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Asian option as a fixed-point

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Abstract. We characterize the price of an Asian option, a financial contract, as a fixed-point of a non-linear operator. In recent years, there has been interest in incorporating changes of regime into the parameters describing the evolution of the underlying asset price, namely the interest rate and the volatility, to model sudden exogenous events in the economy. Asian options are particularly interesting because the payoff depends on the integrated asset price. We study the case of both floating- and fixed-strike Asian call options with arithmetic averaging when the asset follows a regime-switching geometric Brownian motion with coefficients that depend on a Markov chain. The typical approach to finding the value of a financial option is to solve an associated system of coupled partial differential equations. Alternatively, we propose an iterative procedure that converges to the value of this contract with geometric rate using a classical fixed-point theorem.

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1. Introduction

In this paper we use a fixed-point theorem to characterize the price of both floating- and fixed-strike Asian call options (defined below) when the interest rate and volatility of the underlying asset are subject to changes of regime during the pricing period. We next formulate the problem precisely.

Let $(\Omega, \mathcal{F}, P)$ be a probability space which supports a Brownian motion $B = (B_t)_{t\geq 0}$ and a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$ independent of B with finite state space $\mathcal{M} = \{1, 2, \dots, m\}$ and generator $Q = (q_{ij})_{m\times m}$,

$$q_{ij} \geq 0 \quad \text{for } i \neq j, \qquad \sum_{j\in\mathcal{M}} q_{ij} = 0, \qquad q_i := -q_{ii} \geq 0.$$

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Suppose that under P, the underlying asset price follows the regime-switching geometric Brownian motion,

$$\mathrm{d}X_t = X_t[(r(Y_t) - \delta)\,\mathrm{d}t + \sigma(Y_t)\mathrm{d}B_t\,], \qquad 0 \le t \le T,$$

where $r(i) > 0$ and $\sigma(i) > 0$ denote the risk-free interest rate and the volatility at regime i, respectively, and $\delta \ge 0$ is the dividend rate. Denote by $\mathcal{F}_t$ the sigma-algebra generated by $\{(X_u, Y_u) : 0 \le u \le t\}$.

Throughout this paper we fix a time $t_0 \in [0, T)$, and define the integrated process,

$$A_t := \int_{t_0}^{t} X_u\,\mathrm{d}u, \qquad t_0 \le t \le T$$

and for convenience, we extend the definition of A to $A_t = 0$ for $t \in [0, t_0]$.

The European call option has payoff $(X_T - K)^+$ at time T, where $K > 0$ is a fixed strike. An Asian option is a path-dependent European-style option, where the payoff depends on the average of past prices during the time interval $[t_0, T]$. Asian options are mainly classified as *fixed-strike* (when X_T is replaced by A_T and the strike K is fixed) or *floating-strike* (when K is replaced by A_T). In this paper we study both cases.

More precisely, the price at time $s \in [0, T]$ of an Asian call option with floating-strike expiring at T is given by:

$$C(s, x, a, i) = \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(X_T - \frac{A_T}{T - t_0}\right)^+\right], \tag{1.1}$$

while an Asian call option with fixed-strike expiring at T with strike price K is:

$$C_K(s, x, a, i) = \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(\frac{A_T}{T - t_0} - K\right)^+\right], \tag{1.2}$$

where we use the notation $\mathbb{E}_{s,x,a,i}[\cdot]$ for $\mathbb{E}[\cdot \mid X_s = x, A_s = a, Y_s = i]$, $x > 0, a \ge 0$. The options are referred to as *starting* when $s = t_0$, *in-progress* when $s > t_0$, and *forward-starting* when $s < t_0$.

Regime-switching processes in finance were initially proposed by Hamilton in his economic studies with discrete time models on the effect of incorporating shifts in the parameters of the model via an unobserved discrete time two-state Markov chain (see [10, 11]). Since then, several pricing methods for financial instruments have emerged under the assumption of regime-switching coefficients. Such models successfully incorporate sudden changes in the economy and compensate some of the drawbacks of the classical Black–Scholes model due to the constancy of the drift and volatility parameters. To mention some literature, Buffington and Elliott [2], Yao et al. [21], and Zhu et al. [24] concentrate on vanilla European options; Guo and Zhang [9] study perpetual American put options; and Chan and Zhu [4] deal with barrier options.

