J. Fixed Point Theory Appl. (2018) 20:85
https://doi.org/10.1007/s11784-018-0568-8
Published online May 5, 2018

Journal of Fixed Point Theory and Applications

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On fixed points of a linear operator of polynomial form of order 3

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Abstract. Let X be a complex linear space, endowed with an extended (that is, admitting infinite values) norm. We prove a fixed point theorem for operators of the form $p_3\mathcal{L}^3 + p_2\mathcal{L}^2 + p_1\mathcal{L}$, where $\mathcal{L} : X \to X$ is linear and p_1, p_2, p_3 are fixed scalars. That result has been motivated by some issues arising in Ulam stability. One of the tools is the Diaz-Margolis fixed point alternative.

Mathematics Subject Classification. 34K20, 39B82, 47A63, 54H25.

Keywords. Fixed point, operator of polynomial form, linear operator, ulam stability.

1. Introduction

In this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := (0, \infty)$, and $\mathbb{R}_0^+ := [0, \infty)$. We prove a fixed point theorem motivated by some issues arising in Ulam stability. In this way, we obtain an extension of the classical Diaz–Margolis fixed point alternative [15].

Let us recall that the notion of Hyers–Ulam stability originated from the response of Hyers [17] to the interesting question of Ulam concerning approximate homomorphisms of groups. Later, that notion has been generalized in several various directions, which by now are very often collectively called the *Ulam type stability*. Numerous papers on this subject have been published so far and we refer to [1–3, 6, 13, 18–20, 25] for more details, some discussions, recent results and further references. Let us also mention here that the problem of stability of functional equations is connected to the notions of shadowing (see [16, 26, 27]) and the theory of perturbation (see [11, 23]).

Various definitions of that type of stability are possible for particular equations (see, e.g., [1, 18, 25]), but roughly speaking, the following one describes our considerations to some extent: given a metric space (Y, d), a set $S \neq \emptyset$, nonempty classes of functions $\mathcal{D}_0 \subset \mathcal{D} \subset Y^S$ and $\mathcal{E} \subset (\mathbb{R}_0^+)^S$, and operators $\mathcal{T} : \mathcal{D} \to Y^S$ and $\mathcal{S} : \mathcal{E} \to (\mathbb{R}_0^+)^S$, we say that the equation

$$\mathcal{T}(\psi) = \psi$$

is $\mathcal{S}$–stable in $\mathcal{D}_0$ provided for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{E}$ with

$$d\big(\mathcal{T}(\psi)(t), \psi(t)\big) \le \delta(t), \qquad t \in S,$$

there is a solution $\phi \in \mathcal{D}$ of the equation, such that

$$d\big(\phi(t), \psi(t)\big) \le (\mathcal{S}\delta)(t), \qquad t \in S,$$

where A^B denotes the family of all functions mapping a set B into a set A.

There are several papers showing how to deal with the problem of stability of various linear equations of higher orders (see, e.g., [7,8,12,21,22,24,28–30]) of the form:

$$\sum_{i=0}^{m} b_i \mathcal{L}^i \phi = H, \tag{1}$$

where $b_0, \ldots, b_m$ are scalars, H is a given function, $\mathcal{L}$ is a suitable (e.g., difference, differential, functional, integral) operator acting on suitable space of functions ϕ, $\mathcal{L}^0\phi \equiv \phi$ and $\mathcal{L}^i = \mathcal{L} \circ \mathcal{L}^{i-1}$ for $i \in \mathbb{N}$.

It seems that the most general result of this type, in the form of a fixed point theorem, has been proved in [30] (see also [9]), with suitable examples of applications to stability of differential and functional equations. Moreover, in [7], a result has been given, which is much weaker (because only for $m = 2$), but provides estimations of different type than in [30], that in some significant situations are better.

