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Homoclinic solutions for a second-order singular differential equation

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Abstract. In this paper, the problem of existence of homoclinic solutions is studied for the second-order singular differential equation

$$x''(t) + f(x(t))x'(t) - g(x(t)) - \frac{\alpha(t)x(t)}{1-x(t)} = h(t),$$

where $f, g, h, \alpha : R \to R$ are continuous and $\alpha(t+T) \equiv \alpha(t)$ for all $t \in R$. Using the continuation theorem of coincidence degree theory given by Mawhin and Manásevich, a new result on the existence of homoclinic solutions to the equation is obtained.

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1. Introduction

Consider the existence of homoclinic solutions for the equation

$$x''(t) + f(x(t))x'(t) - g(x(t)) - \frac{\alpha(t)x(t)}{1-x(t)} = h(t), \tag{1.1}$$

where $f, g, h, \alpha : R \to R$ are continuous and $\alpha(t+T) \equiv \alpha(t)$ with $\alpha(t) > 0$ for all $t \in R$. We will say that a solution u of Eq. (1.1) is a homoclinic equation, if $u(t) \to 0$ as $t \to \pm\infty$. When such a solution satisfies in addition to $u'(t) \to 0$ as $t \to \pm\infty$, then it is usually called a homoclinic solution or a pulse, although here, 0 is not a stationary solution of Eq. (1.1). In [1], by Leray–Schauder fixed point theorem, Faure has studied the T-periodic solutions of equation

$$x''(t) + cx'(t) - \frac{x(t)}{1-x(t)} = e(t),$$

where $c > 0$ is a constant and $e(t)$ is a continuous T-periodic solution.

The study of singular systems is perhaps as old as the Kepler classical problem in mechanics. In recent years, the problem of periodic solutions has been studied widely for some second-order differential equations with singularity [2–10]. This is due to the fact that periodic solution for the singular equation possesses a significant role in many practical situations (see [5,9,11–15]) and the references therein). Compared with the problem of periodic solution, the problem of homoclinic solution for second-order differential equations with singularity is studied less often. In the case of singular Hamiltonian systems, we find that there were some papers on the study of existence of homoclinic solutions [16–19]. For example, the first result on existence of a homoclinic orbit to autonomous singular Hamiltonian systems

$$u'' + V_u(u) = 0, -\infty < t < +\infty$$

was obtained by Tanaka [16] using variational methods. Costa and Tehrani [17] further studied the problem of homoclinic solutions to a class of non-autonomous singular Hamiltonian systems

$$u'' + V_u(t, u) = 0, -\infty < t < +\infty,$$

where $u = (u_1, u_2, \ldots, u_N) \in R^N$, $V : R\times R^N$ has a singularity at $u = q \in R^N$ and $q \neq 0$. Under the assumption that $V(t, u)$ satisfies strong-force condition, the existence of infinitely many homoclinic solutions is obtained. Bonheure and Torres [20] considered the problem of homoclinic-like solutions to the singular equation

$$-x''(t) + f(t, x(t), x'(t)) = \frac{b(t)}{u^p(t)}, \tag{1.2}$$

where $b \in C(R, R)$ is nonzero nonnegative, $p > 0$ is a constant. The arguments are based upon a well-known fixed point theorem on cones, which is different from the variational methods used in [16–19]. The reason for this is that there is a first-order derivative term in Eq. (1.2). This implies that Eq. (1.2) is not the Euler–Lagrange equation associated with some functional, and then, the variational methods cannot be applied to Eq. (1.2) for obtaining homoclinic-like solution. However, the function $f(t, x, y)$ is required to be linear with respect to the variables x and y. In detail, $f(t, x, y) = a(t)x + c(t)y$, where $a, c \in C(R, R)$ with $a(t) > \tilde{a} > 0$ for all $t \in R$. This is due to the fact that $f(t, x, y)$ in such a way can guarantee the Green function $G(t, s)$ associated with boundary value problem $-x''(t) + c(t)x'(t) + a(t)x(t) = 0, x(-\infty) = x(+\infty) = 0$ satisfying $G(t, s) > 0$ for all $(t, s) \in R^2$; then, for every $h \in C(R, R)$ with $\frac{h}{a}$ being bounded, the nonhomogeneous equation

