^{i} \nu_j \Psi(t_i,t_j) y(u,t_j). \end{aligned} \tag{4.4}$$

Substituting the following notations:

$$y(u,t_i)=y_i(u),\quad g(u,t_i)=g_i(u),\quad \Phi(t_i,t_j)=\Phi_{i,j},\quad \Psi(t_i,t_j)=\Psi_{i,j},$$

we can rewrite Eq. (4.4) in the form

$$\delta_i y_i(u)=g_i(u)+\gamma\sum_{j=0}^{i}\mu_j\Phi_{i,j}\int_0^1 w(u,v)y_j(v)\mathrm{d}v+\gamma\sum_{j=0}^{i-1}\nu_j\Psi_{i,j}\;y_j(u), \quad (4.5)$$

where $\delta_i=(1-\gamma_i),\quad \gamma_i=\gamma\nu_i\Psi_{i,i}$.

Equation (4.5), for $\delta_i\neq 0$, represents a system of linear Fredholm integral equations of the second kind, while for $\delta_i=0$, we have system of linear Fredholm integral equations of the first kind. The solution of the system (4.5), for $\delta_i\neq 0$, can be obtained see [6,14]. If we obtain, first, the value of $y_0(x)$, and let $i=0$ in the system (4.5), we get

$$\delta_0 y_0(u)=g_0(u)+\gamma\mu_0\Phi_{0,0}\int_0^1 w(u,v)y_0(v)\mathrm{d}v,\quad \delta_0=(1-\gamma\nu_0\Psi_{0,0}). \quad (4.6)$$

After obtaining the solution of Eq. (4.6), we can use the mathematical induction to obtain the general solution of (4.5).

4.2. The procedure of solution using degenerate kernel method

Here, we can find that the solution of the linear algebraic integral system (4.5) by applying the degenerate kernel method [11] naturally leads one to consider replacement the given $w(u,v)$ approximately by a degenerate kernel $w_n(u,v)$, that is

$$w_n(u,v)=\sum_{l=1}^{n}M_l(u)N_l(v), \quad (4.7)$$

where the set of functions $\{M_l(u)\}$ and $\{N_l(v)\}$ are assumed to be linearly independent and the degenerate kernel $w_n(u,v)$ should satisfy the condition

$$\left\{\int_0^1\int_0^1|w(u,v)-w_n(u,v)|^2\mathrm{d}u\mathrm{d}v\right\}^{\frac{1}{2}}\to 0,\quad \text{as}\quad n\to\infty. \quad (4.8)$$

Hence, the solution of Eq. (4.5) related to the degenerate kernels $w_n(u,v)$ takes the form:

$$\delta_i y_i^n(u)=g_i(u)+\gamma\sum_{j=0}^{i}\mu_j\Phi_{i,j}\int_0^1 w_n(u,v)y_j^n(v)\mathrm{d}v$$
$$+\gamma\sum_{j=0}^{i-1}\nu_j\Psi_{i,j}\;y_j^n(u),\quad \delta_i\neq 0. \quad (4.9)$$

Using Eq. (4.7) in Eq. (4.9), we have

$$\delta_i y_i^n(u)=g_i(u)+\gamma\sum_{j=0}^{i}\sum_{l=1}^{n}\mu_j\Phi_{i,j}M_l(u)\int_0^1 N_l(v)y_j^n(v)\mathrm{d}v+\gamma\sum_{j=0}^{i-1}\nu_j\Psi_{i,j}\;y_j^n(u). \quad (4.10)$$

We introduce the notations

$$\alpha_j^l = \int_0^1 N_l(v) y_j^n(v) \mathrm{d}v, \tag{4.11}$$

where α_j^l are unknown constants. Then, Eq. (4.10) takes the form:

$$\delta_i y_i^n(u) = g_i(u) + \gamma \sum_{j=0}^{i} \sum_{l=1}^{n} \mu_j \Phi_{i,j} M_l \alpha_j^l + \gamma \sum_{j=0}^{i-1} \nu_j \Psi_{i,j} \ y_j^n(u), \qquad (\delta_i \neq 0). \tag{4.12}$$

