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Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces

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Abstract. In signal processing and image reconstruction, the split feasibility problem (SFP) has been now investigated extensively because of its applications. A classical way to solve the SFP is to use Byrne's CQ-algorithm. However, this method requires the computation of the norm of the bounded linear operator or the matrix norm in a finite-dimensional space. In this work, we aim to propose an iterative scheme for solving the SFP in the framework of Banach spaces. We also introduce a new way to select the step-size which ensures the convergence of the sequences generated by our scheme. We finally provide examples including its numerical experiments to illustrate the convergence behavior. The main results are new and complements many recent results in the literature.

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Keywords. Split feasibility problem, strong convergence, self-adaptive method, optimization, Banach space.

1. Introduction

Let E and F be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A which is defined by

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E, \bar{y} \in F^*.$$

The split feasibility problem (SFP) is to find a point $x \in C$ such that $Ax \in Q$. We denote by $\Omega = C \cap A^{-1}(Q) = \{y \in C : Ay \in Q\}$ the solution set of SFP. Then we have that Ω is a closed and convex subset of E.

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [15] for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms and some interesting results have been studied in order to solve it (see, for example [3–5,13,22,24–26,33])).

In Hilbert spaces, a classical way to solve the SFP is to employ the CQ-algorithm which was introduced by Byrne [12], which is defined in the following manner:

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad n \geq 1, \tag{1.1}$$

where the step-size $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively. We note that this algorithm is found to be a gradient-projection method in convex minimization as a spacial case. It was proved that $\{x_n\}$ generated by (1.1) converges weakly to a solution of SFP.

However, it is noted that the operator norm $\|A\|$ may not be calculated easily in general. To overcome this difficulty, López et al. [22] suggested the following self-adaptive method, which permits step-size μ_n being selected self-adaptively in such a way:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{1.2}$$

where $\rho_n \in (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of SFP.

Also, employing the idea of Halpern's iteration, López et al. [22] proposed the following iteration method:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C(x_n - \mu_n \nabla f(x_n)), \quad n \geq 1, \tag{1.3}$$

where u is fixed in C, $\{\alpha_n\} \subset [0, 1]$ and the step-size μ_n is chosen as above. It was shown that $\{x_n\}$ defined by (1.3) converges strongly to a solution of SFP provided $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Subsequently, there have been many modifications of the CQ-algorithm and the self-adaptive method established in the recent years (see also [37,38]).

For solving the SFP, in the framework of p-uniformly convex and uniformly smooth real Banach spaces, Schöpfer [29] proposed the following algorithm: $x_1 \in E$ and

$$x_{n+1} = \Pi_C J_E^*[J_E(x_n) - \mu_n A^* J_F(Ax_n - P_Q(Ax_n))], \quad n \geq 1, \tag{1.4}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly, the above algorithm covers the CQ-algorithm as a special case. It was proved that the sequence $\{x_n\}$ defined by (1.4) converges weakly to a solution of SFP provided the duality mapping J is weak-to-weak continuous and $\mu_n \in \left(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}\right)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and C_q is the uniform smoothness coefficient of E . See some modifications in [30,31].

In this work, motivated by the previous works, we introduce a Halpern-type iteration process and prove its strong convergence of the sequence generated by our scheme for solving the SFP without prior knowledge of the operator norm in the framework of Banach spaces. Numerical experiments are included to illustrate the convergence behavior. Our main results complement the results of López et al. [22] (from Hilbert spaces to Banach spaces) and Schöpfer [29]. Moreover, our results improve many other results in the literature. We note that the obtained results seem to be new in this direction.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$, and E^* denotes the Banach dual of E endowed with the dual norm $\|\cdot\|_*$. We write $\langle x, j\rangle$ for the value of a functional j in E^* at x in E. As usual, $x_\nu \to x$ and $x_\nu \rightharpoonup x$ stand for the norm and weak convergence of a net $\{x_\nu\}$ to x in E, respectively. The *modulus of convexity* $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \geq \epsilon\right\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$ and p-uniformly convex if there is a $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0,2]$. The *modulus of smoothness* $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

Then E is called *uniformly smooth* if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$ and q-uniformly smooth if there is a $C_q > 0$ such that $\rho_E(\tau) \leq C_q\tau^q$ for any $\tau > 0$. Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is known (see, for example, [1,18]) that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth. Furthermore, Hilbert spaces, $L_p(or\ \ l_p)$ spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are uniformly smooth while

$$L_p(or\ \ell_p) \text{ or } W_m^p \text{ is } \begin{cases} p - \text{uniformly smooth if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth if } p \geq 2. \end{cases}$$

A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a gauge if

$$\varphi(0) = 0, \quad \lim_{t\to+\infty}\varphi(t) = +\infty.$$

The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{j \in E^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \varphi(\|x\|)\}, \quad x \in E,$$

is called the duality mapping with gauge φ. When $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is the normalized duality mapping. In the case $\varphi(t) = t^{p-1}$ where $p > 1$, the duality mapping $J_\varphi = J_p$ is called the generalized duality mapping and it is defined by

$$J_p(x) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|^{p-1}\}, \quad x \in E.$$

Example 2.1. Let $x = (x_1, x_2, \ldots) \in \ell_p$ $(1 < p < \infty)$. Then the generalized duality mapping J_p in ℓ_p is given by

$$J_p(x) = (|x_1|^{p-1}\mathrm{sgn}(x_1), |x_2|^{p-1}\mathrm{sgn}(x_2), \ldots).$$

Example 2.2. Let $f \in L_p([\alpha, \beta])$ $(1 < p < \infty)$. Then the generalized duality mapping J_p is given by

$$J_p(f)(t) = |f(t)|^{p-1}\mathrm{sgn}(f(t)).$$

For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s$$

is a continuous convex strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi^{'}(t) = \varphi(t)$ and $\lim_{t\to+\infty} \Phi(t)/t = +\infty$. When E is uniformly smooth, the duality mapping J_φ on E is norm to norm uniformly continuous on bounded subsets of E (see [1,18]).

We know the following inequality which was proved by Xu [34].

