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Positive solutions to a fractional thermostat model in Banach spaces via fixed point results

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Abstract. The motive behind this manuscript is to set up the existence
and uniqueness of positive solutions to a fractional thermostat model
for certain values of the parameter $\lambda > 0$. We accomplish sufficient
conditions for the existence of positive solutions to the model, and af-
terwards formulate a couple of non-trivial examples to authenticate the
grounds of our obtained results. Our findings are based on certain fixed
point results for contractions depending on a couple of altering distance
functions ϕ and ψ in the setting of Banach spaces.

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sequence, thermostat model.

1. Introduction and preliminaries

Metric fixed point theory is extensively employed in different mathematical
branches as well as in real-world problems originating in applied sciences. The
results on fixed points of contractive maps considered on different underlying
spaces are mostly applied on the validation of the existence and uniqueness
of solutions of functional, differential or integral equations. The plurality of
these types of problems elicits the probe to more and better techniques, which
is a salient feature of the recent research works in this literature.

The dawning of fixed point theory on a complete metric space is inte-
grated with the Banach contraction principle due to Banach [6].

Theorem 1.1. *Let (X, d) be a complete metric space and T be a self-mapping
on X satisfying*

$$d(Tx, Ty) \leq kd(x, y)$$

*for all $x, y \in X$ and $k \in [0, 1)$. Then T has a unique fixed point $z \in X$, and
for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to z.*

Because of its inferences and huge usability in mathematical theory, Banach contraction principle has been improved and generalized in metric spaces, partially ordered metric spaces, Banach spaces and many other spaces, see [1,3,4,7,11–14,17,21,24].

In 1962, Rakotch [23] proved that the Theorem 1.1 still holds if the constant k is replaced by a contraction monotone decreasing function. He proved the following theorem as a corollary.

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \alpha(x, y)d(x, y)$$

for all $x, y \in X$, where α is a function defined on $[0, \infty)$ satisfying the following conditions:

(i) $\alpha(x, y) = \alpha(d(x, y))$, *i.e., α is dependent on the distance of x and y only;*
(ii) $0 \leq \alpha(\tau) < 1$ *for all $\tau > 0$;*
(iii) $\alpha(\tau)$ *is monotonically decreasing function of τ.*

Then T has a unique fixed point.

In his research article, Jaggi [19] used the continuity and some different contractive conditions on the mapping to attain the succeeding result.

Theorem 1.3. *Let f be a continuous self-map defined on a complete metric space (X, d). Further let f satisfy the following condition:*

$$d(f(x), f(y)) \leq \frac{\alpha d(x, f(x))d(y, f(y))}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, with $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then f has a unique fixed point in X.

In this connection, the readers are referred to the pertinent papers [25, 26] for more interesting results.

Thereafter, Khan et al. [20] extended and generalized the Banach principle using a control function, known as altering distance function.

Definition 1.4. A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it satisfies the following conditions:

(i) φ is monotone increasing and continuous;
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

In [20], the authors also proved the following fixed point theorem by means of the newly originated concept of control functions.

Theorem 1.5. *Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$. Also suppose that $f : X \to X$ is a mapping satisfying*

$$\psi(d(fx, fy)) \leq a\psi(d(x, y))$$

for all $x, y \in X$ and for some $0 \leq a < 1$. Then f has a unique fixed point.

Alber and Guerre-Delabriere [2] introduced the notion of weak contractions in a Hilbert space.

Definition 1.6 [2]. Let (X, d) be a metric space. A mapping $T : X \to X$ is called weakly contractive if and only if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

for all $x, y \in X$, where ϕ is an altering distance function.

In 2015, Salazar and Reich [28] proved that a self-mapping defined on a bounded set is of Rakotch type contractive map iff it is a weak contraction in the sense of Guerre-Delabriere. Rhoades [27] generalized the weak contraction condition in metric spaces and proved the following fixed point result in complete metric spaces.

Theorem 1.7. *Let (X, d) be a complete metric space. If $T : X \to X$ is a weakly contractive map, then T has a unique fixed point.*

In their research paper, Dutta and Choudhury [16] generalized Theorems 1.5 and 1.7 to obtain the following theorem.

Theorem 1.8. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$, where ψ and ϕ are two altering distance functions. Then T has a unique fixed point.

Fractional calculus has been explored for many decades mostly as a pure analytic mathematical branch. Though in recent times, many authors are showing a lot of interest in its applications for solving ordinary differential equations. Fractional differential equations appear in different engineering and scientific branches as the mathematical modeling of systems and techniques in the domains of physics, chemistry, aerodynamics, robotics and many more. For a few recent articles in this direction, see [5, 8, 9, 15, 18, 22, 29] and the references in that respect.

Considering exclusively positive solutions are effective for several applications, inspired by the aforementioned works, in our draft, we set up an existence and uniqueness theorem to find positive solutions to a fractional thermostat model with a positive parameter. With a view to inspect the solutions, we enquire into some new fixed point results in a Banach space by considering a pair of altering distance functions in a more adequate appearance. We also extend our results in a Banach space which is equipped with an arbitrary binary relation and keeps the order-preserving property of the mappings. Finally, some suitable constructive examples are furnished to substantiate the effectiveness of our results.

2. Fixed point results

This section deals with the results on the existence and uniqueness of fixed points of maps satisfying a contractive condition with a pair of control functions in a Banach space and also their proofs. Moreover, we formulate an example to elucidate our attained results.

Theorem 2.1. *Let $(X, \|.\|)$ be a Banach space and C be a closed subset of X. Let $T : C \to C$ be a mapping. Assume that there exist two altering distance functions $\phi, \psi : [0, \infty) \to [0, \infty)$ such that*

$$\phi(\|Tx - Ty\|) \le \phi(\|Tx - y\|) - \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in C$. Then T has a unique fixed point in C.

Proof. Let $x_0 \in C$ be arbitrary but fixed. Consider, the iterated sequence $\{x_n\}$ where $x_n = T^n x_0$ for each natural number n.

Therefore, by the given condition we have

$$\phi(\|Tx_{n-1} - Tx_{m-1}\|) \le \phi(\|Tx_{n-1} - x_{m-1}\|) - \psi(\|x_{n-1} - x_{m-1}\|)$$
$$\Rightarrow \phi(\|x_n - x_m\|) \le \phi(\|x_n - x_{m-1}\|) - \psi(\|x_{n-1} - x_{m-1}\|), \tag{2.2}$$

which implies that

$$\phi(\|x_n - x_m\|) \le \phi(\|x_n - x_{m-1}\|)$$

for all $n, m \in \mathbb{N}$. Since ϕ is monotone increasing, we have

$$\|x_n - x_m\| \le \|x_n - x_{m-1}\|$$

for all $n, m \in \mathbb{N}$.