In this paper, we use a somewhat similar approach as in [7], to obtain analogous results, for a particular form of (1) with $m = 3$, that is, for the equation:

$$p_3\mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi = \psi \tag{2}$$

with unknown $\psi \in X$, where $p_1, p_2, p_3 \in \mathbb{C}$ (complex numbers), $p_3 \neq 0$, and X is a complex linear space, endowed with an extended norm (see the next section for a description) that is complete. Namely, we prove a fixed point theorem, which is the main result of this paper and corresponds to the results in [3–5,7,10,14,30]. We also describe some applications of it to the Ulam stability.

As an auxiliary tool we need the following Diaz-Margolis fixed point alternative (see [15]).

Theorem 1. *Let (X, d) be a complete extended metric space and $\mathcal{T} : X \to X$ be a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists $k \in \mathbb{N}$ and $x \in X$, such that $d(\mathcal{T}^k x, \mathcal{T}^{k-1}x) < \infty$, then:*

(a) *The sequence $\{\mathcal{T}^n x\}$ converges to a fixed point x^* of $\mathcal{T}$.*

(b) *x^* is the unique fixed point of $\mathcal{T}$ in*

$$X^* = \{y \in X : d(\mathcal{T}^k x, y) < \infty\}.$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1-L} d(\mathcal{T}y, y).$$

2. Fixed point theorem

Let us recall that a pair $(X, \|\cdot\|)$ is an extended complex normed space if X is a complex linear space and $\|\cdot\|$ is a function mapping X into $[0,\infty]$ (i.e., $\|\cdot\|$ may take the value $+\infty$), such that, for every $\alpha \in \mathbb{C}$ and $x, y \in X$ with $\|x\|, \|y\| \in [0,\infty)$:

$$\|x+y\| \leq \|x\| + \|y\|, \qquad \|\alpha x\| = |\alpha|\,\|x\|,$$

and the equality $\|x\| = 0$ means that x is the zero vector.

In what follows we assume that X is an extended complex Banach space, i.e., an extended complex normed space in which every Cauchy sequence is convergent (in X).

Remark 1. Let Y be a complex Banach space, S be a nonempty set and Y^S denote the family of all functions mapping S into Y. Clearly, Y^S is a linear space over $\mathbb{C}$ with the operations given by the usual formulas:

$$(f+h)(x) := f(x) + h(x), \qquad (\alpha f)(x) := \alpha f(x), \qquad x \in X,$$

for all $f, h \in X^S$, $\alpha \in \mathbb{C}$. Define an extended norm in Y^S by

$$\|f\| = \sup_{s\in S} \|f(s)\|, \qquad f \in Y^S.$$

Then, Y^S (endowed with that extended norm) is a good natural example of such extended Banach space.

Let $\mathcal{L} : X \to X$ be an additive operator, that is

$$\mathcal{L}(f+g) = \mathcal{L}f + \mathcal{L}g, \qquad f, g \in X.$$

Define operator $\mathcal{P} : X \to X$ by

$$\mathcal{P}\psi := p_3\mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi, \qquad \psi \in X. \tag{3}$$

Let $a_1, a_2, a_3 \in \mathbb{C}$ be the roots of the characteristic polynomial of the equation:

$$p_3\mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi = \psi, \tag{4}$$

that is, of

$$P(x) = p_3x^3 + p_2x^2 + p_1x - 1, \qquad x \in \mathbb{C}.$$

Then, $a_i \neq 0$ for $i \in \{1,2,3\}$ and

$$p_3 = \frac{1}{a_1a_2a_3}, \qquad -p_2 = \frac{1}{a_1a_2} + \frac{1}{a_1a_3} + \frac{1}{a_2a_3}, \qquad p_1 = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}.$$

We assume in addition that

$$a_i \neq a_j, \qquad i, j \in \{1,2,3\}, i \neq j,$$

and $\mathcal{L}$ satisfies the Lipschitz condition

$$\|\mathcal{L}f - \mathcal{L}g\| \ \leq \ L\|f-g\|, \qquad f, g \in,$$

with some positive constant L, such that

$$L < \min\{|a_1|, |a_2|, |a_3|\}. \tag{5}$$

Now, we are in a position to prove the main result.