Substituting from Eq. (4.12) into Eq. (4.11), we get

$$\alpha_j^r = \int_0^1 \frac{N_r(v)}{\delta_j} \left[g_j(v) + \gamma \sum_{m=0}^{j} \sum_{l=0}^{n} \mu_m \Phi_{j,m} M_l(v) \alpha_m^l + \gamma \sum_{m=0}^{j-1} \nu_m \Psi_{j,m} y_m^n(v) \right] \mathrm{d}v,$$
$$\delta_j \neq 0, \quad j = 0, 1, \ldots, i \quad r = 1, 2, \ldots, n. \tag{4.13}$$

Define the function

$$K_j^r(\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^n) = \int_0^1 \frac{N_r(v)}{\delta_j} \left[g_j(v) + \gamma \sum_{m=0}^{j} \sum_{l=1}^{n} \mu_m \Phi_{j,m} M_l(v) \alpha_m^l \right.$$
$$\left. + \gamma \sum_{m=0}^{j-1} \nu_m \Psi_{j,m} y_m^n(v) \right] \mathrm{d}v, \tag{4.14}$$

Equation (4.13) represents a system of linear algebraic equations which can be written in the following form [9]:

$$\begin{pmatrix} \alpha_j^1 \\ \alpha_j^2 \\ . \\ . \\ . \\ \alpha_j^n \end{pmatrix} = \begin{pmatrix} K_j^1(\alpha_j^1, \alpha_j^2, \cdots, \alpha_j^n) \\ K_j^2(\alpha_j^1, \alpha_j^2, \cdots, \alpha_j^n) \\ . \\ . \\ . \\ K_j^n(\alpha_j^1, \alpha_j^2, \cdots, \alpha_j^n) \end{pmatrix}. \tag{4.15}$$

Formula (4.15) represents a system of linear algebraic equations and we can solve it numerically. After we get the values of α_j^r, we can immediately determine the values of the functions $y_i(u)$.

Lemma 4.1. *Let $w_n(u,v) \in C([0,1] \times [0,1])$ with the condition (4.8), then the following condition is satisfied*

$$\left\{ \int_0^1 \int_0^1 |w_n(u,v)|^2 \mathrm{d}u\mathrm{d}v \right\}^{\frac{1}{2}} \leq C, \quad \forall n > n_0, \quad n_0 \in N, \quad s.t \quad C \quad is \quad a\ constant. \tag{4.16}$$

Proof. To prove this lemma, we use the formula

$$\left\{ \int_0^1 \int_0^1 |w_n(u,v)|^2 \mathrm{d}u\mathrm{d}v \right\}^{\frac{1}{2}} \leq \left\{ \int_0^1 \int_0^1 [|w(u,v) - w_n(u,v)| + |w(u,v)|]^2 \mathrm{d}u\mathrm{d}v \right\}^{\frac{1}{2}},$$

using condition (4.8), we get

$$\forall\varepsilon > 0, \quad \exists \quad n_0 \in N : \quad \left\{\int_0^1\int_0^1 |w(u,v)-w_n(u,v)|^2 \mathrm{d}u\mathrm{d}v\right\}^{\frac{1}{2}} < \varepsilon, \quad \forall n > n_0.$$

Applying Minkowski's inequality and using condition (i), we get

$$\forall\varepsilon > 0, \quad \exists \quad n_0 \in N : \quad \left\{\int_0^1\int_0^1 |w_n(u,v)|^2 \mathrm{d}u\mathrm{d}v\right\}^{\frac{1}{2}} < C, \quad \forall n > n_0.$$

This completes the proof. □

4.3. The existence and uniqueness of the numerical solution

In this subsection, we will present the proof of the existence and uniqueness of the numerical solution of the system under study. This aim will be achieved through the following theorems:

Theorem 4.2. *The integral equation*

$$y_n(u,t) = g(u,t) + \gamma\int_0^t\int_0^1 \Phi(t,\tau)w_n(u,v)y(v,\tau)\mathrm{d}v\mathrm{d}\tau + \gamma\int_0^t \Psi(t,\tau)y(u,\tau)\mathrm{d}\tau, \tag{4.17}$$

has an unique solution $y_n(u,t)$ *in* $L_2[0,1]\times C[0,T]$, *under the condition:*

$$|\gamma| < \frac{2}{T^2[AC+B]}.$$

Proof. Defining for each $n > n_0$, the integral operator

$$\overline{V}y_n = g(u,t) + Vy_n(u,t), \quad Vy_n = \Phi Wy_n + \Psi y_n, \tag{4.18}$$

where

$$\Phi Wy_n = \gamma\int_0^t\int_0^1 \Phi(t,\tau)w_n(u,v)y_n(v,\tau)\mathrm{d}v\mathrm{d}\tau, \quad \Psi y_n = \gamma\int_0^t \Psi(t,\tau)y_n(u,\tau)\mathrm{d}\tau. \tag{4.19}$$

Firstly, for the normality, we use Eq. (4.18) to get

$$\|Vy_n\| \le |\gamma|\left\|\int_0^t\int_0^1 |\Phi(t,\tau)||w_n(u,v)|y_n(v,\tau)\mathrm{d}v\mathrm{d}\tau\right\| + |\gamma|\left\|\int_0^t |\Psi(t,\tau)|y_n(u,\tau)\mathrm{d}\tau\right\|.$$

Using condition (4.8), we get

$$\|Vy_n\| \le \frac{1}{2}|\gamma|T^2[AC+B]\|y_n\|,$$

then

$$\|Vy_n(u,t)\| \le \eta\|y_n(u,t)\|, \quad \eta = \frac{1}{2}|\gamma|T^2[AC+B] < 1,$$

where

$$|\gamma| < \frac{2}{T^2[AC+B]}.$$

Therefore, the integral operator V has a normality, which leads immediately after using the condition (iii) to the normality of the operator $\overline{V}$.

Secondly, for the continuity, we assume the two functions $y_{n1}(u,t)$, $y_{n2}(u,t)$ satisfy Eq. (4.19), then

$$\overline{V}y_{n1} = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau) w_n(u,v) y_{n1}(v,\tau) \mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) y_{n1}(u,\tau)\mathrm{d}\tau, \tag{4.20}$$

$$\overline{V}y_{n2} = g(u,t) + \gamma \int_0^t \int_0^1 \Phi(t,\tau) w_n(u,v) y_{n2}(v,\tau) \mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) y_{n2}(u,\tau)\mathrm{d}\tau. \tag{4.21}$$

Using conditions (4.16) and (ii), we get

$$\|\overline{V}[y_{n1} - y_{n2}]\| \le |\gamma| AC \int_0^t \int_0^1 \|[y_{n1}(v,\tau) - y_{n2}(v,\tau)]\| \mathrm{d}v\mathrm{d}\tau + |\gamma| B \int_0^t \|[y_{n1}(u,\tau) - y_{n2}(u,\tau)]\| \mathrm{d}\tau,$$

where

$$\|t\| = \frac{1}{2}T^2, \quad T = \max_{0\le t\le T} |t|,$$

so that the last inequality becomes

$$\|\overline{V}[y_{n1} - y_{n2}]\| \le \eta \|y_{n1} - y_{n2}\|, \quad \eta = \frac{1}{2}|\gamma| T^2 [AC + B] < 1, \tag{4.22}$$

with

$$|\,\gamma\,| < \frac{2}{T^2[CA + B]}.$$

Hence, $\overline{V}$ is a contraction in the space $L_2[0,1] \times C[0,T]$. Therefore, by Banach's fixed point theorem, there is an unique fixed point $y_n(u,t)$, which is the solution of the linear V–FIE (4.17). □

Theorem 4.3. *Under the condition (4.8), the unique solution $y_n(u,t)$ of integral Eq. (4.17) converges to the unique solution $y(u,t)$ of integral Eq. (1.1).*