Despite the prominence of regime-switching models, the literature on Asian options within this context is scarce. Some work has been done for the class of fixed-strike Asian options, see for instance Boyle and Draviam [1] and Dan et al. [6]. The pricing methods typically require solving a system of coupled PDEs. In this paper, we explore an alternative approach to pricing

Asian call options with regime-switching, based on the fixed-point theorem for Banach spaces with the supremum norm and when the number of states m is arbitrary. The initial value in the algorithm is precisely the price of a fixed-strike Asian option without regime-switching, which has been more extensively studied in the literature. For instance, Geman and Yor [8] were able to give an expression for the Laplace transform of a *normalized* fixed-strike Asian call option by exploiting probabilistic properties of Bessel processes. The normalized Asian option involves the expectation of a function of Yor's process A_t^ν, see (3.4) below. Then the price of the option can be obtained by inversion of the Laplace transform, although they noted that such inversion was not easy. Later on, Carr and Schröder [5] built on Laplace transform techniques and provided an explicit integral representation of the price. The same year, Linetsky [17] took a different approach and showed that the normalized price is the limit of up-and-out options on the diffusion X, each of which is given as a series representation of known special functions. More recently, Cai et al. [3] obtained an algorithm to price Asian options based on an approximating continuous-time Markov chain sequence that converges to the underlying asset price process. Other authors have provided price bounds, see for instance Rogers and Shi [19] who use iterated conditional expectation.

The paper is organized as follows: In Sect. 2, convenient upper bounds of the call options are derived as well as a symmetry relationship, in the context of no regime switching, between the starting floating-strike call and a fixed-strike put. These upper bounds are crucial in order to ensure the relevant functions belong to the domain of a certain contraction operator after scaling. We summarize the main results of the paper at the end of this section.

Next, in Sect. 3 we split the functions C and C_K into two parts, one that restricts the payoff to the event that the Markov chain jumps before maturity and the other to the complementary event where the Markov chain does not jump in the lifetime of the option. In Sect. 3.1 we find the joint density of the pair (Z_t, A_t), where $Z_t = \log(X_t/x)$, given the information up to time s for $t > s$ and given that the first jump time of the Markov chain after s happens at time t. We use this density in Sect. 4, where we characterize the functions C and C_K as the limit (in the supremum norm) of a sequence whose initial point is in terms of the price of an Asian call option without regime-switching. The contraction operator is a nonlinear operator expressed as a triple integral that accounts for the jumps to different states before maturity.

The ideas in this paper are motivated by the method used by Yao et al. [21] applied to price vanilla European options. The difficulty in our context stems from the fact that Asian options are path-dependent and the joint density of geometric Brownian motion and its integrated process is required. Nonetheless, the fixed-point theorem approach works well in this setup and we are able to show that the rate of convergence of the sequence is geometric. Proof of preliminary lemmas appears in the Appendix.

2. Preliminaries and main results

It is known that European call options are bounded above by the current price of the underlying process. This is also true for the floating-strike option in (1.1) and for the fixed-strike option in (1.2) up to a constant, and will be used in the fixed-point approximation. The proofs of the next lemmas are presented in the Appendix.

Lemma 2.1. *For any initial condition* (s, x, a, i) *with* $s \in [0, T]$, $x > 0$, $a \geq 0$,

$$C(s, x, a, i) \leq x. \tag{2.1}$$

Define the call option conditional on the chain having no jump in the interval $[s, T]$,

$$C^0(s, x, a, i) := \mathbb{E}_{s,x,a,i}\left[e^{-r(i)(T-s)}\left(X_T - \frac{A_T}{T - t_0}\right)^+ \mid Y_t = i, \forall t \in [s, T]\right].$$

When the option is starting or forward-starting, it is possible to establish a symmetry between the associated floating-strike call option $C^0(s, x, 0, i)$ and a fixed-strike Asian put option, for each $i \in \mathcal{M}$. When the option is in-progress, it is equivalent to a *generalized starting option* (see (2.2) below). This type of symmetry results were studied, for instance, by Henderson and Wojakowski [12] and Henderson et al. [13] in the classical setup without regime switching.

The proof of the next lemma is included for completeness of presentation but similar arguments are used in [13].

Lemma 2.2. *For any initial condition* (s, x, a, i) *with* $s \in [0, T]$, $x > 0$, $a \geq 0$,

$$C^0(s, x, a, i) = \mathbb{E}^*_{s,x,a,i}\left[e^{-\delta(T-s)}\left(x - \lambda X^*_T - \beta \frac{1}{T - s}\int_s^T X^*_u du\right)^+\right] \tag{2.2}$$

where $\lambda = \frac{a}{x(T-t_0)}$ *and* $\beta = \frac{T-s}{T-t_0}$ *and the expectation* $\mathbb{E}^*$ *is with respect to an equivalent martingale measure* P^* *under which* X^* *solves the stochastic differential equation,*

$$dX^*_t = X^*_t[(\delta - r(i))\,dt + \sigma(i)\,dB^*_t], \quad X^*_s = x, \qquad t \geq s.$$

In particular, if the option is starting ($s = t_0$) *or forward-starting* ($s < t_0$) *then* $a = 0$ *and the floating-strike call option* C^0 *is equivalent to a fixed-strike put option. Specifically,*

$$C^0(s, x, 0, i) = \mathbb{E}^*_{s,x,0,i}\left[e^{-\delta(T-t_0)}\left(x - \beta\frac{A^*_T}{T - t_0}\right)^+\right] \tag{2.3}$$

where $A^*_T = \int_{t_0}^T X^*_u\, du$, *and* $\beta \geq 1$.