Theorem 2. *For every $\varphi \in X$ with*

$$\varepsilon := \left\|\mathcal{P}\varphi - \varphi\right\| < \infty, \tag{6}$$

$\mathcal{P}$ has a unique fixed point $\psi \in X$, such that

$$\|\varphi - \psi\| < \infty;$$

moreover

$$\|\varphi - \psi\| \le C\varepsilon,$$

where

$$C = \left(\frac{1}{|a_2 - a_1|\,|a_3 - a_1|\,(|a_1| - L)} + \frac{1}{|a_1 - a_2|\,|a_3 - a_2|(|a_2| - L)} + \frac{1}{|a_1 - a_3|\,|a_2 - a_3|\,(|a_3| - L)}\right)|a_1||a_2||a_3|. \tag{7}$$

Proof. Take $\varphi \in X$, such that (6) holds. Define operators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : X \to X$ by

$$\mathcal{T}_j f := \frac{1}{a_j}\mathcal{L}f, \qquad f \in X, j = 1, 2, 3,$$

and write

$$h_1 := \frac{1}{a_2 a_3}\mathcal{L}^2\varphi - \left(\frac{1}{a_2} + \frac{1}{a_3}\right)\mathcal{L}\varphi + \varphi,$$
$$h_2 := \frac{1}{a_1 a_3}\mathcal{L}^2\varphi - \left(\frac{1}{a_1} + \frac{1}{a_3}\right)\mathcal{L}\varphi + \varphi,$$
$$h_3 := \frac{1}{a_1 a_2}\mathcal{L}^2\varphi - \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\mathcal{L}\varphi + \varphi.$$

Then

$$\|\mathcal{T}_j f - \mathcal{T}_j g\| \le \frac{L}{|a_j|}\|f - g\|, \qquad f, g \in X, j = 1, 2, 3.$$

Hence, the operator $\mathcal{T}_j$ is strictly contractive, since $0 < L/|a_j| < 1$ for $j = 1, 2, 3$.

Next

$$\|\mathcal{T}_j h_j - h_j\| = \left\|p_3\mathcal{L}^3\varphi + p_2\mathcal{L}^2\varphi + p_1\mathcal{L}\varphi - \varphi\right\| \le \varepsilon, \qquad j = 1, 2, 3.$$

Therefore, according to Theorem 1, for each $j \in \{1, 2, 3\}$, the sequence $\{\mathcal{T}_j^n h_j\}$ converges to a fixed point F_j of $\mathcal{T}_j$ and

$$\|h_j - F_j\| \le \frac{|a_j|}{|a_j| - L}\,\varepsilon.$$

Moreover, $\mathcal{L}F_j = a_j F_j$, which means that F_j is an eigenvector of $\mathcal{L}$ for $j = 1, 2, 3$.

Let us note that

$$\varphi = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3,$$

where

$$\alpha_1 = \frac{a_2 a_3}{(a_2 - a_1)(a_3 - a_1)}, \qquad \alpha_2 = \frac{a_1 a_3}{(a_1 - a_2)(a_3 - a_2)},$$
$$\alpha_3 = \frac{a_1 a_2}{(a_1 - a_3)(a_2 - a_3)}.$$

Put

$$\psi := \alpha_1 F_1 + \alpha_2 F_2 + \alpha_3 F_3.$$

Since F_i is an eigenvector of $\mathcal{L}$, we have

$$p_3 \mathcal{L}^3 F_i + p_2 \mathcal{L}^2 F_i + p_1 \mathcal{L} F_i - F_i = (p_3 a_i^3 + p_2 a_i^2 + p_1 a_i - 1) F_i = 0 \qquad (8)$$

for $i = 1, 2, 3$. Therefore, ψ, as a linear combination of F_1, F_2 and F_3, fulfils

$$\mathcal{P}\psi = p_3 \mathcal{L}^3 \psi + p_2 \mathcal{L}^2 \psi + p_1 \mathcal{L}\psi = \psi.$$