Lemma 2.3. [34] *Let $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

Let C be a nonempty, closed and convex subset of E. The metric projection $P_C : E \to C$ is defined by

$$P_C x = \mathrm{argmin}_{y\in C} \frac{1}{2}\|x - y\|^2, \quad x \in E.$$

It has been employed successfully in optimization, optimal control, approximation theory, and fixed point theory. In the framework of Hilbert spaces, the metric projection P_C is nonexpansive (i.e., $\|P_C x - P_C y\| \leq \|x - y\|$ for all x, y in H). However, we note that this is no longer true in the framework of Banach spaces. Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E, and let $x \in E$ and $z \in C$. Then $z = P_C x$ if and only if $\langle z - y, J(x - z)\rangle \geq 0$ for all $y \in C$; see [32].

We next recall the definition of Bregman distance studied in [6]. Let E be a real smooth Banach space. The Bregman distance $D_\varphi(x, y)$ between x and y in E is defined by

$$D_\varphi(x, y) = \Phi(\|x\|) - \Phi(\|y\|) - \langle x - y, J_\varphi(y)\rangle.$$

We note that $D_\varphi(x, y) \geq 0$ and $D_\varphi(x, y) = 0$ if and only if $x = y$ (see [21]). It is easily seen by definition that

$$D_\varphi(x, y) + D_\varphi(y, z) - D_\varphi(x, z) = \langle x - y, J_\varphi(z) - J_\varphi(y)\rangle \tag{2.1}$$

and

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle x - y, J_\varphi(x) - J_\varphi(y)\rangle \tag{2.2}$$

for all $x, y, z \in E$. In the case $\varphi(t) = t^{p-1}$ where $p > 1$, the distance $D_\varphi = D_p$ is called the p-Lyapunov functional studied in [7] and it is given by

$$D_p(x, y) = \frac{\|x\|^p}{p} + \frac{\|y\|^p}{q} - \langle x, J_p(y) \rangle, \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\phi(x, y) := 2D_2(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

is the Lyapunov functional. See [8,9,16]. Let E be a strictly convex, smooth and reflexive Banach space. Following [2], we make use of the function $V_p : E \times E^* \to [0, +\infty)$, which is defined by

$$V_p(x, \bar{x}) = \frac{1}{p}\|x\|^p - \langle x, \bar{x} \rangle + \frac{1}{q}\|\bar{x}\|^q, \quad \forall x \in E, \bar{x} \in E^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then V_p is nonnegative and

$$V_p(x, \bar{x}) = D_p(x, J_p^{-1}(\bar{x}))$$

for all $x \in E$ and $\bar{x} \in E^*$. For a proper, lower semicontinuous and convex function $f : E \to (-\infty, \infty]$, the subdifferential ∂f of f at $x \in E$ is defined by

$$\partial f(x) = \{\bar{x} \in E^* : f(x) + \langle y - x, \bar{x} \rangle \le f(y) \quad \forall y \in E\}.$$

We see that, for each $x \in E$, the mapping g defined by $g(\bar{x}) = V_p(x, \bar{x})$ for all $\bar{x} \in E^*$ is a continuous and convex function from E^* into $\mathbb{R}$. So, by the subdifferential of g, we obtain the following inequality:

$$V_p(x, \bar{x}) + \langle \bar{y}, J_p^{-1}(\bar{x}) - x \rangle \le V_p(x, \bar{x} + \bar{y}) \tag{2.4}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see also [20]). Indeed, we have

$$\begin{aligned} \partial g(\bar{x}) &= \partial \left(-\langle x, \cdot \rangle + \frac{1}{q}\| \cdot \|^q \right)(\bar{x}) \\ &= -x + J_p^{-1}(\bar{x}) \end{aligned}$$

for all $\bar{x} \in E^*$. So we obtain

$$g(\bar{x}) + \langle J_p^{-1}(\bar{x}) - x, \bar{y} \rangle \le g(\bar{x} + \bar{y}),$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ which consequently implies (2.4).

Proposition 2.4. [10,21] *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $D_\varphi(x_n, y_n) \to 0$. If $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Proposition 2.5. [11,21] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Let $x \in E$. Then there exists a unique element x_0 in C such that*

$$D_\varphi(x_0, x) = \inf \{D_\varphi(z, x) : z \in C\}.$$

In this case, we denote the generalized projection from E onto C by $\Pi_C^{\varphi}(x) = x_0$. When $\varphi(t) = t$, we have Π_C^{φ} coincides with the generalized projection studied in [2]. Let $p > 1$ and $\varphi(t) = t^{p-1}$. Then Π_C^{φ} becomes the generalized projection with respect to p and is also denoted by Π_C.

We also know the following results.

Proposition 2.6. [21] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Let $x_0 \in C$ and $x \in E$. Then the following assertions are equivalent:*

(a) $x_0 = \Pi_C^{\varphi}(x)$;
(b) $\langle z - x_0, J_{\varphi}(x_0) - J_{\varphi}(x)\rangle \geq 0, \ \forall z \in C$.

Moreover, we have

$$D_{\varphi}(y, \Pi_C^{\varphi}(x)) + D_{\varphi}(\Pi_C^{\varphi}(x), x) \leq D_{\varphi}(y, x), \quad \forall y \in C.$$

We also need the following tools in analysis which will be used in the sequel.

Lemma 2.7. [23] *Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ which satisfies $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : s_k < s_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ *and* $\tau(n) \to \infty$;
(ii) $s_{\tau(n)} \leq s_{\tau(n)+1}$ *and* $s_n \leq s_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.8. [35] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \geq 1,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

3. Main results

In this section, we prove strong convergence of the sequence generated by our scheme for solving the split feasibility problem in Banach spaces. Throughout this paper, let $1 < q \leq 2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and denote by J_X^p and $J_{X^*}^q$ the duality mappings of a smooth Banach space X and its dual space, respectively.

Employing the method of proof given by Xu in [36], we prove the following fixed point formulation of SFP in a reflexive, strictly convex and smooth Banach space.

Lemma 3.1. *Let E and F be two reflexive, strictly convex and smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be* *the adjoint of A. Let $x^* \in E$. Then x^* solves the SFP (i.e., $x^* \in C \cap A^{-1}(Q)$) if and only if x^* solves the fixed point equation*

$$x^* = \Pi_C J^q_{E^*}\Big[J^p_E(x^*) - \gamma A^* J^p_F(Ax^* - P_Q(Ax^*))\Big]. \tag{3.1}$$

Proof. Suppose x^* solves the SFP. We show that x^* solves (3.1). Now, x^* solves SFP implies that $x^* \in C$ and $Ax^* \in Q$. Therefore,

$$Ax^* = P_Q(Ax^*) \Rightarrow Ax^* - P_Q(Ax^*) = 0.$$

Thus,

$$J^p_F(Ax^* - P_Q(Ax^*) = 0$$

and this implies

$$\gamma A^* J^p_F(Ax^* - P_Q(Ax^*)) = 0.$$

So

$$J^q_{E^*}\Big[J^p_E(x^*) - \gamma A^* J^p_F(Ax^* - P_Q(Ax^*))\Big] = J^q_{E^*}\Big(J^p_E(x^*)\Big) = x^*.$$

Hence,

$$\Pi_C J^q_{E^*}\Big[J^p_E(x^*) - \gamma A^* J^p_F(Ax^* - P_Q(Ax^*))\Big] = \Pi_C x^* = x^*.$$

Therefore, x^* solves (3.1).