Interchanging the role of x_n and x_m in the above equation, we get

$$\|x_n - x_m\| \le \|x_{n-1} - x_m\| \tag{2.3}$$

for all $n, m \in \mathbb{N}$.

Thus, for each fixed $n \in \mathbb{N}$, we can conclude that the sequence $\{s_m^{(n)}\}_{m \in \mathbb{N}}$ of non-negative real numbers is monotone decreasing, where $s_m^{(n)} = \|x_n - x_m\|$ for each $m \in \mathbb{N}$. So, $\{s_m^{(n)}\}_{m \in \mathbb{N}}$ is convergent for each $n \in \mathbb{N}$.

Let

$$\lim_{m \to \infty} s_m^{(n)} = a^{(n)}$$

for each $n \in \mathbb{N}$.

Now from Eq. (2.2), we have

$$\phi(\|x_n - x_m\|) + \psi(\|x_{n-1} - x_{m-1}\|) \le \phi(\|x_n - x_{m-1}\|).$$

Keeping n fixed, taking limit as $m \to \infty$ on both sides of the above equation and using the continuity of ϕ, ψ on $[0, \infty)$, we get

$$\lim_{m \to \infty} \phi(\|x_n - x_m\|) + \lim_{m \to \infty} \psi(\|x_{n-1} - x_{m-1}\|) \le \lim_{m \to \infty} \phi(\|x_n - x_{m-1}\|)$$
$$\Rightarrow \phi(\lim_{m \to \infty} \|x_n - x_m\|) + \psi(\lim_{m \to \infty} \|x_{n-1} - x_{m-1}\|) \le \phi(\lim_{m \to \infty} \|x_n - x_{m-1}\|)$$
$$\Rightarrow \phi(a^{(n)}) + \psi(a^{(n-1)}) \le \phi(a^{(n)})$$
$$\Rightarrow \psi(a^{(n-1)}) \le 0$$

$$\Rightarrow \psi(a^{(n-1)}) = 0$$
$$\Rightarrow a^{(n-1)} = 0 \text{ [since, } \psi(t) = 0 \text{ if and only if } t = 0].$$

Therefore, $a^{(n)} = 0$ for all $n \in \mathbb{N}$, i.e., $\lim_{m \to \infty} \|x_n - x_m\| = 0$ for all $n \in \mathbb{N}$.

Now, we consider the sequence of functions $\{f_m\}$ defined on C by

$$f_m(x) = \begin{cases} \|x_n - x_m\|, & \text{if } x = x_n \text{ for some } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\lim_{m \to \infty} f_m(x) = 0$ for all $x \in C$. Thus, the limit function f of the sequence of functions $\{f_m\}$ is given by

$$f(x) = 0 \quad \text{for all } x \in C.$$

Now, let

$$M_m = \sup_{x \in C} |f_m(x) - f(x)|.$$

Therefore,

$$M_m = \sup_{x \in C} |f_m(x)| \text{ [since, } \quad f(x) = 0 \quad \text{for all} \quad x \in C]$$
$$= \sup_n |f_m(x_n)|$$
$$= \sup_n \|x_n - x_m\|.$$

But, we know from (2.3) that

$$\|x_n - x_m\| \le \|x_{n-1} - x_m\| \le \|x_{n-2} - x_m\| \le \cdots \le \|x_1 - x_m\|,$$

which implies that

$$\sup_n \|x_n - x_m\| \le \|x_1 - x_m\|$$
$$\Rightarrow M_m \le \|x_1 - x_m\|$$
$$\Rightarrow \lim_{m \to \infty} M_m \le \lim_{m \to \infty} \|x_1 - x_m\| = 0$$
$$\Rightarrow \lim_{m \to \infty} M_m = 0.$$

Let $\epsilon > 0$ be arbitrary. Since, $\lim_{m \to \infty} M_m = 0$, there exists a natural number N such that

$$|M_m| < \epsilon \text{ for all } m \ge N$$
$$\Rightarrow \sup_{x \in C} |f_m(x) - f(x)| < \epsilon \text{ for all } m \ge N$$
$$\Rightarrow |f_m(x) - f(x)| < \epsilon \text{ for all } m \ge N \text{ and for all } x \in C$$
$$\Rightarrow |f_m(x)| < \epsilon \text{ for all } m \ge N \text{ and for all } x \in C.$$

In particular, we have

$$|f_m(x_n)| < \epsilon \text{ for all } m \ge N \text{ and for all } n \in \mathbb{N}.$$

Therefore, we can write

$$\left| \|x_n - x_m\| - 0 \right| < \epsilon \tag{2.4}$$

for all $n, m \ge N$.

Next, we consider the double sequence $\{s_{nm}\}_{n,m\in\mathbb{N}}$ of real numbers, where

$$s_{nm} = \|x_n - x_m\|$$

for all $n, m \in \mathbb{N}$. Here using (2.4), we have

$$|s_{nm} - 0| < \epsilon \quad \text{for all } n, m \geq N.$$

This implies the double sequence $\{s_{nm}\}_{n,m\in\mathbb{N}}$ converges to 0, i.e,

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in C. C being complete, $\{x_n\}$ must converge to some $z \in C$.

Now from (2.1), we have

$$\phi(\|x_{n+1} - Tz\|) \leq \phi(\|x_{n+1} - z\|) - \psi(\|x_n - z\|)$$
$$\Rightarrow \phi(\|x_{n+1} - Tz\|) \leq \phi(\|x_{n+1} - z\|)$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|x_{n+1} - Tz\|) \leq \lim_{n\to\infty} \phi(\|x_{n+1} - z\|)$$
$$\Rightarrow \phi\left(\lim_{n\to\infty} \|x_{n+1} - Tz\|\right) \leq \phi\left(\lim_{n\to\infty} \|x_{n+1} - z\|\right)$$
$$\Rightarrow \phi\left(\lim_{n\to\infty} \|x_{n+1} - Tz\|\right) \leq \phi(0) = 0$$
$$\Rightarrow \phi\left(\lim_{n\to\infty} \|x_{n+1} - Tz\|\right) = 0$$
$$\Rightarrow \lim_{n\to\infty} \|x_{n+1} - Tz\| = 0.$$

The above equation shows that the sequence $\{x_n\}$ converges to Tz. Thus, $Tz = z$ and z is a fixed point of T.

Finally, we check the uniqueness of the fixed point z. To check this, let z_1 be another fixed point of T, i.e., $Tz_1 = z_1$.

From (2.1) and using Definition 1.4, we have

$$\phi(\|Tz - Tz_1\|) \leq \phi(\|Tz - z_1\|) - \psi(\|z - z_1\|)$$
$$\Rightarrow \phi(\|z - z_1\|) + \psi(\|z - z_1\|) \leq \phi(\|z - z_1\|)$$
$$\Rightarrow \psi(\|z - z_1\|) \leq 0$$
$$\Rightarrow \psi(\|z - z_1\|) = 0$$
$$\Rightarrow z = z_1.$$

Therefore, z is the only fixed point of T. $\square$

Now, we generalize Theorem 2.1 in a Banach space which is equipped with an arbitrary binary relation and state the subsequent theorem.