Moreover

$$\begin{aligned} \|\varphi - \psi\| \leq \left|\frac{a_2 a_3}{(a_2 - a_1)(a_3 - a_1)}\right| \|h_1 - F_1\| \\ + \left|\frac{a_1 a_3}{(a_1 - a_2)(a_3 - a_2)}\right| \|h_2 - F_2\| \\ + \left|\frac{a_1 a_2}{(a_1 - a_3)(a_2 - a_3)}\right| \|h_3 - F_3\|, \end{aligned}$$

whence

$$\|\varphi - \psi\| \leq C\varepsilon.$$

To prove the uniqueness, suppose that ψ_1 and ψ_2 are fixed points of $\mathcal{P}$ (that is solutions of Eq. (4)), such that $\|\psi_1 - \varphi\| < \infty$ and $\|\psi_2 - \varphi\| < \infty$. Then, $\|\psi_1 - \psi_2\|$ is finite. Let

$$G_i := \mathcal{L}^2 \psi_i - (a_1 + a_2)\mathcal{L}\psi_i + a_1 a_2 \psi_i, \qquad i \in \{1, 2\}.$$

Fix an $i \in \{1, 2\}$. Then

$$\mathcal{L}G_i = \mathcal{L}^3 \psi_i - (a_1 + a_2)\mathcal{L}^2 \psi_i + a_1 a_2 \mathcal{L}\psi_i.$$

Since ψ_i satisfies (4), we have

$$\begin{aligned} \mathcal{L}G_i = (a_1 + a_2 + a_3)\mathcal{L}^2 \psi_i - (a_1 a_2 + a_1 a_3 + a_2 a_3)\mathcal{L}\psi_i + a_1 a_2 a_3 \psi_i \\ -(a_1 + a_2)\mathcal{L}^2 \psi_i + a_1 a_2 \mathcal{L}\psi_i. \end{aligned}$$

Consequently

$$\mathcal{L}G_i = a_3 \mathcal{L}^2 \psi_i - (a_1 a_3 + a_2 a_3)\mathcal{L}\psi_i + a_1 a_2 a_3 \psi_i,$$

whence $\mathcal{L}G_i = a_3 G_i$.

Since $\mathcal{L}$ is linear, we have

$$G_1 - G_2 = \mathcal{L}^2(\psi_1 - \psi_2) - (a_1 + a_2)\mathcal{L}(\psi_1 - \psi_2) + a_1 a_2 (\psi_1 - \psi_2).$$

Hence

$$\|G_1 - G_2\| \leq (L^2 + |a_1 + a_2| L + |a_1 a_2|)\|\psi_1 - \psi_2\|,$$

which means that $\|G_1 - G_2\|$ is finite.

Furthermore, $\mathcal{L}G_i = a_3 G_i$ for $i = 1, 2$ and $L < |a_3|$ (see (5)), so we have

$$\|G_1 - G_2\| = \frac{1}{|a_3|}\|\mathcal{L}G_1 - \mathcal{L}G_2\| \leq \frac{L}{|a_3|}\|G_1 - G_2\| < \|G_1 - G_2\|.$$

Hence, $G_1 = G_2$, and by the definition of G_1 and G_2, we obtain

$$\mathcal{L}^2(\psi_1 - \psi_2) - (a_1 + a_2)\mathcal{L}(\psi_1 - \psi_2) + a_1 a_2(\psi_1 - \psi_2) = 0.$$

This means that the function $\Psi := \psi_1 - \psi_2$ satisfies the equation:

$$\frac{1}{a_1 a_2}\mathcal{L}^2\Psi - \Big(\frac{1}{a_1} + \frac{1}{a_2}\Big)\mathcal{L}\Psi + \Psi = 0. \tag{9}$$