There are well-known methods for fixed-strike options without switching coefficients as in (2.3), some works have been cited in the introduction. Using any of such methods, in conjunction with the so-called put-call parity for fixed-strike Asian options (see [14, p.220]), the value of C^0 in (2.3) can be computed. In contrast, methods to compute (2.2) explicitly are less accessible. In a recent work by Funahashi and Kijima [7], they provide an approximation method for generalized Asian options by applying a so-called chaos expansion approach. Monte Carlo methods can be used as a benchmark when there are no closed-form formulas, see [15].

Now we turn to the case of fixed-strike options.

Lemma 2.3. *Fix an initial condition* (s, x, a, i) *with* $s \in [0, T]$, $x > 0$, $a \geq 0$.

(i) *If* $s \leq t_0$ *then* $a = 0$ *and* $C_K(s, x, 0, i) \leq x$.

(ii) *If* $s > t_0$ *then*

$$C_K(s, x, a, i) \leq \frac{a}{T - t_0} + x.$$

We conclude this section with a summary of the main results in the paper, without stating the technical details which we examine in the subsequent sections. Consider the Banach space $\mathcal{S}$ of all bounded measurable functions $H : E \mapsto \mathbb{R}$, $E = [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathcal{M}$, with the supremum norm,

$$||H|| := \sup_{(s,z,a,i) \in E} |H(s, z, a, i)|.$$

Let $F : \mathcal{S} \mapsto \mathcal{S}$ be defined by:

$$F(H)(s, z, a, i) := \sum_{j \neq i} q_{ij} \int_s^T e^{-[q_i + r(i)](t-s)} \int_a^\infty \int_{-\infty}^\infty$$

$$e^{z'} H(t, z + z', a', j)\, \psi(z', a')\, \mathrm{d}z'\, \mathrm{d}a'\, \mathrm{d}t \tag{2.4}$$

where ψ is a joint density function to be derived later (see Proposition 3.2 below).

We now state the main result.

Theorem 2.4. (Contraction)

(i) F *is a contraction mapping on* $\mathcal{S}$.

(ii) *If* $H_0, H \in \mathcal{S}$ *and* H *solves the equation*

$$H(s, z, a, i) = F(H)(s, z, a, i) + H_0(s, z, a, i)$$

then H *is the only solution.*

(iii) *The sequence* $\{H_n\}_{n=0}^\infty$, *with*

$$H_{n+1}(s, z, a, i) = F(H_n)(s, z, a, i) + H_0(s, z, a, i),$$

converges to the fixed-point H *with geometric rate of converge*

$$\rho = \max_{i \in \mathcal{M}} \sum_{j \neq i} \frac{q_{ij}}{q_i + \delta} \left(1 - e^{-(q_i + \delta)(T - s)}\right) < 1. \tag{2.5}$$

We specialize to the floating-strike Asian options below.

Theorem 2.5. (Floating-strike option as a fixed-point) *Define the functions* $g, g_0 \in \mathcal{S}$ *by:*

$$g(s, z, a, i) := e^{-z} C(s, e^z, a, i),$$

$$g_0(s, z, a, i) := e^{-q_i(T-s)} e^{-z} C^0(s, e^z, a, i).$$

Then g is the fixed-point of,

$$g(s,z,a,i) = F(g)(s,z,a,i) + g_0(s,z,a,i).$$

Moreover, the sequence $\{g_n\}_{n=0}^{\infty}$, with $g_{n+1} = F(g_n) + g_0$ converges to g with geometric rate of converge ρ in (2.5).

For fixed-strike Asian options, the initial condition of the approximating sequence depends on whether the option is starting, forward-starting or in-progress. More precisely, we have the following statement. Below, $C_K^0(s,x,a,i)$ is defined similarly to $C^0(s,x,a,i)$, as the fixed-strike call option conditional on the chain having no jump in the interval $[s,T]$.