Theorem 2.2. *Let $(X, \|.\|)$ be a Banach space and $\mathcal{R}$ be an equivalence relation on X. Assume that X has the property that if $\{x_n\}$ be any sequence in X converging to $z \in X$ and $x_n \mathcal{R} x_m$ for each pair of natural numbers n and m, then $x_n \mathcal{R} z$ for each natural number n. Let C be a closed subset of X and $T : C \to C$ be a mapping such that T satisfies the following conditions:*

  (i) *T is order-preserving with respect to $\mathcal{R}$,*

(ii) $\phi(\|Tx - Ty\|) \leq \phi(\|Tx - y\|) - \psi(\|x - y\|)$ *for all* $x, y \in C$ *such that* $x\mathcal{R}y$,

where $\phi, \psi : [0, \infty) \to [0, \infty)$ *are two altering distance functions. Then* T *has a unique fixed point in* C *if there exists* $x_0 \in X$ *such that* $x_0\mathcal{R}Tx_0$.

Proof. The proof of this theorem is analogous to the previous one and so omitted. $\qquad\square$

In the next portion of this section, we present a result which not only gives the guarantee of existence of fixed point but also properly points out the fixed point.

Theorem 2.3. *Let* $(X, \|.\|)$ *be a Banach space and* C *be a closed subspace of* X. *Let* $T : C \to C$ *be a mapping. Also assume that there exist two altering distance functions* $\phi, \psi : [0, \infty) \to [0, \infty)$ *such that* T *satisfies the following conditions:*

(i) $\phi(\|Tx - Ty\|) \leq \phi(\|x - y\|) - \psi(\|x - y\|)$,
(ii) $\phi(\|Tx - y\|) \leq \phi(\|x - y\|) - \psi(\|x - y\|)$

for all $x, y \in C$. *Then the null vector of* X *is the only fixed point of* T.

Proof. Let $x_0 \in C$ be arbitrary but fixed and consider the iterated sequence $\{x_n\}$ where $x_n = T^n x_0$ for all $n \in \mathbb{N}$.

Let $s_n = \|x_n - x_{n+1}\|$ for all $n \in \mathbb{N}$. Now, by condition (i) we get

$$\phi(\|Tx_n - Tx_{n+1}\|) \leq \phi(\|x_n - x_{n+1}\|) - \psi(\|x_n - x_{n+1}\|)$$
$$\Rightarrow \phi(\|x_{n+1} - x_{n+2}\|) \leq \phi(\|x_n - x_{n+1}\|)$$
$$\Rightarrow \phi(s_{n+1}) \leq \phi(s_n)$$
$$\Rightarrow s_{n+1} \leq s_n.$$

This is true for all natural numbers n, which implies that $\{s_n\}$ is a decreasing sequence of non-negative reals and hence this sequence must converge. Let

$$\lim_{n \to \infty} s_n = a.$$

Again, from (i) we have

$$\phi(s_{n+1}) \leq \phi(s_n) - \psi(s_n)$$
$$\Rightarrow \lim_{n \to \infty} \phi(s_{n+1}) \leq \lim_{n \to \infty} \phi(s_n) - \lim_{n \to \infty} \psi(s_n)$$
$$\Rightarrow \phi(a) \leq \phi(a) - \psi(a)$$
$$\Rightarrow \psi(a) \leq 0$$
$$\Rightarrow \psi(a) = 0$$
$$\Rightarrow a = 0$$
$$\Rightarrow \lim_{n \to \infty} s_n = 0.$$

Therefore,

$$\lim_{n \to \infty} \|x_n - x_{n+1} - \theta\| = 0.$$

This shows that the sequence $\{u_n\}$ in C converges strongly to θ, where θ is the null vector in X and $u_n = x_n - x_{n+1}$ for all natural numbers n. Now,

$$\phi(\|Tu_n - T\theta\|) \leq \phi(\|u_n - \theta\|) - \psi(\|u_n - \theta\|)$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - T\theta\|) \leq \lim_{n\to\infty} \phi(\|u_n - \theta\|) - \lim_{n\to\infty} \psi(\|u_n - \theta\|)$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - T\theta\|) \leq 0$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - T\theta\|) = 0$$
$$\Rightarrow \phi(\lim_{n\to\infty} \|Tu_n - T\theta\|) = 0$$
$$\Rightarrow \lim_{n\to\infty} \|Tu_n - T\theta\| = 0.$$

Again, by condition (ii) we get

$$\phi(\|Tu_n - \theta\|) \leq \phi(\|u_n - \theta\|) - \psi(\|u_n - \theta\|)$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - \theta\|) \leq \lim_{n\to\infty} \phi(\|u_n - \theta\|) - \lim_{n\to\infty} \psi(\|u_n - \theta\|)$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - \theta\|) \leq 0$$
$$\Rightarrow \lim_{n\to\infty} \phi(\|Tu_n - \theta\|) = 0$$
$$\Rightarrow \phi\left(\lim_{n\to\infty} \|Tu_n - \theta\|\right) = 0$$
$$\Rightarrow \lim_{n\to\infty} \|Tu_n - \theta\| = 0.$$

Therefore, by the uniqueness of limit, we obtain

$$T\theta = \theta,$$

i.e., θ is a fixed point of T.

Finally, suppose z be another fixed point of T. Therefore,

$$\phi(\|Tz - T\theta\|) \leq \phi(\|z - \theta\|) - \psi(\|z - \theta\|)$$
$$\Rightarrow \phi(\|z - \theta\|) \leq \phi(\|z - \theta\|) - \psi(\|z - \theta\|)$$
$$\Rightarrow \psi(\|z - \theta\|) \leq 0$$
$$\Rightarrow \psi(\|z - \theta\|) = 0$$
$$\Rightarrow z = \theta.$$

Therefore, θ is the only fixed point of T in C. $\qquad\square$

Example 2.4. Consider the Banach space $\mathbb{R}$ endowed with the usual norm and define a relation $\mathcal{R}$ on $\mathbb{R}$ by: for $x, y \in \mathbb{R}$ $x\mathcal{R}y$ if and only if either $x, y \in [-(n+1), -n]$ or $x, y \in [n, n+1]$ for some $n \in \mathbb{N}$ or $x = y$. Then clearly $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.

Now, let $C = C_1 \cup C_2 \cup C_3$, where $C_1 = [-2, -1]$, $C_2 = [1, 2]$, $C_3 = \{0\}$. Then C is a closed subset of $\mathbb{R}$.

Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} -x, & \text{if } x \in C_1; \\ 0, & \text{if } x \in C_2 \cup C_3. \end{cases}$$

Therefore,

$$\|Tx - Ty\| = \begin{cases} |x - y|, & \text{if } x, y \in C_1; \\ |x|, & \text{if } x \in C_1 \text{ and } y \in C_2 \cup C_3; \\ 0, & \text{if } x, y \in C_2 \cup C_3, \end{cases}$$

and

$$\|Tx - y\| = \begin{cases} |x + y|, & \text{if } x \in C_1; \\ |y|, & \text{if } x \notin C_1. \end{cases}$$

Consider the functions $\phi, \psi : [0, \infty) \to [0, \infty)$ defined by

$$\phi(t) = t^2 \text{ and}$$
$$\psi(t) = \frac{t^2}{100000}$$

for all $t \in [0, \infty)$.

Then, clearly ϕ, ψ are two altering distance functions. Let $x, y \in C$ be arbitrary such that $x \mathcal{R} y$. Then the following cases arise.

Case 1 Let $x, y \in C_1$. Then

$$\phi(\|Tx - Ty\|) + \psi(\|x - y\|) - \phi(\|Tx - y\|)$$
$$= |x - y|^2 + \frac{|x - y|^2}{100000} - |x + y|^2$$
$$= -4xy + \frac{(x - y)^2}{100000}$$
$$\leq 0$$
$$\Rightarrow \phi(\|Tx - Ty\|) \leq \phi(\|Tx - y\|) - \psi(\|x - y\|).$$

Case 2 Let $x, y \in C_2$. Then

$$\phi(\|Tx - Ty\|) + \psi(\|x - y\|) - \phi(\|Tx - y\|)$$
$$= 0 + \frac{(x - y)^2}{100000} - y^2$$
$$\leq 0$$
$$\Rightarrow \phi(\|Tx - Ty\|) \leq \phi(\|Tx - y\|) - \psi(\|x - y\|).$$

Case 3 Let $x, y \in C_3$. Then clearly the equality holds.

Thus,

$$\phi(\|Tx - Ty\|) \leq \phi(\|Tx - y\|) - \psi(\|x - y\|)$$

for all $x, y \in C$ with $x \mathcal{R} y$.

Also it is easily seen that T is order-preserving and by Theorem 2.2, 0 is the only fixed point of T.

3. Application to fractional thermostat model

The motivation of this section is to provide an application of the results discussed in this manuscript. For this purpose, we consider the following fractional thermostat model:

$$^C D^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad t \in [0, 1], \tag{3.1}$$

subject to the boundary conditions:

$$u'(0) = 0, \beta^C D^{\alpha-1} u(1) + u(\eta) = 0, \tag{3.2}$$

where $^C D^\alpha$ stands for Caputo fractional derivative of order α, λ is a positive constant and $1 < \alpha \leq 2$, $0 \leq \eta \leq 1$, $\beta > 0$ such that the following conditions hold:

(1) $\beta \Gamma(\alpha) - (1 - \eta)^{(\alpha-1)} > 0$;
(2) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function;
(3) $u : [0, 1] \to \mathbb{R}$ is continuous.

Our aim is to derive some sufficient conditions under which the problem (3.1) with the boundary conditions (3.2) possesses a unique positive solution for certain values of the parameter λ. To proceed further, we first recall the following lemmas.

Lemma 3.1 [22]. *Assume $f \in C[0, 1]$. A function $u \in C[0, 1]$ is a solution of the boundary value problem*

$$^C D^\alpha u(t) + \lambda f(t, u(t)) = 0, t \in [0, 1], \tag{3.3}$$

$$u'(0) = 0, \beta^C D^{\alpha-1} u(1) + u(\eta) = 0, \tag{3.4}$$

if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t, s) f(s) \mathrm{d}s, \tag{3.5}$$

where $G(t, s)$ is the Green's function (depending on α) given by

$$G(t, s) = \beta + H_\eta(s) - H_t(s)$$

and for $r \in [0, 1]$, $H_r(s) : [0, 1] \to \mathbb{R}$ is defined as $H_r(s) = \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \leq r$ and $H_r(s) = 0$ for $s > r$, i.e.,

$$G(t, s) = \begin{cases} \beta - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \leq s \leq \eta, \ s \leq t; \\ \beta + \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \leq s \leq \eta, \ s \geq t; \\ \beta - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } \eta \leq s \leq 1, \ s \leq t; \\ \beta, & \text{if } \eta \leq s \leq 1, \ s \geq t. \end{cases}$$

Lemma 3.2 [30]. *The function $G(t, s)$ arising in Lemma 3.1 satisfies the following conditions:*

(i) *$G(t, s)$ is a continuous map defined on $[0, 1] \times [0, 1]$;*
(ii) *for $t, s \in (0, 1)$, we have $G(t, s) > 0$.*

Now we prove the following lemma.

Lemma 3.3. *The Green's function $G(t, s)$ derived in Lemma 3.1 satisfies*

$$\sup_{t \in [0,1]} \int_0^1 G(t, s)ds = \beta + \frac{\eta^\alpha}{\Gamma(\alpha + 1)}$$

and

$$\inf_{t \in [0,1]} \int_0^1 G(t, s)ds = \beta + \frac{\eta^\alpha - 1}{\Gamma(\alpha + 1)}.$$

Proof. Let us consider the function φ defined on $[0, 1]$ by

$$\varphi(t) = \int_0^1 G(t, s)ds$$

for all $t \in [0, 1]$.

Now, for $t \in [0, 1]$ and $t \le \eta$, $s \ge \eta$, we have $t \le s$ and thus,

$$
\begin{aligned}
\varphi(t) &= \int_0^1 G(t, s)ds \\
&= \int_0^\eta G(t, s)ds + \int_\eta^1 G(t, s)ds \\
&= \int_0^t G(t, s)ds + \int_t^\eta G(t, s)ds + \int_\eta^1 G(t, s)ds \\
&= \int_0^t \left\{ \beta - \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} + \frac{(\eta - s)^{\alpha - 1}}{\Gamma(\alpha)} \right\} ds \\
&\quad + \int_t^\eta \left\{ \beta + \frac{(\eta - s)^{\alpha - 1}}{\Gamma(\alpha)} \right\} ds + \int_\eta^1 \beta ds \\
&= \beta + \frac{\eta^\alpha - t^\alpha}{\Gamma(\alpha + 1)}.
\end{aligned}
$$

Again, for $t \in [0, 1]$ and $t \ge \eta$, $s \le \eta$, we have $t \ge s$ and so

$$
\begin{aligned}
\varphi(t) &= \int_0^1 G(t, s)ds \\
&= \int_0^\eta G(t, s)ds + \int_\eta^1 G(t, s)ds \\
&= \int_0^\eta G(t, s)ds + \int_\eta^t G(t, s)ds + \int_t^1 G(t, s)ds \\
&= \int_0^\eta \left\{ \beta - \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} + \frac{(\eta - s)^{\alpha - 1}}{\Gamma(\alpha)} \right\} ds \\
&\quad + \int_\eta^t \left\{ \beta - \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \right\} ds + \int_t^1 \beta ds \\
&= \beta + \frac{\eta^\alpha - t^\alpha}{\Gamma(\alpha + 1)}.
\end{aligned}
$$

Thus, from the above calculations we get,

$$\varphi(t) = \beta + \frac{\eta^\alpha - t^\alpha}{\Gamma(\alpha + 1)}$$

for all $t \in [0, 1]$.