Conversely, assume that x^* solves the fixed point equation (3.1). We next show that $x^* \in C,\ Ax^* \in Q$. Now, if

$$x^* = \Pi_C J^q_{E^*}\Big[J^p_E(x^*) - \gamma A^* J^p_F(Ax^* - P_Q(Ax^*))\Big],$$

then by Proposition 2.6 (b) we have

$$\langle J^p_E(x^*) - \gamma A^* J^p_F(Ax^* - P_Q(Ax^*)) - J^p_E(x^*), z - x^*\rangle \leq 0, \quad \forall z \in C.$$

That is,

$$\langle \gamma A^* J^p_F(Ax^* - P_Q(Ax^*)), z - x^*\rangle \geq 0, \quad \forall z \in C.$$

Hence,

$$\langle J^p_F(Ax^* - P_Q(Ax^*)), Ax^* - Az\rangle \leq 0, \quad \forall z \in C. \tag{3.2}$$

On the other hand, we have from the characterization of metric projection P_Q that

$$\langle J^p_F(Ax^* - P_Q(Ax^*)), v - Ax^*\rangle \leq 0, \quad \forall v \in Q. \tag{3.3}$$

Adding up (3.2) and (3.3), we obtain

$$\langle J^p_F(Ax^* - P_Q(Ax^*)), v - Az\rangle \leq 0, \quad \forall v \in Q, z \in C.$$

Putting $z = x^* \in C$ and $v = P_Q(Ax^*) \in Q$ gives us $Ax^* = P_Q(Ax^*) \in Q$. This completes the proof. □

Remark 3.2. Our Lemma 3.1 extends the fixed point equivalence of SFP given by Xu in [36] from real Hilbert spaces to reflexive, strictly convex and smooth Banach spaces. This fixed point formulation of SFP allows us to construct a fixed point iteration method to solve the SFP in Banach spaces and this iterative method is given below in (3.4).

Theorem 3.3. *Let E be a p-uniformly convex and uniformly smooth Banach space and F a reflexive, strictly convex and smooth Banach space. Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that $\Omega = C \cap A^{-1}(Q) \neq \emptyset$. Define a sequence $\{x_n\}$ by $u, x_1 \in E$ and*

$$x_{n+1} = \Pi_C J^q_{E^*}\Big[\alpha_n J^p_E(u) + (1-\alpha_n)\Big(J^p_E(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\nabla f(x_n)\Big)\Big], \quad n \geq 1, \tag{3.4}$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$. If $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\rho_n\} \subset (0,\infty)$ satisfies

$$\inf_n \rho_n\Big(pq - C_q\rho_n^{q-1}\Big) > 0, \tag{3.5}$$

then $x_n \to \Pi_\Omega u$.

Proof. We note that $\nabla f(x) = A^* J^p_F(I - P_Q)Ax$ for all $x \in E$ (see Proposition 5.7 in [19]). Set

$$y_n = J^p_E(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\nabla f(x_n)$$

for all $n \in \mathbb{N}$. We see that $(p-1)q = p$. Then, by Lemma 2.3, we have

$$\begin{aligned}
\|y_n\|^q &= \Big\|J^p_E(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\nabla f(x_n)\Big\|^q \\
&\leq \|x_n\|^p - q\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x_n, \nabla f(x_n)\rangle + C_q\rho_n^q \frac{f^{(p-1)q}(x_n)}{\|\nabla f(x_n)\|^{pq}}\|\nabla f(x_n)\|^q \\
&= \|x_n\|^p - q\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x_n, \nabla f(x_n)\rangle + C_q\rho_n^q \frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}.
\end{aligned} \tag{3.6}$$

From Proposition 2.6 and (3.6), it follows that, for each $x^* \in \Omega$,

$$\begin{aligned}
D_p(x^*, x_{n+1}) &\leq D_p(x^*, J^q_{E^*}(\alpha_n J^p_E(u) + (1-\alpha_n)y_n)) \\
&= \frac{\|x^*\|^p}{p} + \frac{1}{q}\|\alpha_n J^p_E(u) + (1-\alpha_n)y_n\|^q - \alpha_n\langle x^*, J^p_E(u)\rangle \\
&\quad - (1-\alpha_n)\langle x^*, y_n\rangle \\
&\leq \frac{\|x^*\|^p}{p} + \frac{1}{q}(\alpha_n\|u\|^p + (1-\alpha_n)\|y_n\|^q) - \alpha_n\langle x^*, J^p_E u\rangle \\
&\quad - (1-\alpha_n)\langle x^*, J^p_E(x_n)\rangle + (1-\alpha_n)\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x^*, \nabla f(x_n)\rangle \\
&= \alpha_n\Big(\frac{\|x^*\|^p}{p} + \frac{\|u\|^p}{q} - \langle x^*, J^p_E u\rangle\Big) \\
&\quad + (1-\alpha_n)\Big(\frac{\|x^*\|^p}{p} + \frac{\|y_n\|^q}{q} - \langle x^*, J^p_E(x_n)\rangle\Big) \\
&\quad + (1-\alpha_n)\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x^*, \nabla f(x_n)\rangle \\
&\leq \alpha_n D_p(x^*, u) \\
&\quad + (1-\alpha_n)\Big(\frac{\|x^*\|^p}{p} + \frac{1}{q}\big(\|x_n\|^p - q\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x_n, \nabla f(x_n)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_q\rho_n^q \frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}\|)\Big)\\
&\quad - (1-\alpha_n)\langle x^*, J_E^p(x_n)\rangle + (1-\alpha_n)\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x^*, \nabla f(x_n)\rangle\\
&= \alpha_n D_p(x^*,u) + (1-\alpha_n)\Big(\frac{\|x^*\|^p}{p} + \frac{\|x_n\|^p}{q} - \langle x^*, J_E^p(x_n)\rangle\Big)\\
&\quad + (1-\alpha_n)\Big(\frac{C_q\rho_n^q}{q}\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x^* - x_n, \nabla f(x_n)\rangle\Big)\\
&= \alpha_n D_p(x^*,u) + (1-\alpha_n)D_p(x^*,x_n)\\
&\quad + (1-\alpha_n)\Big(\frac{C_q\rho_n^q}{q}\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\langle x^* - x_n, \nabla f(x_n)\rangle\Big).
\end{aligned} \tag{3.7}
$$