Theorem 2.6. (Fixed-strike option as a fixed-point)

(i) *If $s \le t_0$, define the functions $h, h_0 \in \mathcal{S}$ by:*

$$h(s,z,a,i) := e^{-z} C_K(s,e^z,a,i),$$
$$h_0(s,z,a,i) := e^{-q_i(T-s)} e^{-z} C_K^0(s,e^z,a,i).$$

Then h is the fixed-point of:

$$h(s,z,a,i) = F(h)(s,z,a,i) + h_0(s,z,a,i).$$

Moreover, the sequence $\{h_n\}_{n=0}^{\infty}$, with $h_{n+1} = F(h_n) + h_0$ converges to h with geometric rate of converge ρ in (2.5).

(ii) *If $s > t_0$, define the functions $\tilde{h}, \tilde{h}_0 \in \mathcal{S}$ by,*

$$\tilde{h}(s,z,a,i) := e^{-z}\left(C_K(s,e^z,a,i) - \frac{a}{T-t_0}\right),$$
$$\tilde{h}_0(s,z,a,i) := e^{-q_i(T-s)} e^{-z}\left(C_K^0(s,e^z,a,i) - \frac{a}{T-t_0}\right) + \tilde{h}^1(s,z,a,i)$$

where

$$\tilde{h}^1(s,z,a,i) = \frac{ae^{-z}}{T-t_0}\left[\frac{q_i}{q_i + r(i)}\left(1 - e^{(q_i + r(i))(T-s)}\right) + e^{-q_i(T-s)} - 1\right]. \tag{2.6}$$

Then $\tilde{h}$ is the fixed-point of,

$$\tilde{h}(s,z,a,i) = F(\tilde{h})(s,z,a,i) + \tilde{h}_0(s,z,a,i).$$

Moreover, the sequence $\{\tilde{h}_n\}_{n=0}^{\infty}$, with $\tilde{h}_{n+1} = F(\tilde{h}_n) + \tilde{h}_0$ converges to $\tilde{h}$ with geometric rate of converge ρ in (2.5).

3. Conditioning on the first jump time

Let us fix the current time $s \in [0,T]$ throughout the rest of the paper. Conditional on $Y_s = i$, let τ denote the first jump time of the Markov chain Y after time s, that is,

$$\tau = \inf\{t > s : Y_t \ne i\}.$$

We know that τ has exponential distribution with parameter q_i. Plainly,

$$\begin{aligned} C(s,x,a,i) &= e^{-q_i(T-s)}C^0(s,x,a,i) \\ &\quad + \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(X_T - \frac{A_T}{T-t_0}\right)^+ \mathbb{1}(\tau \le T)\right]. \end{aligned} \tag{3.1}$$

Likewise,

$$\begin{aligned} C_K(s,x,a,i) &= e^{-q_i(T-s)}C_K^0(s,x,a,i) \\ &\quad + \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(\frac{A_T}{T-t_0} - K\right)^+ \mathbb{1}(\tau \le T)\right]. \end{aligned} \tag{3.2}$$

Notice that by conditioning the expectation in (3.1) on the jump time $\tau = t$, $t \ge s$, we can write it as:

$$\begin{aligned} &\mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(X_T - \frac{A_T}{T-t_0}\right)^+ \mathbb{1}(\tau \le T)\right] \\ &= \int_s^T q_i e^{-q_i(t-s)}\mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(X_T - \frac{A_T}{T-t_0}\right)^+ \mid \tau = t\right]\mathrm{d}t \\ &= \int_s^T q_i e^{-q_i(t-s)}\mathbb{E}_{s,x,a,i}\left[e^{-r(i)(t-s)}C(t,X_t,A_t,Y_t) \mid \tau = t\right]\mathrm{d}t \end{aligned}$$

by virtue of the Markov property of (t, X_t, A_t, Y_t). A similar argument holds for the fixed-strike case.

In what follows it will be convenient to work with the process,

$$Z_t := \int_s^t \sigma(Y_u)\mathrm{d}B_u + \int_s^t \left(r(Y_u) - \delta - \frac{1}{2}\sigma^2(Y_u)\right)\mathrm{d}u, \qquad t \ge s$$

so that

$$X_t = \exp(z + Z_t), \quad z := \ln(x). \tag{3.3}$$

3.1. Density of (Z_t, A_t)

Conditional on $X_s = e^z, A_s = a, Y_s = i$ and $\tau = t$, it follows that,

$$Z_t \overset{law}{=} \sigma(i)B_{t-s} + \nu(i)(t-s), \qquad \nu(i) := r(i) - \delta - \frac{1}{2}\sigma^2(i)$$

and

$$A_t = a + \int_s^t X_u \mathrm{d}u \overset{law}{=} a + e^z \int_0^{t-s} e^{\sigma(i)B_u + \nu(i)u}\mathrm{d}u.$$

The pair (Z_t, A_t) is independent of Y_t and its distribution can be explicitly computed. To this end, define,

$$A_t^\nu := \int_0^t e^{2(B_u + \nu u)}\mathrm{d}u, \qquad \nu \in \mathbb{R}. \tag{3.4}$$

The following preliminary result is due to Yor [23].

