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Hyperstability results for a generalized radical cubic functional equation related to additive mapping in non-Archimedean Banach spaces

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Abstract. The aim of this paper is to introduce and solve the following generalized radical cubic functional equation

$$f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) = \sum_{i=1}^{k} f(x_i), \quad k \in \mathbb{N}_2.$$

We also investigate some hyperstability results for this equation in non-Archimedean Banach spaces.

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1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_m$ the set of natural numbers greater than or equal to m, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ and we write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B. Let us recall (see, for instance, [22]) some basic definitions and facts concerning non-Archimedean normed spaces.

Definition 1.1. By a *non-Archimedean* field, we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$,
(2) $|rs| = |r||s|$,
(3) $|r + s| \leq \max\{|r|, |s|\}$.

The pair $(\mathbb{K}, |\cdot|)$ is called a *valued field.*

In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \le 1$ for $n \in \mathbb{N}_0$. In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_* : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(1) $\|x\|_* = 0$ if and only if $x = 0$,
(2) $\|rx\|_* = |r|\ \|x\|_*\ \ (r \in \mathbb{K}, x \in X)$,
(3) The strong triangle inequality (ultrametric); namely

$$\|x + y\|_* \le \max\left\{\|x\|_*, \|y\|_*\right\}\ \ x, y \in X.$$

Then $(X, \|\cdot\|_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space.*

Definition 1.3. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(1) A sequence$\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* if and only if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;
(2) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x\|_* \le \varepsilon$, for all $n \ge N$. Then, the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $lim_{n\to\infty} x_n = x$;
(3) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space* or an *ultrametric Banach space.*

The stability of functional equations has been a very popular subject of investigations for nearly the last fifty years. There are hundreds of papers and many books published over the last fifty years on this important issue. Its main motivation was given by Ulam [28] in his talk at the University of Wisconsin.

Ulam's Problem 1940:

Let $(G_1, *_1)$ *be a group and let* $(G_2, *_2)$ *be a metric group with a metric* $d(.,.)$*. Given* $\varepsilon > 0$*, does there exist a* $\delta > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$\mathrm{d}\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

for all $x, y \in G_1$*, then there exists a homomorphism* $H : G_1 \to G_2$ *with*

$$\mathrm{d}\big(h(x), H(x)\big) < \varepsilon$$

for all $x \in G_1$*?*

In 1941, Hyers [20] gave the first answer to this question, in the case of Banach space. The next theorem is an example of the most classical results.

Theorem 1.4. [20] *Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then, the limit

$$A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive mapping such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$.

Aoki [5] and Bourgin [7] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [26] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. It was shown by Gajda [17], as well as by Rassias and Šemrl [27] that one cannot prove a stability theorem of the additive equation for a specific function. Găvruţa [18] obtained a generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function.

The hyperstability is the notion which is strictly connected with the well-known issue of Ulam's stability for various equations.

Definition 1.5. Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{X^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.1}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$\mathrm{d}\left(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\right) \leq \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X)$$

fulfills (1.1).

For further information concerning the notion of hyperstability, we refer to the survey paper [14]. Recently, the stability problem of the radical functional equations in various spaces was proved in [1–3,15,16,21,23].

In this paper, we achieve the general solutions of the following generalized radical cubic functional equation:

$$f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) = \sum_{i=1}^{k} f(x_i), \quad k \in \mathbb{N}_2 \tag{1.2}$$

and discuss the Hyers–Ulam stability problem in non-Archimedean Banach spaces using Brzdęk's fixed point result.

2. Fixed point theorem

Brzdęk et al. [8] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers–Ulam stability of some functional equations in non-Archimedean metric spaces. They also obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 2.1. [8] *Let X be a nonempty set, (Y, d) be a complete metric space, and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty} \delta_n = 0.$$

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \le \Lambda\big(\Delta(\xi, \mu)\big)(x), \quad \xi, \mu \in Y^X, \quad x \in X,$$

where $\Delta : Y^X \times Y^X \to \mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi, \mu)(x) := \mathrm{d}\big(\xi(x), \mu(x)\big) \quad \xi, \mu \in Y^X, \quad x \in X.$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\mathrm{d}\Big((\mathcal{T}\varphi)(x), \varphi(x)\Big) \le \varepsilon(x)$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n\varepsilon\big)(x) < \infty$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} \big((\mathcal{T}^n\varphi)\big)(x) \tag{2.1}$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} \big((\mathcal{T}^n\varphi)\big)(x) \tag{2.2}$$

is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(x), \psi(x)\big) \le \varepsilon^*(x) \tag{2.3}$$

for all $x \in X$.