$$-x''(t) + c(t)x'(t) + a(t)x(t) = h(t)$$

with boundary condition $x(-\infty) = x(+\infty) = 0$ has a unique bounded solution $u(t) = \int_R G(t, s)\mathrm{d}s$, which is crucial in [20] for applying some fixed point theorems on cones. Motivated by [16–20], as well as [21,22], we continue to study the existence of homoclinic-like solution for Eq. (1.2).

The work of present paper for investigating the existence of homoclinic solutions to (1.1) is divided three parts. First, for each $k \in N$, we investigate the existence of $2kT$-periodic solutions $u_k(t)$ for the following equation

$$x''(t) + f(x(t))x'(t) - g(x(t)) - \frac{\alpha(t)x(t)}{1 - x(t)} = h_k(t), \tag{1.3}$$

where $h_k : R \to R$ are two $2kT$-periodic solutions with

$$h_k(t) = \begin{cases} h(t), & t \in \left[-kT, kT - \frac{T}{2}\right), \\ h\left(kT - \frac{T}{2}\right) + \frac{2(h(-kT) - h(kT - \frac{T}{2}))}{T}\left(t - kT + \frac{T}{2}\right), & t \in \left[kT - \frac{T}{2}, kT\right]. \end{cases} \quad (1.4)$$

Using a known continuation theorem of coincidence degree theory, we obtain that for each $k \in N$, there is at least one positive $2kT$-periodic solution $u_k(t)$ to Eq. (1.3). Second, we will show that the sequence $\{u_k(t)\}$ satisfies

$$\int_{-kT}^{kT} |u_k(t)|^{n+1} \mathrm{d}t \le M_0, \quad \int_{-kT}^{kT} |u_k'(t)|^2 \mathrm{d}t \le M_1$$

and

$$-\infty < \rho_0 < u_k(t) \le \rho_1 \in (0, 1), \quad \max_{t \in [-kT, kT]} |u_k'(t)| \le \rho_2,$$

where n, M_0, M_1, ρ_0, ρ_1 and ρ_2 are positive constants independent of k. Finally, a homoclinic solution for Eq. (1.1) is obtained as a limit of a certain subsequence of $\{u_k(t)\}$.

By contrast, our approach to Eq. (1.1) is neither based on variational theory used in [16–19], because there is a first derivative term $f(x(t))x'(t)$ in Eq. (1.1), and then Eq. (1.1) has no variational structure, nor based on the methods used in [20], since the terms of $f(x)y$ and $g(x)$ may be generally nonlinear with respect to variables of x and y.

2. Preliminary lemmas

Throughout this paper, the set of all positive integers is denoted by N, and for $\omega > 0$ being a constant, let $C_\omega = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm defined by $|x|_\infty = \max_{t \in [0, \omega]} |x(t)|$.

Let $y(t) = 1 - x(t)$, then (1.3) is converted to the equation

$$y''(t) + f(1 - y(t))y'(t) + g(1 - y(t)) + \frac{\alpha(t)}{y(t)} = -h_k(t) + \alpha(t). \quad (2.1)$$

Clearly, the problem of searching for $2kT$-periodic solution $u(t)$ to (1.1) with $u(t) < 1$ is reduced to the question to investigate positive $2kT$-periodic solution for (2.1). Now,we embed (2.1) into the following equation family with a parameter $\lambda \in (0, 1]$

$$y''(t) + \lambda f(1 - y(t))y'(t) + \lambda g(1 - y(t)) + \frac{\lambda\alpha(t)}{y(t)} = \lambda(-h_k(t) + \alpha(t)), \quad \lambda \in (0, 1]. \quad (2.2)$$

To study the existence of $2kT$-periodic solution to (2.1) for each $k \in N$, we give the following Lemma which is an easy consequence of main result in [23] and [24].