Write

$$\Psi_1 := \frac{1}{a_2}\mathcal{L}\Psi - \Psi.$$

Then, in view of (9), it is easy to check that $\mathcal{L}\Psi_1 = a_1\Psi_1$, and analogously as in the case of $G_1 - G_2$, from (5) we deduce that $\Psi_1 = 0$. Consequently, $\mathcal{L}\Psi = a_2\Psi$, whence (5) yields $\Psi = 0$. This implies that $\psi_1 = \psi_2$. □

3. Applications to Ulam stability

Let Y be a complex Banach space, S be a nonempty set, $\mathcal{C}$ be a linear subspace of Y^S and $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ be a linear operator. We assume that Y^E is endowed with the extended supremum norm (cf. Remark 1) and $\mathcal{C}$ is closed with respect to that extended norm.

It is easily seen that Theorem 2 yields the following stability result for Eq. (2).

Theorem 3. *Assume that* $a_i \neq a_j \neq 0$ *for* $i, j \in \{1, 2, 3\}$, $i \neq j$, *and* $\mathcal{L}$ *satisfies the Lipschitz condition:*

$$\|\mathcal{L}f - \mathcal{L}g\| \ \leq \ L\|f - g\|, \qquad f, g \in \mathcal{C}, \tag{10}$$

with a positive constant $L < \min\{|a_1|, |a_2|, |a_3|\}$. *Then, for every function* $\varphi \in \mathcal{C}$ *with*

$$\varepsilon := \Big\| p_3\mathcal{L}^3\varphi + p_2\mathcal{L}^2\varphi + p_1\mathcal{L}\varphi - \varphi \Big\| < \infty,$$

there is a unique solution $\psi \in \mathcal{C}$ *of* (2) *with* $\|\varphi - \psi\| < \infty$*; moreover*

$$\|\varphi - \psi\| \leq C\varepsilon,$$

where C *is given by* (7).

Below, we provide two simple and natural examples of linear operators $\mathcal{L}$ fulfilling (10) with suitable a_1, a_2, a_3.

- Let $\mathcal{C} = Y^S$, $n \in \mathbb{N}$, and $\mathcal{L}f = \sum_{i=1}^n \Psi_i \circ f \circ \xi_i$, where $\Psi_i : Y \to Y$ is linear and bounded and $\xi_i : S \to S$ is fixed for $i = 1, \ldots, n$. Then

  $$\|\mathcal{L}f(x) - \mathcal{L}h(x)\| \leq \sum_{i=1}^n \lambda_i \|f(\xi_i(x)) - h(\xi_i(x))\|$$

  for every $f, h \in X^S$ and $x \in S$, with

  $$\lambda_i := \inf\{L \in \mathbb{R} : \|\Psi_i(w)\| \leq L\|w\| \text{ for } w \in X\}.$$

  Hence (10) is valid for $L := \sum_{i=1}^n \lambda_i$.

- Let $a, b \in \mathbb{R}$, $a < b$, $S = [a, b]$, $\mathcal{C}$ be the family of all continuous functions mapping the interval $[a, b]$ into $\mathbb{C}$, $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathbb{C}$, $\xi_1, \ldots, \xi_n : S \to S$ be continuous and

$$\mathcal{L}f(x) = \sum_{i=1}^{n} \int_a^x A_i f(\xi_i(t))\mathrm{d}t, \qquad f \in \mathcal{C}, x \in S.$$

Then it is easily seen that (10) is fulfilled with $L := (b - a) \sum_{j=1}^{n} |A_j|$.