In 2013, Brzdęk [9] gave the fixed point result by applying Theorem 2.1 as follows:

Theorem 2.2. [9] *Let X be a nonempty set, (Y, d) be a complete metric space, $f_1, \ldots, f_r : X \to X$ and $L_1, \ldots, L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$\mathrm{d}\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \le \sum_{i=1}^{r} L_i(x)\mathrm{d}\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big)$$

for all $\xi, \mu \in Y^X$, $x \in X$ *and*

$$\Lambda\delta(x) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big), \quad \delta \in \mathbb{R}_+^X,\ x \in X.$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *such that*

$$\mathrm{d}\Big(\mathcal{T}\varphi(x), \varphi(x)\Big) \leq \varepsilon(x)$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x) < \infty$$

for all $x \in X$*, then the limit* (2.1) *exists for each* $x \in X$*. Moreover, the function* (2.2) *is a fixed point of* $\mathcal{T}$ *with* (2.3) *for all* $x \in X$*.*

Using this theorem, Brzdęk [9] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation. Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers–Ulam stability of a number of functional equations (see, for instance, [4,6,8–10,19,24,25] and references therein).

Thanks to a result due to Brzdęk and Ciepliñski [13, Remark 2], we state an analogue of Theorem 2.2 in non-Archimedean Banach spaces. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 2.3. *Let* X *be a nonempty set,* $\big(Y, \|\cdot\|_*\big)$ *be a non-Archimedean Banach space,* $f_1, \ldots, f_k : X \to X$ *and* $L_1, \ldots, L_k : X \to \mathbb{R}_+$ *be given mappings. Suppose that* $\mathcal{T} : Y^X \to Y^X$ *and* $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ *are two operators satisfying the conditions*

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\big\|_* \leq \max_{1\leq i\leq k} \Big\{L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\Big\|_*\Big\},$$

for all $\xi, \mu \in Y^X$, $x \in X$ *and*

$$\Lambda\delta(x) := \max_{1\leq i\leq k} \Big\{L_i(x)\delta\Big(f_i(x)\Big)\Big\}, \quad \delta \in \mathbb{R}_+^X, \quad x \in X. \tag{2.4}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *such that*

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x)\Big\|_* \leq \varepsilon(x), \quad x \in X, z \in Y$$

and

$$\lim_{n\to\infty} \Lambda^n\varepsilon(x) = 0, \quad x \in X.$$

Then, there exists a unique fixed point $\psi \in Y^X$ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\|_* \leq \sup_{n\in\mathbb{N}_0} \Lambda^n\varepsilon(x), \quad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \quad x \in X.$$

3. Solution of generalized radical cubic functional Eq.(1.2)

In this section, we give the general solution of the functional Eq. (1.2). The following theorem can be derived from the results in [11] (see also [12]).

Theorem 3.1. *Let Y be a linear space. A mapping $f : \mathbb{R} \to Y$ satisfies the functional Eq.* (1.2) *if and only if*

$$f(x) = F(x^3), \quad x \in \mathbb{R}, \tag{3.1}$$

with some additive mapping $F : \mathbb{R} \to Y$.

Proof. Indeed, it is not hard to check without any problem that if $f : \mathbb{R} \to Y$ satisfies (3.1), then it is a solution of (1.2).

On the other hand, if $f : \mathbb{R} \to Y$ is a solution of (1.2), then [11, Theorem 2.1] implies that $f(x) = F_0(x^3)$ for $x \in \mathbb{R}$, where F_0 is a solution to the functional equation

$$F_0\left(\sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} F_0(x_i)$$

for all $x_i \in \mathbb{R}$ with $i = 1, 2, \ldots, k$. It is easily seen that F_0 is additive and this completes the proof. □

4. New hyperstability results for the generalized radical cubic functional Eq. (1.2)

In the following theorem, we use Theorem 2.3 to investigate the Hyers–Ulam stability of the generalized radical cubic functional Eq. (1.2) in non-Archimedean Banach spaces. Hereafter, we assume that $\big(Y, \|\cdot\|_*\big)$ is a non-Archimedean Banach space.