Lemma 2.1. *Assume that there exist positive constants N_0, N_1 and N_2 with $0 < N_0 < N_1$, such that the following conditions hold.*

1. *For each* $\lambda \in (0,1]$, *each possible positive* $2kT$*-periodic solution* x *to the equation*

$$y''(t) + \lambda f(1-y(t))y'(t) + \lambda g(1-y(t)) + \frac{\lambda\alpha(t)}{y(t)} = \lambda(-h_k(t) + \alpha(t))$$

satisfies the inequalities $N_0 < x(t) < N_1$ *and* $|x'(t)| < N_2$ *for all* $t \in [0,T]$.

2. *Each possible positive solution* c *to the equation*

$$g(1-c) + \frac{\bar{\alpha}}{c} + \overline{h_k} - \bar{\alpha} = 0,$$

satisfies the inequality $N_0 < c < N_1$.

3. *It holds*

$$\left(g(1-N_0) + \frac{\bar{\alpha}}{N_0} + \overline{h_k} - \bar{\alpha}\right)\left(g(1-N_1) + \frac{\bar{\alpha}}{N_1} + \overline{h_k} - \bar{\alpha}\right) < 0.$$

Then Eq. (2.1) *has at least one positive* $2kT$*-periodic solution* x *such that* $N_0 < x(t) < N_1$ *for all* $t \in [0,T]$.

Lemma 2.2. *If* $u: R \to R$ *is continuously differentiable on* R, $a > 0, \mu > 1$ *and* $p > 1$ *are constants, then for every* $t \in R$, *the following inequality holds:*

$$|u(t)| \le (2a)^{-\frac{1}{\mu}} \left(\int_{t-a}^{t+a} |u(s)|^{\mu} \mathrm{d}s\right)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} |u'(s)|^{p} \mathrm{d}s\right)^{\frac{1}{p}}.$$

This lemma is a special case of Lemma 2.2 in [25].

Lemma 2.3 [26]. *Let* $\{u_k\} \in C^1_{2kT}$ *be a sequence of* $2kT$*-periodic functions, such that for each* $k \in N$, u_k *satisfies*

$$|u_k|_0 \le A_0, |u_k'|_0 \le A_1,$$

where A_0, A_1 *are constants independent of* $k \in N$. *Then there exist a* $u_0 \in C(R,R)$ *and a subsequence* $\{u_{k_j}\}$ *of* $\{u_k\}_{k\in N}$ *such that for each* $j \in N$,

$$\max_{t\in[-jT,jT]} |u_{k_i}(t) - u_0(t)| \longrightarrow 0, \quad as \quad i \longrightarrow +\infty.$$

Now, we list the following assumptions, which will be used for studying the existence of homoclinic solutions to Eq. (1.1).

[$H1$] $g: R \to R$ is strictly monotone increasing and there are constants $\sigma > 0$ and $n > 0$ such that

$$yg(y) \ge \sigma|y|^{n+1} \quad \text{for all} \quad y \in R;$$

[$H2$] $\sup_{t\in R} |h(t)| := \rho \in (0,+\infty)$ and $\int_R |h(t)|^{\frac{n+1}{n}} \mathrm{d}t := \rho_0 < +\infty$, where n is determined in [$H1$].