4. Final comments

It is easy to observe that from [30, Theorem 2.3] we can derive the following analogue of Theorem 2:

Theorem 4. *Let $a_1, a_2, a_3 \in \mathbb{C}$ be the roots of the characteristic polynomial of Eq. (4) with $p_3 = 1$, $\mathcal{C}$ be as in the previous section, $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ be a linear operator and*

$$\mathcal{P}_0\psi := \mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi, \qquad \psi \in \mathcal{C}.$$

If (10) holds with a positive constant $L < \min\{|a_1|, |a_2|, |a_3|\}$ and $\varphi \in X$ satisfies

$$\varepsilon := \|\mathcal{P}_0 g - g\| < \infty, \tag{11}$$

then $\mathcal{P}_0$ has a unique fixed point $\psi \in \mathcal{C}$, such that

$$\|\varphi - \psi\| \le C_0\varepsilon,$$

where

$$C_0 = \frac{1}{(|a_1| - L)(|a_2| - L)(|a_3| - L)}. \tag{12}$$

Note that, in the situation considered in Theorem 4, we have $a_1a_2a_3 = p_3 = 1$, whence (7) takes the form:

$$C = \left(\frac{1}{|a_2 - a_1|\,|a_3 - a_1|\,(|a_1| - L)} + \frac{1}{|a_1 - a_2|\,|a_3 - a_2|(|a_2| - L)} + \frac{1}{|a_1 - a_3|\,|a_2 - a_3|\,(|a_3| - L)}\right). \tag{13}$$

Clearly, $C = \rho C_0$, where

$$\rho := \frac{(|a_2| - L)(|a_3| - L)}{|a_2 - a_1|\,|a_3 - a_1|} + \frac{(|a_1| - L)(|a_3| - L)}{|a_1 - a_2|\,|a_3 - a_2|} + \frac{(|a_1| - L)(|a_2| - L)}{|a_1 - a_3|\,|a_2 - a_3|}.$$

Hence, $C < C_0$ if and only if $\rho < 1$.

Let $a_1 = -a_2 = \sqrt{2}/2 + \sqrt{2}/2i$ and $a_3 = i$. Then $a_1a_2a_3 = p_3 = 1$ and $|a_i| = 1$ for $i = 1, 2, 3$. Clearly, the closer L is to 1, the smaller is ρ.

Certainly, if $p_3 \neq 1$, then the restriction that $L < 1$ is not necessary in Theorem 2; actually, L can be arbitrarily large provided that $L < \min_{i=1,2,3} |a_i|$.

Note yet that the assumption that

$$p_3 = \frac{1}{a_1a_2a_3} \neq 0$$

is important in the proof of Theorem 2. This means that we cannot take $p_3 = 0$ in Theorem 2 and deduce in this way similar results for the operator $\mathcal{P}_2 : X \to X$, given by:

$$\mathcal{P}_2\psi := p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi, \qquad \psi \in X. \tag{14}$$

However, an analogous outcome can be easily derived from a somewhat involved [7, Theorem 2.1] and has the following form (a_1 and a_2 are the roots of the polynomial $P(x) = p_2x^2 + p_1x - 1$).

Theorem 5. *Let $\mathcal{C}$ be as in the previous section, $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ be a linear operator and*

$$\mathcal{P}_2\psi := p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi, \qquad \psi \in \mathcal{C}.$$

Assume that $a_1a_2 \neq 0$ and there is a positive constant $L < \min\{|a_1|, |a_2|\}$, such that

$$\|\mathcal{L}f - \mathcal{L}h\| \leq L\|f - h\|, \qquad f, h \in \mathcal{C}. \tag{15}$$

Then, for every $g \in \mathcal{C}$ with

$$\varepsilon := \|\mathcal{P}_2 g - g\| < \infty, \tag{16}$$

there exists a unique fixed point ψ of $\mathcal{P}_2$ with

$$\|g - \psi\| < \infty; \tag{17}$$

moreover

$$\|g - F\| \leq \frac{|a_1a_2|\varepsilon}{|a_2 - a_1|}\left(\frac{1}{|a_1| - L} + \frac{1}{|a_2| - L}\right). \tag{18}$$

In connection with this observation, there arises a natural question if analogous results can be obtained, for $n > 3$, also for operators $\mathcal{P}_n : X \to X$, of the form:

$$\mathcal{P}_n\psi := \sum_{i=1}^{n} p_i\mathcal{L}^i\psi, \qquad \psi \in X. \tag{19}$$

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