Therefore,

$$\varphi'(t) = \frac{-\alpha t^{\alpha-1}}{\Gamma(\alpha+1)} < 0$$

for all $t \in [0, 1]$.

This implies that the function φ is a decreasing function on $[0, 1]$. So,

$$\sup_{t \in [0,1]} \int_0^1 G(t,s)\mathrm{d}s = \sup_{t \in [0,1]} \varphi(t)$$
$$= \varphi(0)$$
$$= \beta + \frac{\eta^\alpha}{\Gamma(\alpha+1)},$$

and

$$\inf_{t \in [0,1]} \int_0^1 G(t,s)\mathrm{d}s = \inf_{t \in [0,1]} \varphi(t)$$
$$= \varphi(1)$$
$$= \beta + \frac{\eta^\alpha - 1}{\Gamma(\alpha+1)}.$$

This completes the proof of the lemma.   □

As a special case of Proposition 1 of [10], we have the following lemma.

Lemma 3.4. *For the Green's function $G(t, s)$ derived in Lemma* 3.1,

$$\beta - \frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)} \le G(t,s) \le \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha)}$$

for all $t, s \in [0, 1]$ holds.

Now we prove the following theorems concerning the existence and uniqueness of a positive solution to the fractional thermostat model given by Eqs. (3.1) and (3.2).

Theorem 3.5. *Let us consider the fractional thermostat model with parameter $\lambda > 0$ given by Eqs. (3.1) and (3.2). Assume that the following conditions hold:*

(i) $\beta\Gamma(\alpha+1) + \eta^\alpha > 1$;

(ii) *for all $s \in [0, 1]$,*

$$\lambda|f(s, u(s)) - f(s, v(s))| \le \lambda|f(s, u(s))| - \lambda \sup_{t \in [0,1]} |v(t)| - \psi(\sup_{t \in [0,1]} |u(t) - v(t)|)$$

for some altering distance function ψ and for all real-valued continuous functions $u(s), v(s)$ defined on $[0, 1]$;

(iii) *f is non-decreasing with respect to the second argument and there exists $t_0 \in (0, 1)$ such that $f(t_0, 0) > 0$.*

Then the fractional thermostat model with parameter λ given by Eqs. (3.1) and (3.2) has a unique positive solution for $\lambda \ge \frac{1}{k}$, where $k = \beta + \frac{\eta^\alpha - 1}{\Gamma(\alpha+1)}$.

Proof. Consider the Banach space $C[0,1]$ of all real-valued continuous functions defined on $[0,1]$ equipped with the sup norm.

Define a mapping $T : C[0,1] \rightarrow C[0,1]$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$

for all $u \in C[0,1]$, where $G(t,s)$ is defined as in Lemma 3.1.

From Lemma 3.1, it is obvious that the thermostat model given by Eqs. (3.1) and (3.2) has $u(t)$ as a solution if and only if $u(t)$ is a fixed point of T.

Now, by condition (ii), we have

$$\lambda|f(s,u(s)) - f(s,v(s))| \leq \lambda|f(s,u(s))| - \lambda \sup_{t\in[0,1]} |v(t)|$$
$$- \psi(\sup_{t\in[0,1]} |u(t) - v(t)|)$$
$$= \lambda|f(s,u(s))| - \lambda\|v\| - \psi(\|u-v\|).$$

Multiplying both sides by $|G(t,s)|$, we get

$$\lambda|f(s,u(s)) - f(s,v(s))||G(t,s)| \leq \lambda|f(s,u(s))||G(t,s)|$$
$$- \lambda\|v\||G(t,s)|$$
$$- \psi(\|u-v\|)|G(t,s)|$$

$$\Rightarrow \lambda \int_0^1 |f(s,u(s)) - f(s,v(s))||G(t,s)|\mathrm{d}s \leq \lambda \int_0^1 |f(s,u(s))||G(t,s)|\mathrm{d}s$$
$$- \lambda \int_0^1 \|v\||G(t,s)|\mathrm{d}s$$
$$- \int_0^1 \psi(\|u-v\|)|G(t,s)|\mathrm{d}s$$
$$= \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$
$$- \lambda\|v\| \int_0^1 G(t,s)\mathrm{d}s$$
$$- \psi(\|u-v\|) \int_0^1 G(t,s)\mathrm{d}s$$
$$\leq \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$
$$- \lambda\|v\| \inf_{t\in[0,1]} \int_0^1 G(t,s)\mathrm{d}s$$
$$- \psi(\|u-v\|) \inf_{t\in[0,1]} \int_0^1 G(t,s)\mathrm{d}s$$
$$\leq \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$
$$- \lambda k\|v\| - k\psi(\|u-v\|).$$

Now if $\lambda k \geq 1$, then from the above equation we obtain,

$$\lambda \int_0^1 |f(s,u(s)) - f(s,v(s))||G(t,s)|\mathrm{d}s \leq \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - \|v\|$$
$$- k\psi(\|u - v\|)$$
$$\leq \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - |v(t)|$$
$$- k\psi(\|u - v\|)$$
$$\leq \left| \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - v(t) \right|$$
$$- k\psi(\|u - v\|). \tag{3.6}$$

Therefore, using Eq. (3.6) we get

$$|Tu(t) - Tv(t)| = \left| \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - \lambda \int_0^1 G(t,s)f(s,v(s))\mathrm{d}s \right|$$
$$= \left| \lambda \int_0^1 G(t,s)(f(s,u(s)) - f(s,v(s)))\mathrm{d}s \right|$$
$$\leq \left| \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - v(t) \right| - k\psi(\|u - v\|).$$

The above inequality holds for all $t \in [0,1]$ and so we have

$$\sup_{t\in[0,1]} |Tu(t) - Tv(t)| \leq \sup_{t\in[0,1]} \lambda \left| \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - v(t) \right| - k\psi(\|u - v\|)$$
$$\Rightarrow \|Tu - Tv\| \leq \|Tu - v\| - k\psi(\|u - v\|). \tag{3.7}$$

It is easily perceived by condition (i) that $k > 0$.