3. Main result

Theorem 3.1. *Suppose that assumptions of [H_1] and [H_2] hold. Then Eq. (1.1) has at least one nontrivial homoclinic solution.*

Proof. Suppose that $v(t)$ is an arbitrary positive $2kT$-periodic solution to (2.2), then

$$v'' + \lambda f(1-v(t))v'(t) + \lambda g(1-v(t)) + \frac{\lambda\alpha(t)}{v(t)} = \lambda(-h_k(t)+\alpha(t)), \quad \lambda \in (0,1]. \tag{3.1}$$

Let t_1 and t_2 be the maximum point and the minimum point of $v(t)$ on $[-kT, kT]$. This implies that $v'(t_1) = v'(t_2) = 0$, $v''(t_1) \le 0$ and $v''(t_2) \ge 0$, which together with (3.1) gives that

$$g(1-v(t_1)) + \frac{\alpha(t_1)}{v(t_1)} \ge -h_k(t_1) + \alpha(t_1) \ge -|h_k|_\infty + \alpha(t_1) \tag{3.2}$$

and

$$g(1-v(t_2)) + \frac{\alpha(t_2)}{v(t_2)} \le -h_k(t_2) + \alpha(t_2) \le |h_k|_\infty + \alpha(t_2). \tag{3.3}$$

Using the monotonicity property of $g(x)$, we have from (3.2) that

$$v(t_1) < 1 - g^{-1}(-\rho), \tag{3.4}$$

where ρ is determined in $[H2]$. In fact, if

$$v(t_1) \ge 1 - g^{-1}(-\rho), \tag{3.5}$$

then $v(t_1) > 1$; and it follows from (3.2) that

$$g(1-v(t_1)) > -|h_k|_\infty \ge -\rho,$$

i.e.,

$$v(t_1) < 1 - g^{-1}(-\rho),$$

which contradicts to (3.5). This contradiction implies that (3.4) holds. Also, we can conclude from (3.3) that

$$v(t_2) > \frac{\alpha_l}{\rho + \alpha_l}, \tag{3.6}$$

where $\alpha_l := \min_{t\in[0,T]} \alpha(t)$. If (3.6) does not hold, then

$$v(t_2) \le \frac{\alpha_l}{\rho + \alpha_l}. \tag{3.7}$$

It follows from (3.3) that

$$\begin{aligned} g(1-v(t_2)) &\le \rho+\alpha(t_2)-\frac{\alpha(t_2)}{v(t_2)} \\ &= \rho+\alpha(t_2)\left(1-\frac{1}{v(t_2)}\right) \\ &\le \rho+\alpha(t_2)\left(1-\frac{\rho+\alpha_l}{\alpha_l}\right) \\ &\le \rho-\frac{\alpha_l\rho}{\alpha_l} \\ &= 0, \end{aligned}$$

which together with assumption $[H1]$ yields that

$$1-v(t_2)\le 0,$$

i.e.,

$$v(t_2)\ge 1,$$

which contradicts to (3.7), (3.4) and (3.6) give that

$$\gamma_0:=\frac{\alpha_l}{\rho+\alpha_l}<v(t)<1-g^{-1}(-\rho):=\gamma_1,\quad \text{for all}\ \ t\in[-kT.kT]. \tag{3.8}$$

Let $w_\lambda(t)=v'(t)+\lambda F(v(t)),\lambda\in(0,1]$, where $F(x)=\int_0^x f(1-s)\mathrm{d}s$, then from (3.1) that

$$w_\lambda'(t)=-\lambda g(1-v(t))-\frac{\lambda\alpha(t)}{v(t)}+\lambda(-h_k(t)+\alpha(t)),\quad \lambda\in(0,1],$$

and then

$$\max_{t\in[-kT,kT]}|w_\lambda'(t)|\le g_{\gamma_0,\gamma_1}+\frac{\alpha_\infty}{\gamma_0}+\alpha_\infty+\rho:=\gamma_2,\quad \lambda\in(0,1], \tag{3.9}$$

where $\alpha_\infty=\max_{t\in[-kT,kT]}\alpha(t)$ and $g_{\gamma_0,\gamma_1}=\max_{\gamma_0\le x\le\gamma_1}|g(1-x)|$. Furthermore, for each $t\in[-kT,kT]$, it is easy to see that there is an integer $i\in\{-k,-k+1,\dots,k-1\}$ such that $t\in[iT,(i+1)T]$. From the continuity of $v'(t)$ on $[iT,(i+1)T]$, we have $t_i\in[iT,(i+1)T]$ such that