Lemma 3.1. *We have:*

$$P(A_t^\nu \in dw \mid B_t + \nu t = z) = f(t, z, w)\, dw$$

where,

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right) f(t, z, w) = \frac{1}{w} \exp\left(-\frac{1}{2w}(1 + e^{2z})\right) \theta_{e^z/w}(t)$$

and

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} \exp\left(\frac{\pi^2}{2t}\right) \int_0^\infty e^{-y^2/2t}\, e^{-r\cosh(y)} \sinh(y) \sin(\pi y/t)\, dy.$$

We refer to Proposition 2 in [23] for a proof.

Proposition 3.2. *The joint density $\psi(z', a')$ of the pair (Z_t, A_t), conditional on $X_s = x = e^z, A_s = a, Y_s = i$, and $\tau = t$, is given by:*

$$\psi(z', a') = \frac{\sigma^2(i)}{4} e^{-z} f(t', z', w(a'))\, \phi\left(\frac{z' - 2\nu t'}{2\sqrt{t'}}\right) 1_{\{\mathbb{R}\times[a,\infty]\}},$$

with $w(a') = \frac{\sigma^2(i)}{4} e^{-z}(a' - a)$ and $t' = \frac{\sigma^2(i)}{4}(t - s)$.

Proof. A direct consequence of Lemma 3.1 is that,

$$P(\, 2(B_t + \nu t) \in \mathrm{d}z, A_t^\nu \in \mathrm{d}w) = f\left(t, \frac{z}{2}, w\right) \phi\left(\frac{z - 2\nu t}{2\sqrt{t}}\right) \mathrm{d}w\, \mathrm{d}z$$

where $\phi(\cdot)$ is the density of a standard normal distribution.

Henceforth, conditional on $X_s = e^z, A_s = a, Y_s = i$, and $\tau = t$, and writing $P(\cdot \mid X_s = e^z, A_s = a, Y_s = i, \tau = t) = P(\cdot)$ for short, we have:

$$\begin{aligned} P(Z_t &\le z',\ A_t \le a') \\ &= P\left(\sigma(i)B_{t-s} + \nu(i)(t - s) \le z',\ \int_0^{t-s} e^{\sigma(i)B_u + \nu(i)u} \mathrm{d}u \le e^{-z}(a' - a)\right) \\ &= P\left(2(B_{t'} + \nu t') \le z',\ \int_0^{t'} e^{2(B_u + \nu u)}\, \mathrm{d}u \le \frac{\sigma^2(i)}{4} e^{-z}(a' - a)\right) \end{aligned}$$

where we used the scaling property $\sigma(i)B_{t-s} \overset{law}{=} B_{\sigma^2(i)(t-s)}$ and the change of variables,

$$t' \equiv \frac{\sigma^2(i)}{4}(t - s), \qquad \nu \equiv \frac{2\nu(i)}{\sigma^2(i)}.$$

Finally,

$$P(Z_t \le z',\ A_t \le a') = \int_{-\infty}^{z'} \int_0^{w(a')} f\left(t', \frac{z}{2}, w\right) \phi\left(\frac{z - 2\nu t'}{2\sqrt{t'}}\right) \mathrm{d}w\, \mathrm{d}z$$

and a further change of variable from w to a' concludes the proof. □

4. Fixed-point

4.1. The main contraction theorem

The goal of this subsection is to show Theorem 2.4.

Proposition 4.1. *F is a contraction mapping on $\mathcal{S}$.*

Proof. For each (s, z, a, i) and $t \geq s$ fixed, it follows that,

$$\int_a^\infty \int_{-\infty}^\infty e^{z'} \psi(z', a') \,\mathrm{d}z' \,\mathrm{d}a' = e^{(r(i)-\delta)(t-s)},$$

and so

$$\begin{aligned}
\rho(i) &:= \sum_{j\neq i} q_{ij} \int_s^T e^{-[q_i + r(i)](t-s)} \int_a^\infty \int_{-\infty}^\infty e^{z'} \psi(z', a') \,\mathrm{d}z' \,\mathrm{d}a' \,\mathrm{d}t \\
&= \sum_{j\neq i} q_{ij} \int_s^T e^{-(q_i+\delta)(t-s)} \mathrm{d}t = \sum_{j\neq i} \frac{q_{ij}}{q_i + \delta} \int_s^T (q_i + \delta) e^{-(q_i+\delta)(t-s)} \mathrm{d}t \\
&= \sum_{j\neq i} \frac{q_{ij}}{q_i + \delta} \left(1 - e^{-(q_i+\delta)(T-s)}\right) < 1.
\end{aligned}$$

Then,

$$\rho := \max_{i\in\mathcal{M}} \rho(i) < 1$$

which yields the inequality $||F(H)|| \leq \rho ||H||$, as desired. □

Parts (ii) and (iii) of Theorem 2.4 are immediate from Proposition 4.1.