Theorem 4.1. *Let $h_i : \mathbb{R}_0 \to \mathbb{R}_+$ be functions such that*

$$\mathcal{U} := \left\{ n \in \mathbb{N} \colon \alpha_n := \max\left\{ \prod_{i=1}^{k} \lambda_i\big((k-1)n^3+1\big)\ ,\ (k-1)\prod_{i=1}^{k} \lambda_i(n^3) \right\} \right. \\ \left. < \ 1 \right\} \neq \phi \tag{4.1}$$

is an infinite set, where

$$\lambda_i(n) := \inf\left\{ t \in \mathbb{R}_+ \colon h_i(nx^3) \le t\, h_i(x^3), \quad x \in \mathbb{R}_0 \right\}$$

for all $n \in \mathbb{N}$, with $i = 1, 2, \ldots, k$ such that

$$\lim_{n\to\infty} \prod_{i=1}^{k-1} \lambda_i(n) = 0.$$

Assume that $f : \mathbb{R} \to Y$ satisfies $f(0) = 0$ and the inequality

$$\left\| f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k} f(x_i) \right\|_* \le \prod_{i=1}^{k} h_i(x_i^3) \tag{4.2}$$

for all $x_i \in \mathbb{R}_0$ with $i = 1, 2, \ldots, k$, then (1.2) *holds.*

Proof. For $i = 1, 2, \ldots, (k-1)$, replacing x_i with mx and x_k with x, where $x_i \in \mathbb{R}_0$ and $m \in \mathbb{N}$, in (4.2), we get

$$\left\| f\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right) - (k-1)f(mx) - f(x)\right\|_* \le h_k(x^3)\prod_{i=1}^{k-1} h_i(m^3x^3) \tag{4.3}$$

for all $x \in \mathbb{R}_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{\mathbb{R}} \to Y^{\mathbb{R}}$ by

$$\mathcal{T}_m\xi(x) := \xi\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right) - (k-1)\xi(mx), \quad \xi \in Y^{\mathbb{R}}, x \in \mathbb{R}_0.$$

Further, put

$$\varepsilon_m(x) := h_k(x^3)\prod_{i=1}^{k-1} h_i(m^3x^3), \quad x \in \mathbb{R}_0 \tag{4.4}$$

and observe that

$$\varepsilon_m(x) = h_k(x^3)\prod_{i=1}^{k-1} h_i(m^3x^3) \le \prod_{i=1}^{k-1}\lambda_i(m^3)\prod_{i=1}^{k} h_i(x^3), \quad x \in \mathbb{R}_0,\ m \in \mathbb{N}. \tag{4.5}$$

Then, (4.3) takes the form

$$\left\|\mathcal{T}_m f(x) - f(x)\right\|_* \le \varepsilon_m(x), \quad x \in \mathbb{R}_0.$$

Furthermore, for every $x \in \mathbb{R}_0$, $\xi, \mu \in Y^{\mathbb{R}}$, we obtain

$$\begin{aligned}\left\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\right\|_* &= \left\|\xi\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right) - (k-1)\xi(mx)\right.\\ &\qquad \left. - \mu\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right) + (k-1)\mu(mx)\right\|_*\\ &\le \max\left\{\left\|(\xi-\mu)\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right)\right\|_*,\right.\\ &\qquad \left.(k-1)\left\|(\xi-\mu)(mx)\right\|_*\right\}.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ by

$$\begin{aligned}&\Lambda_m\delta(x) := \max\left\{\delta\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right),\ (k-1)\delta(mx)\right\},\\ &\delta \in \mathbb{R}_+^{\mathbb{R}}, x \in \mathbb{R}_0.\end{aligned}$$

For each $m \in \mathbb{N}$, the above operator has the form described in (2.4) with $f_1(x) = \sqrt[3]{\big((k-1)m^3+1\big)x^3}$, $f_2(x) = mx$ and $L_1(x) = 1$, $L_2(x) = k-1$ for all $x \in \mathbb{R}_0$. By the mathematical induction, we will show that for each $x \in \mathbb{R}_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$, we have

$$\Lambda_m^n\varepsilon_m(x) \le \prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^n\prod_{i=1}^{k} h_i(x^3), \tag{4.6}$$