Proof. From Eqs. (1.1) and (4.17), we have

$$y(u,t) - y_n(u,t) = \gamma \int_0^t \int_0^1 \Phi(t,\tau) \left[w(u,v)y(v,\tau) - w_n(u,v)y_n(v,\tau)\right] \mathrm{d}v\mathrm{d}\tau + \gamma \int_0^t \Psi(t,\tau) \left[y(u,\tau) - y_n(u,\tau)\right] \mathrm{d}\tau. \tag{4.23}$$

Using properties of the norm and condition (ii), we get

$$\|y(u,t)-y_n(u,t)\| \leq |\gamma| A \int_0^t \int_0^1 \|w(u,v)-w_n(u,v)\| \|y(v,\tau)\| \mathrm{d}v\mathrm{d}\tau$$
$$+ |\gamma| A \int_0^t \int_0^1 \|(w_n(u,v)\| \|y(v,\tau)-y_n(v,\tau)\| \mathrm{d}v\mathrm{d}\tau$$
$$+ |\gamma| B \int_0^t \|y(u,\tau)-y_n(u,\tau)\| \mathrm{d}\tau.$$

Using condition (4.16), the last inequality becomes

$$\|y(u,t)-y_n(u,t)\| \leq |\gamma| AT \|w(u,v)-w_n(u,v)\| \|y(u,t)\|$$
$$+ |\gamma| T(AC+B) \|y(u,t)-y_n(u,t)\|.$$

The last inequality can be adapted as:

$$\|y-y_n\| \leq \frac{|\gamma| AT \|y\|}{1-\eta} \|w-w_n\|, \quad \eta = \frac{1}{2} |\gamma| T^2 (AC+B) < 1,$$

subsequently, if $\|w-w_n\| \to 0$ as $n \to \infty$, we have $\|y-y_n\| \to 0$ as $n \to \infty$.

Hence, the proof is completed. □

Theorem 4.4. *From Eq. (4.14) and the degenerate kernel $w_n(u,v)$, the linear algebraic system (4.15) has the unique solution $(\alpha_j^1, \alpha_j^2, \cdots, \alpha_j^n)$, so that the Eq. (4.12) has the unique solution $y_i^n(u)$.*

Proof. We use the notations $(\alpha_j^1, \alpha_j^2, \cdots, \alpha_j^n)$, $(\beta_j^1, \beta_j^2, \cdots, \beta_j^n)$ as the two different solutions of (4.15) and from Eq. (4.14), we get

$$K_j^r(\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^n) = \int_0^1 \frac{N_r(v)}{\delta_j} \left[g_j(v) + \gamma \sum_{m=0}^{j} \sum_{l=1}^{n} \mu_m \Phi_{j,m} M_l(v) \alpha_m^l \right.$$
$$\left. + \gamma \sum_{m=0}^{j-1} \nu_m \Psi_{j,m} y_m^n(v) \right] \mathrm{d}v,$$

and

$$K_j^r(\beta_j^1, \beta_j^2, \ldots, \beta_j^n) = \int_0^1 \frac{N_r(v)}{\delta_j} \left[g_j(v) + \gamma \sum_{m=0}^{j} \sum_{l=1}^{n} \mu_m \Phi_{j,m} M_l(v) \beta_m^l \right.$$
$$\left. + \gamma \sum_{m=0}^{j-1} \nu_m \Psi_{j,m} \; y_m^n(v) \right] \mathrm{d}v.$$