Define two functions $\phi, \psi_1 : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = t \ and$$
$$\psi_1(t) = k\psi(t)$$

for all $t \in [0,\infty)$. Then one can easily verify that ϕ, ψ_1 are two altering distance functions and also from Eq. (3.7) we get

$$\phi(\|Tu - Tv\|) \leq \phi(\|Tu - v\|) - \psi_1(\|u - v\|). \tag{3.8}$$

The above inequality holds for all $u, v \in C[0,1]$ and so by Theorem 2.1, T has a unique fixed point $u(t)$, say, in $C[0,1]$.

Note that Eq. (3.8) holds if $\lambda k \geq 1$. So, T has $u(t)$ as a fixed point if $\lambda k \geq 1$, i.e., $u(t)$ is a solution of the thermostat model (3.1) and (3.2) if $\lambda k \geq 1$, i.e., $\lambda \geq \frac{1}{k}$.

Now we have $\lambda > 0$, $G(t,s) > 0$ and $f(s,u(s)) \geq 0$ for all $t,s \in [0,1]$. Therefore, it is clear that

$$\lambda \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \geq 0$$

for all $t \in [0,1]$. This means that $Tu(t) \geq 0$ for all $t \in [0,1]$ and which leads us to the fact that $u(t) \geq 0$ for all $t \in [0,1]$.

Finally, we show that the unique solution $u(t)$ is always positive. To show this, first we show that the zero function 0 is not a fixed point of T.

Suppose to the contrary that the zero function 0 is a fixed point of T. Then, we have

$$0 = \lambda \int_0^1 G(t,s) f(s,0) \mathrm{d}s,$$

for all $t \in [0,1]$. Since $G(t,s) f(s,0) \geq 0$ for all $t \in [0,1]$ and for all $s \in [0,1]$, we have

$$G(t,s) f(s,0) = 0,$$

for all $t \in [0,1]$ and for almost all $s \in [0,1]$. This fact leads us to

$$f(s,0) = 0 \quad \text{for almost all } s \in [0,1]. \tag{3.9}$$

By condition (iii), there exists $t_0 \in (0,1)$ such that $f(t_0,0) > 0$. Again, since f is continuous at $(t_0,0)$, there exists a subset A of $[0,1]$ of positive Lebesgue measure such that $f(s,0) > 0$ for all $s \in A$. This is a contradiction to Eq. (3.9). So the zero function 0 is not a fixed point of T.

Now, let $u(t_1) = 0$ for some $t_1 \in (0,1)$. Therefore, we have

$$\int_0^1 G(t_1,s) f(s,u(s)) \mathrm{d}s = 0. \tag{3.10}$$

But $u(s) \geq 0$ for all $s \in [0,1]$ and f is non-decreasing with respect to the second argument. Hence,

$$0 \geq \int_0^1 G(t_1,s) f(s,u(s)) \mathrm{d}s \geq \int_0^1 G(t_1,s) f(s,0) \mathrm{d}s \geq 0. \tag{3.11}$$

Therefore, from (3.10) and (3.11), we obtain

$$\int_0^1 G(t_1,s) f(s,0) \mathrm{d}s = 0.$$

As $G(t_1,s) f(s,0) \geq 0$, it follows that $G(t_1,s) f(s,0) = 0$ for almost all $s \in [0,1]$. This implies that $f(s,0) = 0$ for almost all $s \in [0,1]$, which is a contradiction.

Hence, it follows that $u(t) > 0$ for all $t \in (0,1)$. Again, since u is continuous on $[0,1]$, we have $u(t) > 0$ for all $t \in [0,1]$. Thus, the fractional thermostat model, given by Eqs. (3.1) and (3.2), has a unique positive solution for $\lambda \geq \frac{1}{k}$, where $k = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha+1)}$.   □

Theorem 3.6. *Let us consider the fractional thermostat model with parameter λ given by Eqs. (3.1) and (3.2). Assume that the following conditions hold:*
(i) $\beta\Gamma(\alpha+1) + \eta^\alpha > 1$;

(ii) *for all $s \in [0, 1]$,*

$$\lambda|f(s, u(s)) - f(s, v(s))| \leq \lambda|f(s, u(s))| - \lambda \sup_{t \in [0,1]} |v(t)| - \psi(\sup_{t \in [0,1]} |u(t) - v(t)|),$$

for some bounded altering distance function ψ and for all $u(s), v(s)$ in the set $C = \{u(s) \in C[0, 1] : R_1 \leq u(s) \leq R, \text{ for all } s \in [0, 1], \text{ where } R, R_1 \text{ are constants with } R > 1, R_1 > 0\}$;

(iii) $k_1 = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha)}$, $k_2 = \beta - \frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)} > 0$, $k = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha+1)} < 1$;

(iv) $\int_0^1 f(s, R)\mathrm{d}s \leq \frac{6R}{\lambda k_1}$ *and* $\int_0^1 f(s, 0)\mathrm{d}s > \frac{R_1}{5\lambda k_2}$;

(v) *f is non-decreasing with respect to the second argument and there exists $t_0 \in (0, 1)$ such that $f(t_0, 0) > 0$.*

Then the fractional thermostat model with parameter λ given by Eqs. (3.1) and (3.2) has a unique positive solution in C for $\lambda \geq \frac{1}{k}$.

Proof. Let us take the Banach space $C[0, 1]$ endowed with the sup norm. We consider the set C' defined as

$$C' = \left\{ u(s) \in C[0, 1] : \frac{R_1}{5} \leq u(s) \leq 6R \text{ for all } s \in [0, 1] \text{ or } u(s) = 0 \text{ for all } s \in [0, 1] \right\}.$$

Then it is easily noticeable that C' is a closed subset of $C[0, 1]$. The fact that f is non-decreasing with respect to the second argument gives us

$$\int_0^1 f(s, u(s))\mathrm{d}s \leq \int_0^1 f(s, R)\mathrm{d}s$$
$$\leq \frac{6R}{\lambda k_1},$$

and

$$\int_0^1 f(s, u(s))\mathrm{d}s \geq \int_0^1 f(s, R_1)\mathrm{d}s$$
$$\geq \int_0^1 f(s, 0)\mathrm{d}s$$
$$> \frac{R_1}{5\lambda k_2}$$

for all $u \in C$. Therefore,

$$\lambda \int_0^1 G(t, s)f(s, u(s))\mathrm{d}s \leq \lambda \left(\beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha)} \right) \int_0^1 f(s, u(s))\mathrm{d}s$$
$$\leq \lambda k_1 \frac{6R}{\lambda k_1}$$
$$= 6R,$$

i.e.,

$$\lambda \int_0^1 G(t, s)f(s, u(s))\mathrm{d}s \leq 6R, \tag{3.12}$$

and

$$\lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \geq \lambda \left(\beta - \frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)} \right) \int_0^1 f(s,u(s))\mathrm{d}s$$

$$\geq \lambda k_2 \frac{R_1}{5\lambda k_2}$$

$$= \frac{R_1}{5},$$

i.e.,

$$\lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \geq \frac{R_1}{5}. \tag{3.13}$$

Next, we define a mapping $T : C' \to C'$ by

$$Tu(t) = \begin{cases} \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s, & \text{for all } u \in C \cup \{0\}; \\ 0, & \text{elsewhere.} \end{cases}$$

From Eqs. (3.12) and (3.13) one can easily check that T is well defined on C'.