Corollary 4.2. *If $H_0, H \in \mathcal{S}$ and H solves the equation,*

$$H(s, z, a, i) = F(H)(s, z, a, i) + H_0(s, z, a, i) \tag{4.1}$$

then H is the only solution.

Proof. Since F is a contraction so is the translation mapping $F(\cdot) + H_0$. Henceforth, $F(\cdot) + H_0$ has a fixed point thanks to the Banach Fixed-Point Theorem. This in turn implies the uniqueness. □

Corollary 4.3. *The sequence $\{H_n\}_{n=0}^\infty$, with,*

$$H_{n+1}(s, z, a, i) = F(H_n)(s, z, a, i) + H_0(s, z, a, i),$$

converges to the fixed-point H with geometric rate of converge,

$$\rho = \max_{i\in\mathcal{M}} \sum_{j\neq i} \frac{q_{ij}}{q_i + \delta} \left(1 - e^{-(q_i+\delta)(T-s)}\right) < 1.$$

Proof. Thanks to Corollary 4.2, $\{H_n\}_{n=0}^\infty$ converges to H in the supremum norm. We have that $H_{n+1} - H = F(H_n) - F(H) = F(H_n - H)$. Then using the fact that F is a contraction,

$$||H_{n+1} - H|| \leq \rho ||H_n - H||$$

and

$$||H_{n+1} - H_n|| \leq \rho^n ||H_1 - H_0||$$

where ρ is defined in Proposition 4.1. □

4.2. Floating-strike case

In this subsection, we show Theorem 2.5.

Consider the functions:

$$g(s,z,a,i)=e^{-z}C(s,e^z,a,i), \qquad g^0(s,z,a,i)=e^{-z}C^0(s,e^z,a,i).$$

Observe that (3.1) can be written as:

$$\begin{aligned} g(s,z,a,i) = e^{-q_i(T-s)}g^0(s,z,a,i) + \int_s^T q_i e^{-q_i(t-s)}\mathbb{E}_{s,x,a,i} \\ \left[e^{-r(i)(t-s)}e^{Z_t}g(t,z+Z_t,A_t,Y_t)\mid \tau=t\right]\mathrm{d}t. \end{aligned} \tag{4.2}$$

Moreover,

$$\begin{aligned} &\mathbb{E}_{s,x,a,i}\left[e^{-r(i)(t-s)}e^{Z_t}g(t,z+Z_t,A_t,Y_t)\mid \tau=t\right] \\ &\quad=\sum_{j\neq i}\frac{q_{ij}}{q_i}\int_a^\infty\int_{-\infty}^\infty e^{-r(i)(t-s)}e^{z'}g(t,z+z',a',j)\,\psi(z',a')\,\mathrm{d}z'\,\mathrm{d}a' \end{aligned}$$

where ψ is the density in Proposition 3.2, and the second term on the right-hand side of equation (4.2) then reads:

$$\sum_{j\neq i}q_{ij}\int_s^T e^{-[q_i+r(i)](t-s)}\int_a^\infty\int_{-\infty}^\infty e^{z'}g(t,z+z',a',j)\,\psi(z',a')\,\mathrm{d}z'\,\mathrm{d}a'\,\mathrm{d}t.$$

This is the mapping F as defined in (2.4) and we can further write:

$$g(s,z,a,i)=F(g)(s,z,a,i)+e^{-q_i(T-s)}g^0(s,z,a,i). \tag{4.3}$$

4.3. Fixed-strike case

In this subsection, we show Theorem 2.6.

For $s\le t_0$ (starting and forward-starting options), consider the functions:

$$h(s,z,a,i)=e^{-z}C_K(s,e^z,a,i), \qquad h^0(s,z,a,i)=e^{-z}C_K^0(s,e^z,a,i).$$

In this case, similar in structure to the floating-strike, we obtain the equation,

$$h(s,z,a,i)=F(h)(s,z,a,i)+e^{-q_i(T-s)}h^0(s,z,a,i), \qquad s\le t_0. \tag{4.4}$$