Using the properties of the norm, we obtain

$$\|K_j^r(\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^n) - K_j^r(\beta_j^1, \beta_j^2, \ldots, \beta_j^n)\|$$
$$\leq QE \left| \frac{\gamma}{\delta_j} \right| \left\| \int_0^1 |N_r(v)| \left[\sum_{m=0}^{j} \sum_{l=1}^{n} |M_l(v)| |(\alpha_m^l - \beta_m^l)| \right] \mathrm{d}v \right\|,$$

where $Q = \left\{\sum_{m=0}^{j} |\mu_m|^2\right\}^{\frac{1}{2}}$ and $E = \left\{\sum_{m=0}^{j} |\Phi_{j,m}|^2\right\}^{\frac{1}{2}}$, which can be written in the vector form:

$$\|\overline{K_j}(\overline{\alpha_j}) - \overline{K_j}(\overline{\beta_j})\| \leq QE\left|\frac{\gamma}{\delta_j}\right|\left\{\sum_{r=0}^{n}\int_0^1 |N_r(v)|^2\mathrm{d}v\right\}^{\frac{1}{2}}.\left\{\sum_{l=0}^{n}\int_0^1 |M_l(v)|^2\mathrm{d}v\right\}^{\frac{1}{2}} .\|(\overline{\alpha_m}) - (\overline{\beta_m})\|.$$

So the operator $\overline{K_j}$ is continuous under the condition:

$$QE\left|\frac{\gamma}{\delta_j}\right|\left\{\sum_{r=0}^{n}\int_0^1 |N_r(v)|^2\mathrm{d}v\right\}^{\frac{1}{2}}.\left\{\sum_{l=0}^{n}\int_0^1 |M_l(v)|^2\mathrm{d}v\right\}^{\frac{1}{2}} < 1.$$

By Banach's fixed point theorem, $\overline{K_j}$ has an unique fixed point $\overline{\alpha_j}$, that is, of course the unique solution of the system (4.15). It is obvious, for the only solution $(\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^n)$, there is only solution $y_i^n(u)$ of Eq. (4.12). □

5. Application and numerical results

In this section, we try to apply some of the numerical methods to approximate the solution of the Volterra–Fredholm integral equation.

Example. Consider the following linear Volterra–Fredholm integral equation:

$$y(u,t) = u^2t^2 - \frac{u^2t^6}{16} - \frac{u^2t^7}{5} + \int_0^t\int_0^1 t^2\tau(u^2v)y(v,\tau)\mathrm{d}v\mathrm{d}\tau + \int_0^t t^2\tau^2 y(u,\tau)\mathrm{d}\tau. \tag{5.1}$$

If $|\gamma| < 16.075102$ (when $T = 0.6$), we find that the numerical solution quickly converges with the exact solution $y(u,t) = u^2t^2$. We divide the interval $[0, T]$, $0 \leq T < 1$, as $0 = t_0 < t_1 < t_2 < t_3 = T$, where, $t = t_i$, $i = 0, 1, 2, 3$, the linear Volterra–Fredholm integral Eq. (5.1) take the form:

$$\delta_i y_i(u) = u^2t_i^2 - \frac{u^2t_i^6}{16} - \frac{u^2t_i^7}{5} + \sum_{j=0}^{i}\mu_j t_i^2 t_j u^2\alpha_j + \sum_{j=0}^{i-1}\nu_j t_i^2 t_j^2 y_j(u),$$

where

$$\alpha_j = \int_0^1 \frac{v}{\delta_j}\left[v^2t_{ij}^2 - \frac{v^2t_j^6}{16} - \frac{v^2t_j^7}{5} + \sum_{m=0}^{j}\mu_m t_j^2 t_m v^2\alpha_m + \sum_{m=0}^{j-1}\nu_m t_j^2 t_m^2 y_m(v)\right]\mathrm{d}v.$$

In Table 1, we presented the absolute error $|y(u,t_i) - y_i(u)|$, $i = 0, 1, 2, 3$, using the introduced numerical method (degenerate kernel) with $N = 3$ in the interval $[0, 0.6]$.

In addition, in Figs. 1, 2, 3, and 4, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method with different values of t_i, $i = 0, 1, 2, 3$ with $N = 3$ in the interval $[0, 1]$.