$$v'(t_i)=\frac{1}{T}\int_{iT}^{(i+1)T}v'(s)\mathrm{d}s,$$

which together with (3.8) yields

$$|v'(t_i)|=\left|\frac{1}{T}\int_{iT}^{(i+1)T}v'(s)\mathrm{d}s\right|=\frac{1}{T}\left|v(iT)-v((i+1)T)\right|<\frac{2\gamma_1}{T}. \tag{3.10}$$

Since

$$\begin{aligned}|w_\lambda(t)| &= |w_\lambda(t_i) + \int_{t_i}^{t} w'_\lambda(s)\mathrm{d}s| \\ &\le |w_\lambda(t_i)| + \int_{iT}^{(i+1)T} |w'_\lambda(s)|\mathrm{d}s \\ &\le |v'(t_i)| + |F(v(t_i)| + \int_{iT}^{(i+1)T} |w'_\lambda(s)|\mathrm{d}s,\end{aligned}$$

it follows from (3.8), (3.9) and (3.10) that

$$|w_\lambda(t)| \le \frac{2\gamma_1}{T} + F_{\gamma_0,\gamma_1} + T\gamma_2,$$

where $F_{\gamma_0,\gamma_1} := \max_{\gamma_0\le x\le\gamma_1} |F(x)|$, i.e.,

$$|v'|_\infty = \max_{t\in[-kT,kT]} |v'(t)| \le \frac{2\gamma_1}{T} + 2F_{\gamma_0,\gamma_1} + T\gamma_2 := \gamma_3. \tag{3.11}$$

Clearly, γ_3 is a positive constant independent of $k \in N$. By (3.8), it is easy to check that

$$g(1-\gamma_0) + \frac{\bar{\alpha}}{\gamma_0} + \overline{h_k} - \bar{\alpha} > 0$$

and

$$g(1-\gamma_1) + \frac{\bar{\alpha}}{\gamma_1} + \overline{h_k} - \bar{\alpha} < 0,$$

and then

$$\left(g(1-\gamma_0) + \frac{\bar{\alpha}}{\gamma_0} + \overline{h_k} - \bar{\alpha}\right)\left(g(1-\gamma_1) + \frac{\bar{\alpha}}{\gamma_1} + \overline{h_k} - \bar{\alpha}\right) < 0.$$

Thus, using Lemma 2.1 for the case of $N_0 = \gamma_0$, $N_1 = \gamma_1$ and $N_2 = \gamma_3$, we have from (3.8) and (3.11) that for each $k \in N$, there is a positive $2kT$-periodic solution $v_k(t)$ to (2.1) such that

$$\frac{\alpha_l}{\rho+\alpha_l} < v_k(t) < 1 - g^{-1}(-\rho),\ |v'_k|_\infty < \gamma_3 \quad \text{for all} \ \ k \in N.$$

It follows from the substitution defined by $y(t) = 1 - x(t)$ that for each $k \in N$, there is a $2kT$-periodic solution $u_k(t)$ to (1.3) such that

$$A_0 := g^{-1}(-\rho) < u_k(t) < \frac{\rho}{\rho+\alpha_l} := A_1 \tag{3.12}$$

and

$$|u'_k|_\infty \le \gamma_3. \tag{3.13}$$

Since $u_k(t)$ is a $2kT$-periodic solution to (1.3), we have

$$u''_k(t) + f(u_k(t))u'_k(t) - g(u_k(t)) - \frac{\alpha(t)u_k(t)}{1-u_k(t)} = h_k(t), \tag{3.14}$$