On the other hand, we see that

$$
\begin{aligned}
\langle \nabla f(x_n), x^* - x_n\rangle &= \langle A^* J_E^p(I-P_Q)Ax_n, x^* - x_n\rangle\\
&= \langle J_E^p(I-P_Q)Ax_n, Ax^* - Ax_n\rangle\\
&= \langle J_E^p(I-P_Q)Ax_n, P_Q Ax_n - Ax_n\rangle\\
&\quad + \langle J_E^p(I-P_Q)Ax_n, Ax^* - P_Q Ax_n\rangle\\
&\le -\|(I-P_Q)Ax_n\|^p = -pf(x_n).
\end{aligned} \tag{3.8}
$$

Using (3.7) and (3.8), we obtain

$$
\begin{aligned}
D_p(x^*, x_{n+1}) &\le \alpha_n D_p(x^*,u) + (1-\alpha_n)D_p(x^*,x_n)\\
&\quad + (1-\alpha_n)\Big(\frac{C_q\rho_n^q}{q} - \rho_n p\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p},
\end{aligned}
$$

which implies, by (3.5)

$$
D_p(x^*, x_{n+1}) \le \alpha_n D_p(x^*,u) + (1-\alpha_n)D_p(x^*,x_n).
$$

Hence, by induction, $\{D_p(x^*,x_n)\}$ is bounded. So we can conclude that $\{x_n\}$ is bounded. Set $v_n = J_{E^*}^q\Big[\alpha_n J_E^p(u) + (1-\alpha_n)\Big(J_E^p(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\nabla f(x_n)\Big)\Big]$ for all $n \in \mathbb{N}$. Using Proposition 2.6 and (2.4), we next consider the following estimation:

$$
\begin{aligned}
D_p(x^*, x_{n+1}) &= D_p(x^*, \Pi_C v_n) \le D_p(x^*, v_n) - D_p(v_n, \Pi_C v_n)\\
&= D_p(x^*, J_{E^*}^q(\alpha_n J_E^p(u) + (1-\alpha_n)y_n)) - D_p(v_n, \Pi_C v_n)\\
&= V_p(x^*, \alpha_n J_E^p(u) + (1-\alpha_n)y_n) - D_p(v_n, \Pi_C v_n)\\
&\le V_p(x^*, \alpha_n J_E^p(u) + (1-\alpha_n)y_n - \alpha_n(J_E^p(u) - J_E^p(x^*))\\
&\quad + \alpha_n\langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle - D_p(v_n, \Pi_C v_n)\\
&= V_p(x^*, \alpha_n J_E^p(x^*) + (1-\alpha_n)y_n)\\
&\quad + \alpha_n\langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle - D_p(v_n, \Pi_C v_n)\\
&\le (1-\alpha_n)V_p(x^*, y_n) + \alpha_n\langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle\\
&\quad - D_p(v_n, \Pi_C v_n)\\
&= (1-\alpha_n)D_p(x^*, J_{E^*}^q(y_n)) + \alpha_n\langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle\\
&\quad - D_p(v_n, \Pi_C v_n)
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha_n)\Big(\frac{\|x^*\|^p}{p} + \frac{\|y_n\|^q}{q} - \langle x^*, y_n\rangle\Big) \\
&\quad + \alpha_n \langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle - D_p(v_n, \Pi_C v_n) \\
&\le (1-\alpha_n) D_p(x^*, x_n) + (1-\alpha_n)\rho_n \Big(\frac{C_q \rho_n^{q-1}}{q} - p\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \\
&\quad + \alpha_n \langle J_E^p(u) - J_E^p(x^*), v_n - x^*\rangle - D_p(v_n, \Pi_C v_n). \qquad (3.9)
\end{aligned}
$$

Let $s_n = D_p(\Pi_\Omega u, x_n)$ for all $n \in \mathbb{N}$. Then, by (3.9), we have

$$
\begin{aligned}
s_{n+1} &\le (1-\alpha_n)s_n + (1-\alpha_n)\rho_n \Big(\frac{C_q \rho_n^{q-1}}{q} - p\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \\
&\quad + \alpha_n \langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_n - \Pi_\Omega u\rangle - D_p(v_n, \Pi_C v_n). \qquad (3.10)
\end{aligned}
$$

Case 1 If $\{s_n\}$ is decreasing, then $(1-\alpha_n)\rho_n\big(\frac{C_q\rho_n^{q-1}}{q} - p\big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \to 0$ and $D_p(v_n, \Pi_C v_n) \to 0$. It follows that $f(x_n) \to 0$ by (3.5). Hence $\|Ax_n - P_Q Ax_n\| \to 0$ and $\|v_n - \Pi_C v_n\| \to 0$ by Lemma 2.4. We also see that

$$
\begin{aligned}
\|J_E^p(v_n) - J_E^p(x_n)\| &\le \alpha_n \|J_E^p(u) - J_E^p(x_n)\| \\
&\quad + (1-\alpha_n)\Big\|\rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\nabla f(x_n)\Big\| \\
&\to 0.
\end{aligned}
$$

Since $J_{E^*}^q$ is norm to norm uniformly continuous on bounded subsets of E^*, $\|v_n - x_n\| \to 0$ as $n \to \infty$. Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \rightharpoonup z$ in $\omega_w(v_n)$. Also, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in \omega_w(x_n)$. From (2.2) we obtain

$$
\begin{aligned}
D_p(z, \Pi_C z) &\le \langle J_E^p(z) - J_E^p(\Pi_C z), z - \Pi_C z\rangle \\
&= \langle J_E^p(z) - J_E^p(\Pi_C z), z - v_{n_i}\rangle + \langle J_E^p(z) - J_E^p(\Pi_C z), v_{n_i} - \Pi_C v_{n_i}\rangle \\
&\quad + \langle J_E^p(z) - J_E^p(\Pi_C z), \Pi_C v_{n_i} - \Pi_C z\rangle \\
&\le \langle J_E^p(z) - J_E^p(\Pi_C z), z - v_{n_i}\rangle + \langle J_E^p(z) - J_E^p(\Pi_C z), v_{n_i} - \Pi_C v_{n_i}\rangle \\
&\to 0. \qquad (3.11)
\end{aligned}
$$