For $s>t_0$ (in-progress options), consider the functions:

$$\begin{aligned} \tilde h(s,z,a,i)&=e^{-z}\left(C_K(s,e^z,a,i)-\frac{a}{T-t_0}\right), \\ \tilde h^0(s,z,a,i)&=e^{-z}\left(C_K^0(s,e^z,a,i)-\frac{a}{T-t_0}\right), \end{aligned}$$

so that (3.2) can be written as

$$\begin{aligned}\tilde{h}(s,x,a,i)+\frac{ae^{-z}}{T-t_0} &= e^{-q_i(T-s)}\left(\tilde{h}^0(s,x,a,i)+\frac{ae^{-z}}{T-t_0}\right)\\ &\quad+\int_s^T q_i e^{-q_i(t-s)}\mathbb{E}_{s,x,a,i}\\ &\quad\times\left[e^{-r(i)(t-s)}\left(e^{Z_t}\tilde{h}(t,z+Z_t,A_t,Y_t)+\frac{ae^{-z}}{T-t_0}\right)\mid \tau=t\right]\mathrm{d}t.\end{aligned}$$

After some algebraic manipulation, we obtain the equation, for $s>t_0$,

$$\tilde{h}(s,z,a,i)=F(\tilde{h})(s,z,a,i)+e^{-q_i(T-s)}\tilde{h}^0(s,z,a,i)+\tilde{h}^1(s,z,a,i),$$

where the extra term is $\tilde{h}^1$ is in (2.6).

4.4. Iteration

Theorem 2.4 provides an iterative method to approximate the Asian option functions. For instance, we can approximate,

$$g(s,z,a,i)=e^{-z}C(s,e^z,a,i)$$

with $z=\ln(x)$ by a fixed small error, say $\epsilon>0$:

$$\begin{aligned}g_0(s,z,a,i)&=e^{-(q_i(T-s)+z)}\,C^0(s,e^z,a,i),\\ g_{n+1}(s,z,a,i)&=F(g_n)(s,z,a,i)+g_0(s,z,a,i),\qquad n\geq 0\\ \text{If}\quad ||g_{n+1}&-g_n||<\epsilon,\quad \text{stop.}\end{aligned}$$

Observe that the larger the dividend rate δ, the faster the convergence. This can be implied from the expression for $\rho(i)$ in equation (2.5).

While the algorithm is theoretically appealing and provides an alternative to solving a certain system of PDEs as it is usual in option pricing, we should mention that in order to approximate the function g (and then C), it is necessary to compute first the initial function C^0 for the iteration. A good estimation of C^0 is important to avoid amplifying the error in the iteration. In this regards, it is known that the so-called Hartman-Watson density appearing in the definition of the density ψ is indeed difficult to implement. To analyze the effect of the error incurred by such approximation, suppose that the initial function for the iteration is, say $\tilde{g}_0\in\mathcal{S}$. Then the mapping $F(\cdot)+\tilde{g}_0$ is also a contraction with the same rate of convergence ρ. Moreover, the fixed-point theorem implies that the sequence $\{\tilde{g}_n\}_{n\geq 0}$ defined by:

$$\tilde{g}_{n+1}:=F(\tilde{g}_n)+\tilde{g}_0,\qquad n\geq 0$$

converges to a fixed-point, say $\tilde{g}$, which solves the equation:

$$\tilde{g}=F(\tilde{g})+\tilde{g}_0.$$

Hence, we can check that,

$$||\tilde{g}-g||\leq\frac{1}{1-\rho}||\tilde{g}_0-g_0||.$$

In other words, the accuracy of the algorithm depends proportionally on the error incurred at the initial step.

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Appendix A: Proofs

Proof of Lemma 2.1. Define the probability measure P^* equivalent to P via the Radon Nikodym derivative,

$$\frac{\mathrm{d}P^*}{\mathrm{d}P}\ |_{\mathcal{F}_T} = \mathcal{E}_T$$

where,

$$\mathcal{E}_t := \exp\left(\int_0^t \sigma(Y_u)\mathrm{d}B_u - \frac{1}{2}\int_0^t \sigma^2(Y_u)\mathrm{d}u\right).$$

The call option satisfies:

$$\begin{aligned}\frac{C(s,x,a,i)}{x} &= \mathbb{E}_{s,x,a,i}\left[\frac{e^{-\int_s^T r(Y_u)\mathrm{d}u}X_T}{x}\left(1-\frac{1}{T-t_0}\frac{A_T}{X_T}\right)^+\right]\\ &= \mathbb{E}_{s,x,a,i}\left[\mathcal{E}_T\mathcal{E}_s^{-1}\, e^{-\delta(T-s)}\left(1-\frac{1}{T-t_0}\frac{A_T}{X_T}\right)^+\right]\\ &= \mathbb{E}^*_{s,x,a,i}\left[e^{-\delta(T-s)}\left(1-\frac{1}{T-t_0}\frac{A_T}{X_T}\right)^+\right] \le 1.\end{aligned}$$

where the expectation $\mathbb{E}^*$ is with respect to P^*. The result is now clear. □