and then by (3.12) and (3.13), we have

$$|u''_k|_\infty \le \gamma_4, \tag{3.15}$$

where $\gamma_4 := \gamma_3 f_{A_0,A_1} + g_{A_0,A_1} + \frac{\alpha_\infty A_1}{1-A_1} + \rho$ is a constant independent of $k \in N$. Using Lemma 2.3, we see that there are a $u_0 \in C^1(R,R)$ and a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that

$$\max_{t\in[-jT,jT]} |u_{k_i}(t) - u_0(t)| \longrightarrow 0, \quad \text{and} \quad \max_{t\in[-jT,jT]} |u'_{k_i}(t) - u'_0(t)| \to 0 \quad \text{as} \quad i \to +\infty. \tag{3.16}$$

For any real numbers a and b satisfying $a < b$, there is a positive integer j_0 such that for $j > j_0, [-k_jT, k_jT) \supset [a,b]$. Thus, if $j > j_0$, then from (1.4) and (3.14), we see that

$$u''_{k_j}(t) + f(u_{k_j}(t))u'_{k_j}(t) - g(u_{k_j}(t)) - \frac{\alpha(t)u_{k_j}(t)}{1-u_{k_j}(t)} = h(t), \quad t \in [a,b], \tag{3.17}$$

Integrating (3.17) over $[a,t] \subset [a,b]$, we get

$$\begin{aligned} u'_{k_j}(t) - u'_{k_j}(a) = &-\int_a^t f(u_{k_j}(s))u'_{k_j}(s)\mathrm{d}s - \int_a^t g(u_{k_j}(s))\mathrm{d}s \\ &- \frac{\alpha(s)u_{k_j}(s)}{1-u_{k_j}(s)}\mathrm{d}s + \int_a^t h(s)\mathrm{d}s, \text{ for } t \in [a,b]. \end{aligned} \tag{3.18}$$

(3.16) implies that $u_{k_j}(t) \to u_0(t)$ uniformly for $t \in [a,b]$ and $u'_{k_j}(t) \to u'_0(t)$ uniformly for $t \in [a,b]$. Let $j \to \infty$ in (3.18), we have

$$\begin{aligned} u'_0(t) - u'_0(a) = &-\int_a^t f(u_0(s))u'_0(s)\mathrm{d}s - \int_a^t g(u_0(s))\mathrm{d}s \\ &- \frac{\alpha(s)u_0(s)}{1-u_0(s)}\mathrm{d}s + \int_a^t h(s)\mathrm{d}s, \text{ for } t \in [a,b]. \end{aligned} \tag{3.19}$$

Considering a and b are two arbitrary constants with $a < b$, it is easy to see from (3.19) that u_0 is a solution to (1.1), i.e.,

$$u''_0(t) + f(u_0(t))u'_0(t) - g(u_0(t)) - \frac{\alpha(t)u_0(t)}{1-u_0(t)} = h(t), \quad t \in R. \tag{3.20}$$

Below, we will show

$$u_0(t) \to 0 \quad \text{and} \quad u'_0(t) \to 0 \quad \text{as} \quad |t| \to +\infty.$$

For each $k \in N$, multiplying (3.14) with $u_k(t)$ and integrating it over the interval $[-kT, kT]$, we have

$$\begin{aligned} &\int_{-kT}^{kT} |u'_k(t)|^2\mathrm{d}t - \int_{-kT}^{kT} f(u_k(t))u_k(t)u'_k(t)\mathrm{d}t \\ &\quad + \int_{-kT}^{kT} g(u_k(t))u_k(t)\mathrm{d}t + \int_{-kT}^{kT} \frac{\alpha(t)u_k(t)}{1-u_k(t)}\mathrm{d}t \\ &= \int_{-kT}^{kT} h_k(t)u_k(t)\mathrm{d}t. \end{aligned}$$