It follows that $z \in C$. Since $x_{n_i} \rightharpoonup z$, $Ax_{n_i} \rightharpoonup Az$ and $\|Ax_{n_i} - P_Q Ax_{n_i}\| \to 0$ as $i \to \infty$, we have

$$
\begin{aligned}
\|Az - P_Q Az\|^p &= \langle J_F^p(Az - P_Q Az), Az - P_Q Az\rangle \\
&= \langle J_F^p(Az - P_Q Az), Az - Ax_{n_i}\rangle \\
&\quad + \langle J_F^p(Az - P_Q Az), Ax_{n_i} - P_Q Ax_{n_i}\rangle \\
&\quad + \langle J_F^p(Az - P_Q Az), P_Q Ax_{n_i} - P_Q Az\rangle \\
&= \langle J_F^p(Az - P_Q Az), Az - Ax_{n_i}\rangle \\
&\quad + \langle J_F^p(Az - P_Q Az), Ax_{n_i} - P_Q Ax_{n_i}\rangle \\
&\to 0,
\end{aligned}
$$

as $i \to \infty$. Then we obtain $Az \in Q$ and therefore $z \in \Omega = C \cap A^{-1}Q$. We next show that

$$
\limsup_{n\to\infty} \langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_n - \Pi_\Omega(u)\rangle \le 0.
$$

To this end, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_n - \Pi_\Omega(u)\rangle = \lim_{i\to\infty}\langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_{n_i} - \Pi_\Omega u\rangle.$$

Since $v_{n_i} \rightharpoonup z \in \Omega$, it follows that

$$\limsup_{n\to\infty}\langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_n - \Pi_\Omega u\rangle \le 0.$$

Using Lemma 2.8, we conclude that $s_n \to 0$, that is, $D_p(\Pi_\Omega u, x_n) \to 0$ as $n \to \infty$. So, by Lemma 2.4, we obtain $x_n \to \Pi_\Omega u$ as $n \to \infty$.
Case 2 Assume that $\{s_n\}$ is not monotonically decreasing and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N} : k \le n, s_k \le s_{k+1}\}.$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $0 \le s_{\tau(n)} \le s_{\tau(n)+1}, \forall n \ge n_0$. So from (3.10) we can show that $\|Ax_{\tau(n)} - P_Q Ax_{\tau(n)}\| \to 0$ and $\|v_{\tau(n)} - \Pi_C v_{\tau(n)}\| \to 0$ as $n \to \infty$. By the similar argument as above in Case 1, we can also show that $\|v_{\tau(n)} - x_{\tau(n)}\| \to 0$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_{\tau(n)} - \Pi_\Omega u\rangle \le 0.$$

Also, from (3.10), we see that

$$s_{\tau(n)} \le \langle J_E^p(u) - J_E^p(\Pi_\Omega u), v_{\tau(n)} - \Pi_\Omega u\rangle.$$

It follows that $\limsup_{n\to\infty} s_{\tau(n)} \le 0$ and thus $\lim_{n\to\infty} s_{\tau(n)} = 0$. We next show that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. To show this, it suffices to prove that $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\begin{aligned}\|x_{\tau(n)+1} - x_{\tau(n)}\| &\le \|x_{\tau(n)+1} - v_{\tau(n)}\| + \|v_{\tau(n)} - x_{\tau(n)}\|\\ &= \|\Pi_C v_{\tau(n)} - v_{\tau(n)}\| + \|v_{\tau(n)} - x_{\tau(n)}\|\\ &\to 0.\end{aligned}$$

From (2.1), it follows that

$$\begin{aligned}&D_p(\Pi_\Omega u, x_{\tau(n)+1}) + D_p(x_{\tau(n)+1}, x_{\tau(n)}) - D_p(\Pi_\Omega u, x_{\tau(n)})\\ &\quad = \langle \Pi_\Omega u - x_{\tau(n)+1}, J_E^p(x_{\tau(n)}) - J_E^p(x_{\tau(n)+1})\rangle.\end{aligned}$$

Hence

$$\begin{aligned}s_{\tau(n)+1} = D_p(\Pi_\Omega u, x_{\tau(n)+1}) &\le D_p(\Pi_\Omega u, x_{\tau(n)})\\ &\quad + \langle \Pi_\Omega u - x_{\tau(n)+1}, J_E^p(x_{\tau(n)}) - J_E^p(x_{\tau(n)+1})\rangle\\ &\to 0.\end{aligned}$$

Thus, by Lemma 2.7, we obtain $s_n \le s_{\tau(n)+1}$, which implies that $\lim_{n\to\infty} s_n = 0$. This shows that $x_n \to \Pi_\Omega u$ as $n \to \infty$. We thus complete the proof. □

We consequently obtain the following result in Hilbert spaces which was studied by Yao et al. [37].

Theorem 3.4. (Yao et al. [37]) *Let H_1 and H_2 be Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Suppose that $\Omega = C \cap A^{-1}(Q) \neq \emptyset$. Define a sequence $\{x_n\}$ by $u, x_1 \in H_1$ and*

$$x_{n+1} = P_C\Big[\alpha_n u + (1-\alpha_n)\Big(x_n - \rho_n \frac{f(x_n)}{\|\nabla f(x_n)\|^2}\nabla f(x_n)\Big)\Big], \quad n \geq 1, \tag{3.12}$$

where $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$. If $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\rho_n\} \subset (0, \infty)$ satisfies

$$\inf_n \rho_n(4 - \rho_n) > 0, \tag{3.13}$$

then $x_n \to P_\Omega u$.

4. Applications

In this section, we apply our result on SFP to split equality problem (SEP) introduced by Moudafi [27,28] in p-uniformly convex real Banach spaces which are also uniformly smooth. As far as we know, this is the first time SEP is being studied in higher Banach spaces outside real Hilbert spaces which has been studied by numerous authors in the literature.

Our interest here is to convert an SEP to SFP in p-uniformly convex real Banach spaces which are also uniformly smooth. To do this, we need the following important lemma.