Table 1. Absolute error of solution of Eq. (5.1) using the degenerate kernel with $N = 3$ and $0 \leq t \leq 0.6$

u	$\|y(u,t_0) - y_0(u)\|$	$\|y(u,t_1) - y_1(u)\|$	$\|y(u,t_2) - y_2(u)\|$	$\|y(u,t_3) - y_3(u)\|$
0.1	4.00078×10^{-9}	7.84282×10^{-8}	1.39661×10^{-7}	1.47198×10^{-7}
0.2	1.60031×10^{-8}	3.13713×10^{-7}	5.58645×10^{-7}	5.88792×10^{-7}
0.3	3.60071×10^{-8}	7.05854×10^{-7}	1.25695×10^{-6}	1.32478×10^{-6}
0.4	6.40125×10^{-8}	1.25485×10^{-6}	2.23458×10^{-6}	2.35517×10^{-6}
0.5	1.00020×10^{-7}	1.96071×10^{-6}	3.49153×10^{-6}	3.67995×10^{-6}
0.6	1.44028×10^{-7}	2.82342×10^{-6}	5.02781×10^{-6}	5.29913×10^{-6}
0.7	1.96038×10^{-7}	3.84298×10^{-6}	6.84340×10^{-6}	7.21270×10^{-6}
0.8	2.56050×10^{-7}	5.01941×10^{-6}	8.93832×10^{-6}	9.42067×10^{-6}
0.9	3.24064×10^{-7}	6.35269×10^{-6}	1.13126×10^{-5}	1.19230×10^{-5}

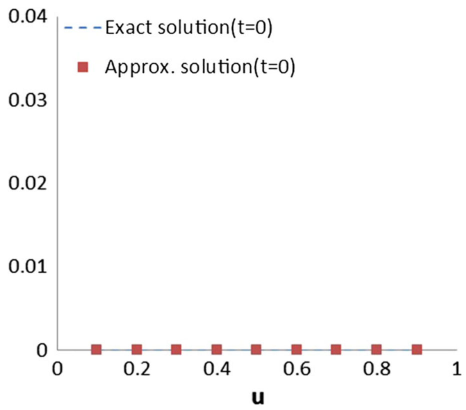

Figure 1. The exact solution $y(u,t_0) = u^2 t_0^2$ and the approximate solution $y_0(u)$

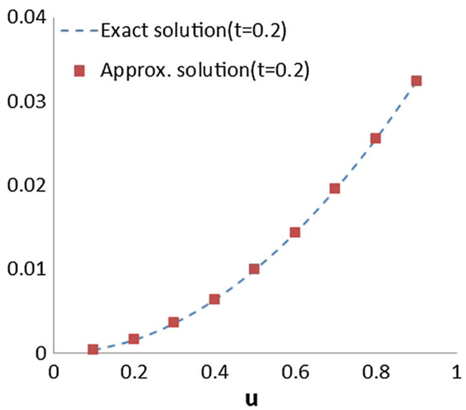

Figure 2. The exact solution $y(u,t_1) = u^2 t_1^2$ and the approximate solution $y_1(u)$

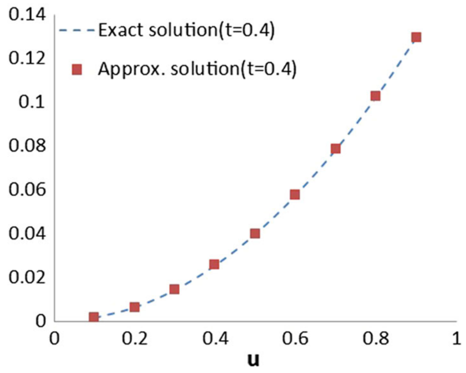

FIGURE 3. The exact solution $y(u,t_2) = u^2t_2^2$ and the approximate solution $y_2(u)$

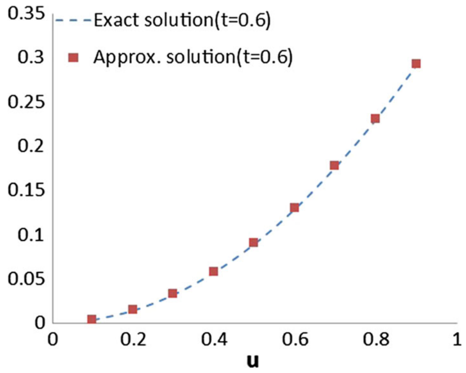

FIGURE 4. The exact solution $y(u,t_3) = u^2t_3^2$ and the approximate solution $y_3(u)$