We now define a relation $\mathcal{R}$ on $C[0,1]$ by the following:

for $u, v \in C[0,1]$, $u\mathcal{R}v$ if and only if

(1) either $u, v \in C'\backslash\{0\}$ and both $Tu, Tv \neq 0$;
(2) or $u, v \in C'\backslash\{0\}$ and both $Tu, Tv = 0$;
(3) or $u = v = 0$.

Then it is clear that $\mathcal{R}$ is an equivalence relation on $C[0,1]$. We claim that T is order-preserving on C' with respect to $\mathcal{R}$.

Let $u, v \in C'$ be arbitrary with $u\mathcal{R}v$. Then the following cases may arise.

Case 1 When $u, v \in C'\backslash\{0\}$ and both $Tu, Tv \neq 0$.

Then

$$Tu \geq \frac{R_1}{5} > 0$$

$$\Rightarrow T(Tu) > 0.$$

Similarly, $T(Tv) > 0$. So, $T(Tu), T(Tv) \neq 0$, and $Tu\mathcal{R}Tv$.

Case 2 When $u, v \in C'\backslash\{0\}$ and both $Tu, Tv = 0$.

Since $\mathcal{R}$ is an equivalence relation, $Tu\mathcal{R}Tv$.

Case 3 When $u = v = 0$.

Then we have $Tu = Tv$ and so $Tu\mathcal{R}Tv$, since $\mathcal{R}$ is an equivalence relation. Therefore, T is order-preserving on C' with respect to $\mathcal{R}$.

Since ψ is bounded, so there exists a constant $M > 0$ such that $\psi(t) \leq M$ for all $t \in [0,\infty)$. Without loss of generality, we may assume that $\frac{R_1}{5} \geq M$. Now we define two altering distance functions by $\phi(t) = (2M+1)kt$ and $\psi_1(t) = (2M+1)k^2\psi(t)$ for all $t \in [0,\infty)$.

Let $u, v \in C'$ be arbitrary with $u\mathcal{R}v$. Then the succeeding cases may arise.

Case 1 When $u, v \in C'\backslash\{0\}$ and both $Tu, Tv \neq 0$.

So $u, v \in C$ and

$$Tu = \lambda \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s$$

and

$$Tv = \lambda \int_0^1 G(t,s) f(s, v(s)) \mathrm{d}s.$$

Then proceeding as in Theorem 3.5, we get

$\|Tu - Tv\| \leq \|Tu - v\| - k\psi(\|u - v\|), \quad \text{if } \lambda \geq \frac{1}{k}$

$\Rightarrow (2M+1)k\|Tu - Tv\| \leq (2M+1)k\|Tu - v\| - (2M+1)k^2\psi(\|u - v\|)$

$\Rightarrow \phi(\|Tu - Tv\|) \leq \phi(\|Tu - v\|) - \psi_1(\|u - v\|), \quad \text{if } \lambda \geq \frac{1}{k}.$

Case 2 When $u, v \in C'\backslash\{0\}$ and both $Tu, Tv = 0$.

Therefore, $u, v \notin C$ and $\phi(\|Tu - Tv\|) = \phi(0) = 0$. Also, $\|v\| > \frac{R_1}{5}$. Therefore,

$$\phi(\|v\|) > \phi\left(\frac{R_1}{5}\right)$$
$$= (2M+1)k\frac{R_1}{5}$$
$$\geq (2M+1)k^2\frac{R_1}{5}$$
$$\geq (2M+1)k^2 M$$
$$\geq (2M+1)k^2\psi(t)$$
$$= \psi_1(t)$$
$$\Rightarrow \phi(\|Tu - v\|) \geq \psi_1(t)$$
$$\Rightarrow \phi(\|Tu - v\|) \geq \psi_1(\|u - v\|)$$
$$\Rightarrow \phi(\|Tu - v\|) - \psi_1(\|u - v\|) \geq \phi(\|Tu - Tv\|).$$

Case 3 When $u = v = 0$.

In this case, clearly we have

$$\phi(\|Tu - Tv\|) \leq \phi(\|Tu - v\|) - \psi(\|u - v\|).$$

Thus, we see that

$$\phi(\|Tu - Tv\|) \leq \phi(\|Tu - v\|) - \psi(\|u - v\|)$$

holds for all $u, v \in C'$ with $u\mathcal{R}v$ if $\lambda \geq \frac{1}{k}$. So by Theorem 2.2, T has a unique fixed point in C' if $\lambda \geq \frac{1}{k}$. But by the definition of T, it can have the fixed point only in C. So T has a unique fixed point in C.

Thus, $u(t)$ is the unique solution of the fractional thermostat model given by Eqs. (3.1) and (3.2), which follows by Lemma 3.1 and the definition of T. Also, since $u \in C$, u is positive. Hence, the fractional thermostat model given by Eqs. (3.1) and (3.2) satisfying the hypotheses of Theorem 3.6 has a unique positive solution in C for $\lambda \geq \frac{1}{k}$. $\qquad\square$

Now, we demonstrate an example which validates the effectiveness of the aforementioned result.

Example 3.7. Let us consider the fractional thermostat model

$$^{C}D^{\alpha}u(t) + \lambda f(t, u(t)) = 0, t \in (0, 1), \qquad (3.14)$$

$$u'(0) = 0, \beta^{C}D^{\alpha-1}u'(1) + u(\eta) = 0. \qquad (3.15)$$

We choose

$$\alpha = \frac{3}{2}, \ \beta = \frac{4}{5}, \ \eta = \frac{1}{2}$$

and

$$f(t, u) = \begin{cases} \ln(3^{20} + t^2) + t^3 + \frac{1}{24-u}, & \text{if } u \le 20; \\ \ln(3^{20} + t^2) + t^3 + \frac{1}{4}, & \text{elsewhere.} \end{cases}$$

Then, $\beta\Gamma(\alpha) - (1-\eta)^{(\alpha-1)} = \frac{4}{5} \cdot \frac{1}{2} \cdot \sqrt{\pi} - (\frac{1}{2})^{\frac{1}{2}} > 0.$

Clearly, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}^{+}$ is a continuous function and also f is non-decreasing with respect to the second argument and there exists $\frac{1}{2} \in (0, 1)$ such that $f(\frac{1}{2}, 0) > 0.$