Lemma 4.1. [17,34]

(i) *A real Banach space X is p-uniformly convex if and only if there exists $c_1 > 0$ such that*

$$\|x + y\|^p \geq \|x\|^p + p\langle y, J_X^p(x)\rangle + c_1\|y\|^p, \quad \forall x, y \in X.$$

(ii) *A real Banach space X is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that for all $x, y \in B_r := \{x \in X : \|x\| \leq r\}$, we have

$$\langle x - y, J_X^p(x) - J_X^p(y)\rangle \leq g(\|x - y\|).$$

We now give the following lemma which is an analogue of Lemma 4.1 in product spaces. Furthermore, this lemma will be crucial in our application.

Lemma 4.2. *For $p > 1$, let X and Y be real p-uniformly convex Banach spaces which are also uniformly smooth. Let $E = X \times Y$ with norm*

$$\|z\|_E = (\|u\|_X^p + \|v\|_Y^p)^{\frac{1}{p}}$$

for every arbitrarily $z = (u, v) \in E$. Let $E^* = X^* \times Y^*$ denote the dual space of E. For each $x = (x_1, x_2) \in E$, define the mapping $J_E^p : E \to E^*$ by

$$J_E^p(x) = J_E^p(x_1, x_2) = (J_X^p(x_1), J_Y^p(x_2)),$$

and for arbitrarily $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2)$ *in* E, *the duality pair* $\langle \cdot, \cdot \rangle$ *is given by*

$$\langle z_1, J_E^p(z_2)\rangle = \langle u_1, J_X^p(u_2)\rangle + \langle v_1, J_Y^p(v_2)\rangle.$$

Then we have

(a) J_E^p *is a duality mapping on* E;
(b) E *is* p*-uniformly convex real Banach space which is also uniformly smooth.*

Proof. (a) Observe that J_E^p is single-valued if and only if E is smooth. For arbitrarily $x = (x_1, x_2) \in E$, let $J_E^p(x) = J_E^p(x_1, x_2) = \psi_p$. Then $\psi_p = (J_X^p(x_1), J_Y^p(x_2)) \in E^*$. Observe that for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \|\psi_p\|_{E^*} &= \|(J_X^p(x_1), J_Y^p(x_2))\|^{\frac{1}{q}} \\ &= (\|J_X^p(x_1)\|_{X^*}^q + \|J_Y^p(x_2)\|_{Y^*}^q)^{\frac{1}{q}} \\ &= (\|x_1\|_X^{(p-1)q} + \|x_2\|_Y^{(p-1)q})^{\frac{1}{q}} \\ &= (\|x_1\|_X^p + \|x_2\|_Y^p)^{\frac{p-1}{p}} \\ &= \|x\|_E^{p-1}. \end{aligned}$$

Hence $\|\psi_p\|_{E^*} = \|x\|_E^{p-1}$. Also, we have

$$\begin{aligned} \langle x, \psi_p\rangle &= \langle (x_1, x_2), (J_X^p(x_1), J_Y^p(x_2))\rangle \\ &= \langle (x_1, J_X^p(x_1)), (x_2, J_Y^p(x_2))\rangle \\ &= \|x_1\|_X^p + \|x_2\|_Y^p \\ &= (\|x_1\|_X^p + \|x_2\|_Y^p)^{\frac{1}{p}} (\|x_1\|_X^p + \|x_2\|_Y^p)^{\frac{p-1}{p}} \\ &= \|x\|_{E^*} \cdot \|\psi_p\|_{E^*} \\ &= \|x\|_E^p. \end{aligned}$$

Hence J_E^p is a single-valued normalized duality mapping on E.

(b) Let $x = (x_1, x_2), y = (y_1, y_2) \in E$. Then

$$\begin{aligned} \|x + y\|_E^p &= \|(x_1 + y_1, x_2 + y_2)\|_E^p \\ &= \|x_1 + y_1\|_X^p + \|x_2 + y_2\|_Y^p \\ &\geq \|x_1\|_X^p + \|x_2\|_Y^p + c(\|y_1\|_X^p + \|y_2\|_Y^p) \\ &\quad + p\{\langle y_1, J_X^p(x_1)\rangle + \langle y_2, J_Y^p(x_2)\rangle\} \end{aligned}$$

for some $c > 0$. Hence

$$\|x + y\|_E^p \geq \|x\|_E^p + p\langle y, J_E^p(x)\rangle + c\|y\|_E^p.$$

Therefore, E is p-uniformly convex from Lemma 4.1 (i) . We next show that E is uniformly smooth. Now,

$$\begin{aligned} \langle x - y, J_E^p(x) - J_E^p(y)\rangle &= \langle (x_1 - y_1, x_2 - y_2), (J_E^p(x_1) - J_E^p(y_1), J_E^p(x_2) - J_E^p(y_2))\rangle \\ &= \langle x_1 - y_1, J_E^p(x_1) - J_E^p(y_1)\rangle + \langle x_2 - y_2, J_E^p(x_2) - J_E^p(y_2)\rangle \\ &\leq g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|), \end{aligned}$$

where g_1, g_2 are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1(0) = g_2(0) = 0$. Therefore,

$$\langle x - y, J_E^p(x) - J_E^p(y)\rangle \leq g(\|x - y\|),$$

where $g(\|x - y\|) = g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|)$. Hence the result follows from Lemma 4.1 (ii) that E is uniformly smooth. □

Let E_1, E_2 and E_3 be real p-uniformly convex which are also uniformly smooth Banach spaces. Suppose $C_1 \subseteq E_1$ and $Q_1 \subseteq E_2$ are nonempty closed and convex sets. Let $A : E_1 \to E_2$ and $B : E_2 \to E_3$ be bounded linear operators. The split equality problem (SEP) [27,28] is defined by

$$\text{find} \quad x \in C_1 \quad \text{and} \quad y \in Q_1 \quad \text{such that} \quad Ax = By. \tag{4.1}$$

Our interest here is to transform (4.1) into the SFP. Now suppose $E = E_1 \times E_2$, $F = E_3 \times E_3$, $C = C_1 \times Q_1 \subset E$ and $Q = \{(z, w) \in F : z = w\}$. We know from Lemma 4.2 that E and F are p-uniformly convex real Banach spaces which are also uniformly smooth.