It follows from $\int_{-kT}^{kT} f(u_k(t))u_k(t)u_k'(t)\mathrm{d}t = 0$, together with assumption $[H1]$ that

$$\begin{aligned}\int_{-kT}^{kT} |u_k'(t)|^2\mathrm{d}t + \sigma\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t\\ \leq \int_{-kT}^{kT} |h_k(t)u_k(t)|\mathrm{d}t\\ \leq \left(\int_{-kT}^{kT} |h_k(t)|^{\frac{n+1}{n}}\mathrm{d}t\right)^{\frac{n}{n+1}}\left(\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t\right)^{\frac{1}{n+1}}. \end{aligned} \quad (3.21)$$

Furthermore, from (1.4) we see that

$$\begin{aligned}\int_{-kT}^{kT} |h_k(t)|^{\frac{n+1}{n}}\mathrm{d}t &= \int_{-kT}^{kT-\frac{T}{2}} |h_k(t)|^{\frac{n+1}{n}}\mathrm{d}t + \int_{kT-\frac{T}{2}}^{kT} |h_k(t)|^{\frac{n+1}{n}}\mathrm{d}t\\ &\leq \int_{-kT}^{kT-\frac{T}{2}} |h(t)|^{\frac{n+1}{n}}\mathrm{d}t + \frac{\rho^{\frac{n+1}{n}}T}{2}\\ &\leq \int_{R} |h(t)|^{\frac{n+1}{n}}\mathrm{d}t + \frac{\rho^{\frac{n+1}{n}}T}{2}\\ &:= \rho_3, \end{aligned} \quad (3.22)$$

which together with (3.21) yields

$$\sigma\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t \leq \rho_3^{\frac{n}{n+1}}\left(\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t\right)^{\frac{1}{n+1}} \quad (3.23)$$

and

$$\int_{-kT}^{kT} |u_k'(t)|^2\mathrm{d}t \leq \rho_3^{\frac{n}{n+1}}\left(\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t\right)^{\frac{1}{n+1}}. \quad (3.24)$$

(3.23) gives

$$\int_{-kT}^{kT} |u_k(t)|^{n+1}\mathrm{d}t \leq \rho_3\sigma^{-\frac{n+1}{n}}, \quad \text{for all} \quad k \in N. \quad (3.25)$$

Substituting (3.25) into (3.24), we get

$$\int_{-kT}^{kT} |u_k'(t)|^2\mathrm{d}t \leq \rho_3\sigma^{-\frac{1}{n}}, \quad \text{for all} \quad k \in N. \quad (3.26)$$

Since

$$\begin{aligned}\int_{-\infty}^{+\infty} (|u_0(t)|^{n+1} + |u_0'(t)|^2)\mathrm{d}t &= \lim_{i\to+\infty}\int_{-iT}^{iT} (|u_0(t)|^{n+1} + |u_0'(t)|^2)\mathrm{d}t\\ &= \lim_{i\to+\infty}\lim_{j\to+\infty}\int_{-iT}^{iT} (|u_{k_j}(t)|^{n+1} + |u_{k_j}'(t)|^2)\mathrm{d}t,\end{aligned}$$

clearly, for every $i \in N$, if $k_j > i$, then by (3.25) and (3.26), we have

$$\int_{-iT}^{iT} (|u_{k_j}(t)|^{n+1} + |u'_{k_j}(t)|^2)\mathrm{d}t \le \int_{-k_jT}^{k_jT} (|u_{k_j}(t)|^{n+1} + |u'_{k_j}(t)|^2)\mathrm{d}t$$
$$\le \rho_3\sigma^{-\frac{1}{n}} + \rho_3\sigma^{-\frac{n+1}{n}}.$$