We take

$$C = \left\{ u(s) \in C[0, 1] : \frac{1}{10} \le u(s) \le 20, \text{ for all } s \in [0, 1] \right\},$$

i.e., here $R_1 = \frac{1}{10}, \ R = 20.$

Now,

$$\beta\Gamma(\alpha+1) + \eta^{\alpha} = \frac{4}{5} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} + \left(\frac{1}{2}\right)^{\frac{3}{2}} \approx 1.4165 > 1,$$

and

$$k = \beta + \frac{\eta^{\alpha} - 1}{\Gamma(\alpha+1)}$$

$$= \frac{4}{5} + \frac{(\frac{1}{2})^{\frac{3}{2}} - 1}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$\approx 0.3135 < 1$$

$$\Rightarrow \frac{1}{k} \approx 3.1897.$$

Also we have

$$k_1 = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha)}$$

$$= \frac{4}{5} + \frac{(\frac{1}{2})^{\frac{1}{2}}}{\frac{1}{2} \cdot \sqrt{\pi}}$$

$$\approx 1.5981 \text{ and}$$

$$k_2 = \beta - \frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)}$$

$$= \frac{4}{5} - \frac{(\frac{1}{2})^{\frac{1}{2}}}{\frac{1}{2} \cdot \sqrt{\pi}}$$

$$\approx 0.0019.$$

We choose $\lambda = 3.2$ and clearly $\lambda \geq \frac{1}{k}$.

Now,

$$\int_0^1 f(s, R)\mathrm{d}s \leq \ln\left(1 + 3^{20}\right) + \frac{1}{4} + \frac{1}{4}$$

$$\approx 22.4722$$

$$\leq \frac{6R}{\lambda k_1}$$

as

$$\frac{6R}{\lambda k_1} \approx \frac{6 \times 20}{3.2 \times 1.5981}$$

$$\approx 23.4653.$$

Also,

$$\int_0^1 f(s, 0)\mathrm{d}s = \ln\left(3^{20}\right) + \frac{1}{4} + \frac{1}{24}$$

$$\approx 22.2639$$

$$> \frac{R_1}{5\lambda k_2}$$

as

$$\frac{R_1}{5\lambda k_2} \approx \frac{6 \times 20}{3.2 \times 1.5981}$$

$$\approx 3.2894.$$

Next, we define a mapping $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} t^2, & \text{if } 0 \leq t < 1; \\ 1, & \text{if } t \geq 1. \end{cases}$$

Then it is an easy task to note that ψ is a bounded altering distance function.

Finally, for any $u(s), v(s) \in C$, we have

$$\lambda|f(s, u(s)) - f(s, v(s))| = 3.2 \left| \frac{1}{24 - u(s)} - \frac{1}{24 - v(s)} \right|$$

$$\leq 3.2 \left(\frac{1}{4} + \frac{1}{4} \right)$$

$$= 1.6.$$

But,

$$\lambda|f(s, u(s))| - \lambda \sup_{t \in [0,1]} |u(t)| - \psi(\sup_{t \in [0,1]} |u(t) - v(t)|) \geq 3.2 \times 21 - 3.2 \times 20 - \psi(40)$$

$$= 2.20.$$

Therefore,

$$\lambda|f(s, u(s)) - f(s, v(s))| \leq \lambda|f(s, u(s))| - \lambda \sup_{t \in [0,1]} |u(t)| - \psi\left(\sup_{t \in [0,1]} |u(t) - v(t)| \right)$$

for all $u(s), v(s) \in C$. So, by Theorem 3.6, the thermostat model given by Eqs. (3.14) and (3.15) has a unique positive solution in C for $\lambda = 3.2$.

Remark 3.8. It is quite interesting to note that if we take $\lambda < \frac{1}{k}$ in Theorem 3.6, then the thermostat model given by Eqs. (3.1) and (3.2) may not have a positive solution in C. We present the following example in support of our claim.

Example 3.9. Let us consider the fractional thermostat model

$$^C D^\alpha u(t) + \lambda f(t, u(t)) = 0, t \in (0, 1), \tag{3.16}$$

$$u'(0) = 0, \beta^C D^{\alpha-1} u'(1) + u(\eta) = 0. \tag{3.17}$$

We take

$$\alpha = \frac{3}{2}, \ \beta = \frac{6}{5}, \ \eta = \frac{1}{2},$$

$$f(t, u) = 6 + 5t^2$$

and

$$C = \left\{ u(s) \in C[0, 1] : \frac{1}{15} \le u(s) \le 5, \text{ for all } s \in [0, 1] \right\},$$

i.e., here $R_1 = \frac{1}{15}$, $R = 5$.

Then, clearly $f : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function and also f is non-decreasing with respect to the second argument and there exists $\frac{1}{2} \in (0, 1)$ such that $f(\frac{1}{2}, 0) > 0$. Also, $\beta\Gamma(\alpha) - (1 - \eta)^{(\alpha-1)} = \frac{6}{5} \cdot \frac{1}{2} \cdot \sqrt{\pi} - (\frac{1}{2})^{\frac{1}{2}} > 0$. Again we have $\beta\Gamma(\alpha + 1) + \eta^\alpha \approx 2.3019 > 1$, $k = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha+1)} = 0.7136 < 1$, $\frac{1}{k} \approx 1.1014$, $k_1 = \beta + \frac{\eta^{\alpha-1}}{\Gamma(\alpha)} \approx 1.9981$, $k_2 = \beta - \frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)} \approx 0.4019$. We choose $\lambda = \frac{1}{230}$, then clearly $\lambda < \frac{1}{k}$. Now,

$$\int_0^1 f(s, R) ds \approx 7.66$$

$$\le \frac{6R}{\lambda k_1}$$

$$\approx 3303.13$$

and

$$\int_0^1 f(s, 0) ds \approx 7.66$$

$$> \frac{R_1}{5\lambda k_2}$$

$$= 7.6304.$$

Now, we consider the bounded altering distance function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} \frac{t}{230}, & \text{if } t \le 1; \\ \frac{1}{230}, & \text{if } t > 1. \end{cases}$$

Therefore, for any $u(s), \ v(s) \in C$ we have

$$\lambda|f(s,u(s))| - \lambda \sup_{t\in[0,1]} |u(t)| - \psi\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right) \geq \frac{1}{230} \times 6 - \frac{1}{230} \times 5 - \psi(10)$$

$$= 0$$

$$= \lambda|f(s,u(s)) - f(s,v(s))|.$$

Thus, all the conditions of Theorem 3.6 are satisfied but the thermostat model does not have a positive solution in C, because if this thermostat model has a positive solution $u(t)$, say in C, then we have

$$T(u(t)) = u(t) = \lambda \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$

$$\leq \frac{1}{230} \times 1.9981 \times \left(6 + \frac{5}{3}\right)$$

$$< \frac{1}{15}$$

which contradicts the fact that $u(t) \in C$.

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