Define an operator $T : E \to F$ by

$$T(x, y) = (Ax, By)$$

for all $(x, y) \in E$. Since if $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, then

$$\begin{aligned}
T(\alpha z_1 + \beta z_2) &= T[(\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)] \\
&= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\
&= (A(\alpha x_1 + \beta x_2), B(\alpha y_1 + \beta y_2)) \\
&= (\alpha A x_1 + \beta A x_2, \alpha B y_1 + \beta B y_2) \\
&= \alpha(Ax_1, By_1) + \beta(Ax_2, By_2) \\
&= \alpha T z_1 + \beta T z_2,
\end{aligned}$$

which shows that T is linear. Also, it is easy to see that T is bounded from the boundedness of A and B. Set $S = \{(x, y) \in C : T(x, y) \in Q\}$. Hence $(x, y) \in E$ solves (4.1) (using Lemma 3.1) if and only if

$$(x, y) = \Pi_C J_{E^*}^q (J_E^p(x, y) + \gamma T^* J_F^p (P_Q - I) T(x, y)),$$

where

$$\begin{aligned}
P_Q(z, w) &= \left(\frac{z + w}{2}, \frac{z + w}{2}\right), (z, w) \in F, \\
T^* J_F^p(z, w) &= (A^* J_{E_3}^p(z), B^* J_{E_3}^p(w)), \\
J_E^p(x, y) &= (J_{E_1}^p(x), J_{E_2}^p(y)) \quad \text{for all} \quad (x, y) \in E,
\end{aligned}$$

and

$$\Pi_C J_{E^*}^q(x, y) = (\Pi_{C_1} J_{E_1^*}^q(x), P_{Q_2} J_{E_2^*}^q(y)) \quad \text{for all} \quad (x, y) \in E^*.$$

Using the fixed point formulation (4.2), we construct an iterative method for solving SEP (4.1) and obtain the following convergence theorem for solving SEP (4.1) by applying the result of Theorem 3.3.

Theorem 4.3. *Let E_1 and E_2 be two real p-uniformly convex which are also uniformly smooth Banach spaces and E_3 a reflexive, strictly convex and smooth Banach space. Suppose $C_1 \subseteq E_1$ and $Q_1 \subseteq E_2$ are nonempty closed and convex sets. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ be the adjoints of A and B respectively. Suppose that Ω denotes the set of solutions of SEP (4.1) and $\Omega \neq \emptyset$. Define a sequence $\{(x_n, y_n)\}$ by $x_1, y_1 \in E_1$ and*

$$\begin{cases} x_{n+1} = \Pi_{C_1} J^q_{E_1{}^*}\Big[\alpha_n J^p_{E_1}(x_1) + (1-\alpha_n)\Big(J^p_{E_1}(x_n) + \frac{\rho_n}{2} A^* J^p_{E_3}(By_n - Ax_n)\Big)\Big], & n \geq 1,\\ y_{n+1} = P_{Q_1} J^q_{E_2{}^*}\Big[\alpha_n J^p_{E_2}(y_1) + (1-\alpha_n)\Big(J^p_{E_2}(y_n) + \frac{\rho_n}{2} B^* J^p_{E_3}(Ax_n - By_n)\Big)\Big], & n \geq 1. \end{cases} \tag{4.2}$$

If $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\rho_n\} \subset (0, \infty)$ satisfies $\inf_n \rho_n\Big(pq - C_q \rho_n^{q-1}\Big) > 0$, then $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*) which simultaneously solves SEP (4.1) and is the nearest point to the initial guess (x_1, y_1).

5. Examples and numerical results

In this section, we present some numerical examples to illustrate the performance of our algorithm. All codes were written in Matlab 2012*b* and run on Hp $i-5$ Dual-Core 8.00 GB (7.78 GB usable) RAM laptop.

Example 5.1. We consider the problem in $(L_2([\alpha, \beta]), ||\cdot||_{L_2})$ and also give numerical examples using Theorem 3.3. Now take

$$C := \{x \in L_2([\alpha, \beta]) : \langle a, x\rangle \leq b\},$$

where $0 \neq a \in L_2([\alpha, \beta])$ and $b \in \mathbb{R}$, then (see [14])

$$\Pi_C(x) = P_C(x) = \begin{cases} \frac{b - \langle a, x\rangle}{||a||^2_{L_2}} a + x, & \langle a, x\rangle > b\\ x, \langle a, x\rangle \leq b. \end{cases}$$

Let

$$Q = \{x \in L_2([\alpha, \beta]) : ||x - d||_{L_2} \leq r\}$$

be a closed ball centered at $d \in L_2([\alpha, \beta])$ with radius $r > 0$, then

$$P_Q(x) = \begin{cases} d + r\frac{x-d}{||x-d||}, x \notin Q\\ x, x \in Q. \end{cases}$$

Define an operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by $Ax(t) = \frac{x(t)}{2},\ t \in [0, 2\pi]$ for all $x \in L_2([0, 2\pi])$. Then it can be easily verified that A is continuous and bounded linear operator.

Now, suppose

$$C = \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} e^t x(t)\mathrm{d}t \leq 1\right\}$$

and

$$Q = \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2 \mathrm{d}t \leq 16\right\}.$$

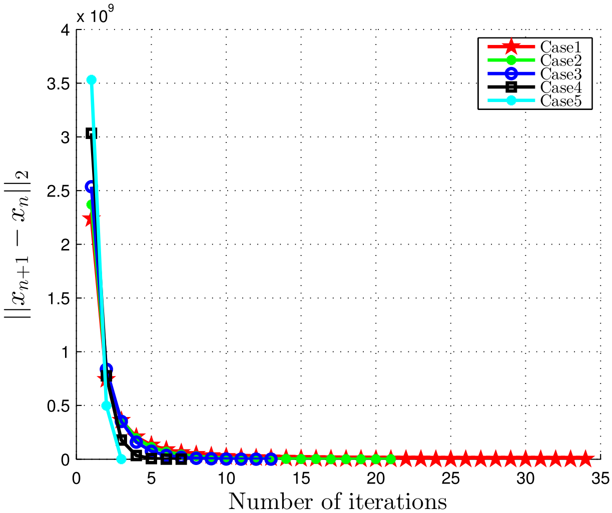

FIGURE 1. Different cases with Choice 1

Let us consider the following problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{5.1}$$

Observe that the set of solutions of problem (5.1) is nonempty (since $x(t) = 0$, *a.e.* is in the set of solutions). Take $\alpha_n = \frac{1}{n+1}$, $\forall n \geq 1$, then our iterative scheme (3.4) becomes

$$x_{n+1} = P_C \left[\frac{1}{n+1}(u) + \left(1 - \frac{1}{n+1}\right)\left(x_n - \rho_n \frac{f(x_n)}{\|\nabla f(x_n)\|^2} \nabla f(x_n)\right)\right], \quad n \geq 1, \tag{5.2}$$

where $f(x_n) = \frac{1}{2}\|Ax_n - P_Q Ax_n\|^2$ for all $n \in \mathbb{N}$.