Let $i \to +\infty$ and $j \to +\infty$, we get

$$\int_{-\infty}^{+\infty} (|u_0(t)|^{n+1} + |u'_0(t)|^2)\mathrm{d}t \le \rho_3\sigma^{-\frac{1}{n}} + \rho_3\sigma^{-\frac{n+1}{n}}, \tag{3.27}$$

and then

$$\int_{|t|\ge r} \left(|u_0(t)|^{n+1} + |u'_0(t)|^2\right) \mathrm{d}t \to 0$$

as $r \to +\infty$. So using Lemma 2.2, we obtain

$$\begin{aligned} |u_0(t)| &\le (2T)^{-\frac{1}{n+1}} \left(\int_{t-T}^{t+T} |u_0(s)|^{n+1}\mathrm{d}s\right)^{\frac{1}{n+1}} + T(2T)^{-\frac{1}{2}} \left(\int_{t-T}^{t+T} |u'_0(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\le \left[(2T)^{-\frac{1}{m}} + T(2T)^{-\frac{1}{2}}\right] \left[\left(\int_{t-T}^{t+T} |u_0(s)|^{n+1}\mathrm{d}s\right)^{1/(n+1)} \right. \\ &\quad \left. + \left(\int_{t-T}^{t+T} |u'_0(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\to 0, \text{ as } |t| \to +\infty, \end{aligned}$$

which implies that

$$u_0(t) \to 0 \quad \text{as} \quad |t| \to +\infty. \tag{3.28}$$

Next, we will prove that

$$u'_0(t) \to 0 \quad \text{as} \quad |t| \to +\infty. \tag{3.29}$$

From (3.12), (3.13) and (3.16), we obtain

$$|u_0(t)| \le \max\left\{\frac{\rho}{\rho + \alpha_l}, |g^{-1}(-\rho)|\right\} := A_0, \quad \text{for} \quad t \in R. \tag{3.30}$$

and

$$|u'_0(t)| \le \gamma_3, \quad \text{for } t \in R. \tag{3.31}$$

It follows from (3.20) that

$$|u''_0(t)| \le f_{A_0,A_1}\gamma_3 + g_{A_0,A_1} + \frac{\alpha_\infty A_1}{1 - A_1} + \rho := A_2, \quad \text{for} \quad t \in R. \tag{3.32}$$

If (3.29) does not hold, then there is a constant $\delta \in (0, \frac{1}{2})$ and a sequence $\{t_k\}$ that

$$|t_1| < |t_2| < |t_3| < \cdots,$$

with $|t_k| + 1 < |t_{k+1}|$, $k = 1, 2, \ldots$, and

$$|u'_0(t_k)| \ge 2\delta, \quad k = 1, 2, \ldots,$$

which results in

$$\begin{aligned}|u_0'(t)| &= \left|u_0'(t_k) + \int_{t_k}^{t} u_0''(s)\mathrm{d}s\right| \\ &\geq |u_0'(t_k)| - \int_{t_k}^{t} |u_0''(s)|\mathrm{d}s \\ &\geq \delta, \quad \text{for} \quad t \in [t_k, t_k + \frac{\delta}{1+A_2}],\end{aligned}$$

and then

$$\int_{-\infty}^{+\infty} |u_0'(t)|^2\mathrm{d}t \geq \sum_{k=1}^{+\infty} \int_{t_k}^{t_k+\frac{\delta}{1+A_2}} |u_0'(t)|^2\mathrm{d}t = +\infty.$$

This contradicts to (3.27). It is easy to see that (3.29) holds. Thus,$u_0(t)$ is just a homoclinic solution to equation (1.1). □

4. Example

In this section, we present an example to demonstrate the main result.

Consider the following equation:

$$x''(t) + f(x(t))x'(t) - x^3(t) - \frac{(1-\frac{1}{2}\sin t)x(t)}{1-x(t)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}, \tag{4.1}$$

where $f : R \to R$ are continuous,$h(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ is a standard normal distribution probability function. Corresponding to (1.1), we have $g(x) = x^3$, $\alpha(t) = 1 - \frac{1}{2}\sin t$. We can easily check that $[H1]$ and $[H2]$ holds for the case of $\sigma = 1$ and $n = 3$. From Theorem 3.1, we know that equation (4.1) has at least one nontrivial homoclinic solution.

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