Example. Consider the following linear Volterra–Fredholm integral equation:

$$y(u,t) = u^2e^t - \frac{u^2t^5}{3} - \frac{1}{36}t^5(4+3u) + \int_0^t\int_0^1 t^2\tau(1+e^uv)y(v,\tau)\mathrm{d}v\mathrm{d}\tau + \int_0^t t^2\tau y(u,\tau)\mathrm{d}\tau. \tag{5.2}$$

The numerical solution quickly converges with the exact solution $y(u,t) = u^2e^t$ if $|\gamma| < 5.451171$ (when $T = 0.6$). If we divide the interval $[0,T], 0 \leq T < 1$, as $0 = t_0 < t_1 < t_2 < t_3 = T$, where, $t = t_i$, $i = 0,1,2,3$, the linear Volterra–Fredholm integral Eq. (5.2) take the form:

$$\delta_i y_i(u) = u^2e^{t_i} - \frac{u^2t_i^5}{3} - \frac{1}{36}t_i^5(4+3u) + \sum_{j=0}^{i}\mu_j t_i^2 t_j(\alpha_j + e^u\beta_j) + \sum_{j=0}^{i-1}\nu_j t_i^2 t_j y_j(u),$$

Table 2. Absolute error of solution of Eq. (5.2) using the degenerate kernel with $N = 3$ and $0 \leq t \leq 0.6$

u	$\lvert y(u,t_0) - y_0(u)\rvert$	$\lvert y(u,t_1) - y_1(u)\rvert$	$\lvert y(u,t_2) - y_2(u)\rvert$	$\lvert y(u,t_3) - y_3(u)\rvert$
0.1	2.00008×10^{-9}	7.84282×10^{-8}	1.39661×10^{-7}	1.47198×10^{-7}
0.2	8.00031×10^{-9}	3.13713×10^{-7}	5.58645×10^{-7}	5.88792×10^{-7}
0.3	1.80007×10^{-8}	7.05854×10^{-7}	1.25695×10^{-6}	1.32478×10^{-6}
0.4	3.20013×10^{-8}	1.25485×10^{-6}	2.23458×10^{-6}	2.35517×10^{-6}
0.5	5.00020×10^{-8}	1.96071×10^{-6}	3.49153×10^{-6}	3.67995×10^{-6}
0.6	7.20028×10^{-8}	2.82342×10^{-6}	5.02781×10^{-6}	5.29913×10^{-6}
0.7	9.80038×10^{-8}	3.84298×10^{-6}	6.84340×10^{-6}	7.21270×10^{-6}
0.8	1.28005×10^{-8}	5.01941×10^{-6}	8.93832×10^{-6}	9.42067×10^{-6}
0.9	1.62006×10^{-8}	6.35269×10^{-6}	1.13126×10^{-5}	1.19230×10^{-5}

where

$$\alpha_j = \int_0^1 \frac{1}{\delta_j}[v^2 e^{t_j} - \frac{v^2 t_j^5}{3} - \frac{1}{36} t_j^5 (4+3v) + \sum_{m=0}^{j} \mu_m t_j^2 t_m (\alpha_m + e^v \beta_m) + \sum_{m=0}^{j-1} \nu_m t_j^2 t_m y_m(v)] \mathrm{d}v, \quad j = 0,1,2,3,$$

and

$$\beta_j = \int_0^1 \frac{v}{\delta_j}[v^2 e^{t_j} - \frac{v^2 t_j^5}{3} - \frac{1}{36} t_j^5 (4+3v) + \sum_{m=0}^{j} \mu_m t_j^2 t_m (\alpha_m + e^v \beta_m) + \sum_{m=0}^{j-1} \nu_m t_j^2 t_m y_m(v)] \mathrm{d}v, \quad j = 0,1,2,3,$$

In Table 2, we presented the absolute error $|y(u,t_i) - y_i(u)|$, $i = 0,1,2,3$, using the introduced numerical method (degenerate kernel) with $N = 3$ in the interval $[0, 0.6]$.