Proof of Lemma 2.2. Following up the proof of Lemma 2.1, we have that,

$$C^0(s,x,a,i) = \mathbb{E}^*_{s,x,a,i}\left[e^{-\delta(T-s)}\left(x-\frac{x}{T-t_0}\frac{A_T}{X_T}\right)^+ \mid Y_t = i, \forall t\in[s,T]\right],$$

and $\hat{B}_u = B_u - \int_0^u \sigma(Y_s)\mathrm{d}s$ is a Brownian motion under P^*. Here,

$$x\frac{A_T}{X_T} = \frac{x}{X_T}\left(a+\int_s^T X_u\,\mathrm{d}u\right).$$

The process $(B^*_u)_{s\le u\le T}$, defined by $B^*_u := B^*_s + \hat{B}_{T+s-u} - \hat{B}_T$ with B^*_s a constant, is also a Brownian motion under P^* starting at B^*_s. Now, conditional on $X_s = x$, $A_s = a$, and $Y_t = i$ for all $t\in[s,T]$,

$$\begin{aligned}\frac{x}{X_T} &= \exp\left(\sigma(i)(\hat{B}_s-\hat{B}_T) + \left[r(i)-\delta+\frac{\sigma^2(i)}{2}\right](s-T)\right)\\ &\overset{\text{law}}{=} \exp\left(\sigma(i)(B^*_T-B^*_s) + \left[\delta-r(i)-\frac{\sigma^2(i)}{2}\right](T-s)\right)\end{aligned}$$

and

$$x\int_s^T \frac{X_u}{X_T} = \int_s^T x\exp\left(\sigma(i)(\hat{B}_u - \hat{B}_T) + \left[r(i) - \delta + \frac{\sigma^2(i)}{2}\right](u-T)\right)\mathrm{d}u$$
$$\overset{\text{law}}{=} \int_s^T x\exp\left(\sigma(i)(B^*_{T+s-u} - B^*_s) + \left[\delta - r(i) - \frac{1}{2}\sigma^2(i)\right](T-u)\right)\mathrm{d}u$$
$$= \int_s^T x\exp\left(\sigma(i)(B^*_w - B^*_s) + \left[\delta - r(i) - \frac{1}{2}\sigma^2(i)\right](w-s)\right)\mathrm{d}w$$

where the third equality is obtained after the change of variable $w = T+s-u$. Therefore, $C^0(s,x,a,i)$ is given by:

$$\mathbb{E}^*_{s,x,a,i}\left[e^{-\delta(T-s)}\left(x - \frac{a}{x(T-t_0)}X^*_T - \frac{T-s}{T-t_0}\frac{1}{T-s}\int_s^T X^*_u\mathrm{d}u\right)^+\right]$$

where the underlying process X^* follows:

$$\mathrm{d}X^*_t = X^*_t[(\delta - r(i))\mathrm{d}t + \sigma(i)\mathrm{d}B^*_t], \quad X^*_s = x, \qquad t \geq s.$$

Defining the parameters $\lambda = \frac{a}{x(T-t_0)}$ and $\beta = \frac{T-s}{T-t_0}$ the proof is complete. □

Proof of Lemma 2.3. Part (i). Let $s \leq t_0$. Then

$$C_K(s,x,0,i) \leq \mathbb{E}_{s,x,0,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\frac{A_T}{T-t_0}\right]$$
$$\leq \frac{1}{T-t_0}\int_{t_0}^T \mathbb{E}_{s,x,0,i}\left[e^{-\int_s^t r(Y_u)\mathrm{d}u}X_t\right]\mathrm{d}t$$
$$\leq \frac{x}{T-t_0}\int_{t_0}^T e^{-\delta(t-s)}\mathrm{d}t \leq x.$$

Part(ii). Let $s > t_0$. Then

$$C_K(s,x,a,i) \leq \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^T r(Y_u)\mathrm{d}u}\left(\frac{a + \int_s^T X_t\mathrm{d}t}{T-t_0}\right)\right]$$
$$\leq \frac{a}{T-t_0} + \left(\frac{T-s}{T-t_0}\right)\frac{1}{T-s}\int_s^T \mathbb{E}_{s,x,a,i}\left[e^{-\int_s^t r(Y_u)\mathrm{d}u}X_t\right]\mathrm{d}t$$
$$\leq \frac{a}{T-t_0} + \left(\frac{T-s}{T-t_0}\right)x \leq \frac{a}{T-t_0} + x.$$

□

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