We now study the effect (in terms of convergence, stability, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, \infty)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases (Figs. 1, 2, 3, 4; Table 1).

Case 1: $\rho_n = \frac{n}{4n+1}$;
Case 2: $\rho_n = \frac{n}{2n+1}$;
Case 3: $\rho_n = \frac{n}{n+1}$;
Case 4: $\rho_n = \frac{2n}{n+1}$;
Case 5: $\rho_n = \frac{3n}{n+1}$.

For each case mentioned above, using stopping criterion $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-4}$, we also consider different choices of x_1 and u as
Choice 1: $x_1 = 2tcos(3t)e^{3t}$ and $3(t^7 - 1)e^{-5t}$;
Choice 2: $x_1 = 2tsin(3t)e^{2t}$ and $t^2 sin(5\pi t)$;
Choice 3: $x_1 = 3t^2 e^{4t-1}$ and $t^3 - t^2 + 4t + 1$;
Choice 4: $x_1 = 2t^3 e^{5t}$ and $t^4 + 3t^2 + 5$.

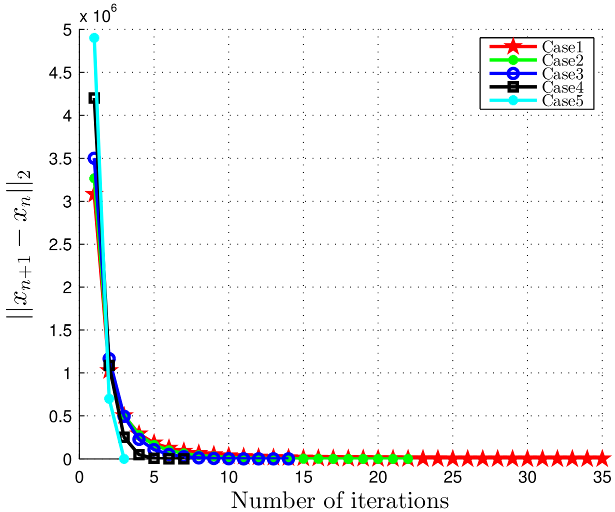

FIGURE 2. Different cases with Choice 2

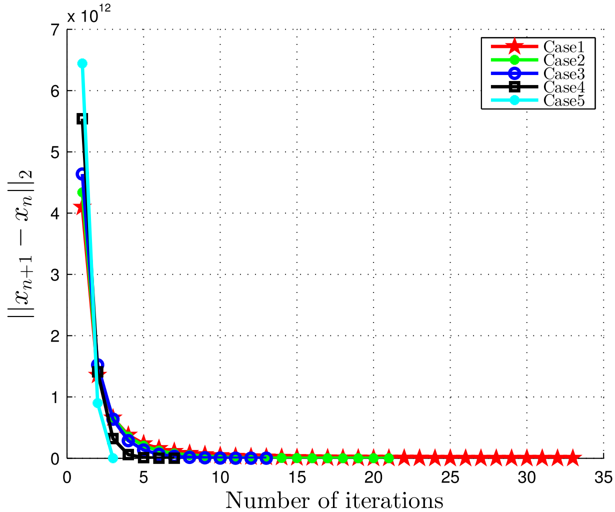

FIGURE 3. Different cases with Choice 3

Remark 5.2. We make the following observations from the numerical results presented above.

1. The numerical results from different Cases and different Choices show that our proposed Algorithm (5.2) is fast, stable, efficient, easy to implement and required small number of iterations.
2. We have that the number of iterations and the cpu run time are decreasing starting from Case 1 to Case 5. However, there is no significant difference in both cpu run time and number of iterations for different

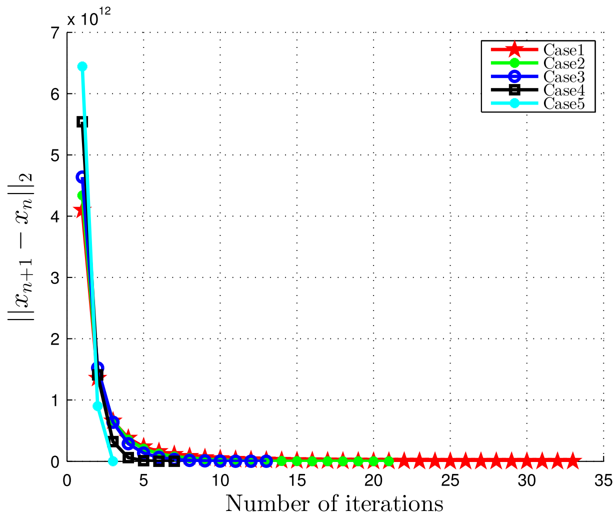

FIGURE 4. Different cases with Choice 4

TABLE 1. Algorithm (5.2) with different cases of ρ_n and different choices of x_1 and u

	Case 1	Case 2	Case 3	Case 4	Case 5
Choice 1					
No. of Iter.	34	21	13	7	3
cpu (time)	5.5602×10^{-3}	3.7686×10^{-3}	2.2286×10^{-3}	1.0621×10^{-3}	3.8957×10^{-4}
Choice 2					
No. of Iter.	35	22	14	7	3
cpu (time)	6.1863×10^{-3}	4.2536×10^{-3}	2.1723×10^{-3}	1.0322×10^{-3}	3.8020×10^{-4}
Choice 3					
No. of Iter.	33	21	13	7	3
cpu (time)	5.7544×10^{-3}	4.2300×10^{-3}	2.1799×10^{-3}	1.0746×10^{-3}	4.1189×10^{-4}
Choice 4					
No. of Iter.	33	21	13	7	3
cpu (time)	5.4273×10^{-3}	3.5084×10^{-3}	2.1014×10^{-3}	1.0581×10^{-3}	3.8690×10^{-4}

Choices of x_1 and u. So, initial guess does not have any significant effect on the convergence of the algorithm.

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