In addition, in Figs. 5, 6, 7, and 8, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method with different values of t_i, $i = 0,1,2,3$ with $N = 3$ in the interval $[0, 1]$.

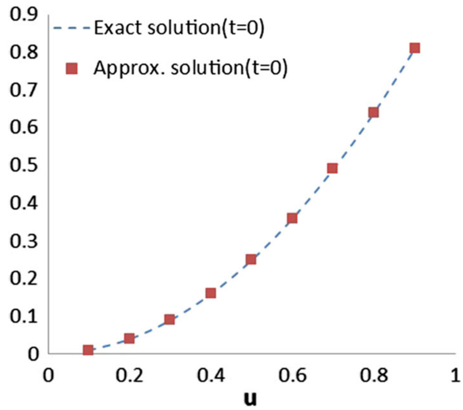

FIGURE 5. The exact solution $y(u, t_0) = u^2 t_0$ and the approximate solution $y_0(u)$

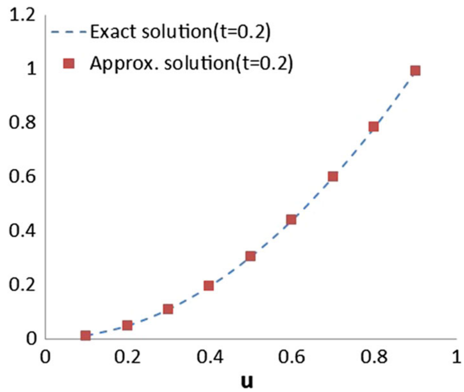

FIGURE 6. The exact solution $y(u, t_1) = u^2 t_1$ and the approximate solution $y_1(u)$

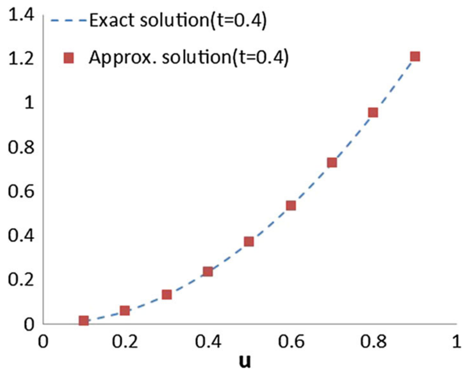

FIGURE 7. The exact solution $y(u, t_2) = u^2 t_2$ and the approximate solution $y_2(u)$

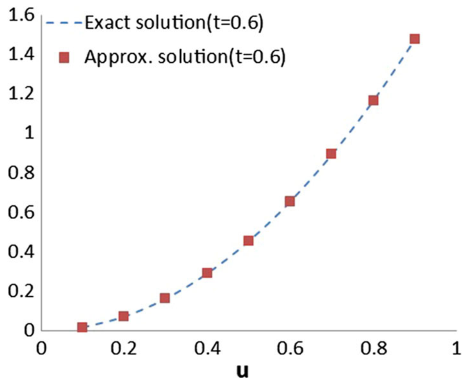

FIGURE 8. The exact solution $y(u, t_3) = u^2 t_3$ and the approximate solution $y_3(u)$

6. Conclusion and remarks

From the above results and discussion, the following may be concluded:

1. Equation (1.1) has a unique solution $y(u, t)$ in the space $L_2[0, 1] \times C^2[0, T]$, under some conditions.
2. The mixed integral equation of the second kind, in time and position, after using quadratic method leads to a system of linear Fredholm integral equations of the second kind in position.
3. The solution of the system of linear Fredholm integral equations is obtained using the degenerate kernel method.
4. The error value increases when it gets closer to the ends points $u = \pm 1$. It decreases at the middle when it gets closer to zero.

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