where α_m^n is given by (4.1). From (4.4) and (4.5), we obtain that the inequality (4.6) holds for $n = 0$. Next, we will assume that (4.6) holds for $n = r$, where $r \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\Lambda_m^{r+1}\varepsilon_m(x) &= \Lambda_m\left(\Lambda_m^r\varepsilon_m(x)\right)\\
&= \max\left\{\Lambda_m^r\varepsilon_m\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right),\ (k-1)\Lambda_m^r\varepsilon_m(mx)\right\}\\
&\le \max\left\{\prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^r\prod_{i=1}^{k}h_i\Big(\big((k-1)m^3+1\big)x^3\Big),\right.\\
&\qquad \left.(k-1)\prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^r\prod_{i=1}^{k}h_i(m^3x^3)\right\}\\
&\le \prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^r\prod_{i=1}^{k}h_i(x^3)\max\left\{\prod_{i=1}^{k}\lambda_i\big((k-1)m^3+1\big),\right.\\
&\qquad \left.\times(k-1)\prod_{i=1}^{k}\lambda_i(m^3)\right\}\\
&= \prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^{r+1}\prod_{i=1}^{k}h_i(x^3)
\end{aligned}$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{U}$. This shows that (4.6) holds for $n = r+1$. Now, we can conclude that the inequality (4.6) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\lim_{n\to\infty}\Lambda_m^n\varepsilon_m(x) = 0$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{U}$. Thus, according to Theorem 2.3, for each $m \in \mathcal{U}$, the mapping $Q_m : \mathbb{R} \to Y$, given by $Q_m(x) := \lim_{n\to\infty}\mathcal{T}_m^n f(x)$ for $x \in \mathbb{R}_0$ and $Q_m(0) = 0$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$Q_m(x) = Q_m\left(\sqrt[3]{\big((k-1)m^3+1\big)x^3}\right) - (k-1)Q_m(mx), \quad x \in \mathbb{R}, m \in \mathcal{U}.$$

Moreover,

$$\big\|f(x) - Q_m(x)\big\|_* \le \sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m(x), \quad x \in X, \quad m \in \mathcal{U}.$$

We show that

$$\left\|\mathcal{T}_m^n f\left(\sqrt[3]{\sum_{i=1}^{k}x_i^3}\right) - \sum_{i=1}^{k}\mathcal{T}_m^n f(x_i)\right\|_* \le \alpha_m^n\prod_{i=1}^{k}h_i(x^3) \tag{4.7}$$

for each $n \in \mathbb{N}_0$, $m \in \mathcal{U}$ and $x_1, x_2, \ldots, x_k \in \mathbb{R}_0$. Since the case $n = 0$ is just (4.2), take $r \in \mathbb{N}$ and assume that (4.7) holds for $n = r$ and every $x_i \in \mathbb{R}_0$ and all $m \in \mathcal{U}$. Then, for each $x_i \in \mathbb{R}$ and $m \in \mathcal{U}$, we get

$$\left\|\mathcal{T}_m^{r+1} f\left(\sqrt[3]{\sum_{i=1}^{k}x_i^3}\right) - \sum_{i=1}^{k}\mathcal{T}_m^{r+1} f(x_i)\right\|_*$$

$$
\begin{aligned}
&= \Bigg\| \mathcal{T}_m^r f\left(\sqrt[3]{\Big((k-1)m^3+1\Big)\sum_{i=1}^{k} x_i^3}\right) \\
&\quad -(k-1)\mathcal{T}_m^r f\left(m\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k}\mathcal{T}_m^r f\left(\sqrt[3]{\Big((k-1)m^3+1\Big)x_i^3}\right) \\
&\quad +(k-1)\sum_{i=1}^{k}\mathcal{T}_m^r f(mx_i)\Bigg\|_* \\
&\leq \max\Bigg\{\Bigg\| \mathcal{T}_m^r f\left(\sqrt[3]{\Big((k-1)m^3+1\Big)\sum_{i=1}^{k} x_i^3}\right) \\
&\quad -\sum_{i=1}^{k}\mathcal{T}_m^r f\left(\sqrt[3]{\Big((k-1)m^3+1\Big)x_i^3}\right)\Bigg\|_*, \\
&\quad (k-1)\left\|\mathcal{T}_m^r f\left(m\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k}\mathcal{T}_m^r f(mx_i)\right\|_*\Bigg\} \\
&\leq \max\left\{\alpha_m^r\prod_{i=1}^{k} h_i\Big(((k-1)m^3+1)x^3\Big),\ (k-1)\alpha_m^r\prod_{i=1}^{k} h_i(m^3x^3)\right\} \\
&\leq \alpha_m^r\prod_{i=1}^{k} h_i(x^3)\max\left\{\prod_{i=1}^{k}\lambda_i\Big(((k-1)m^3+1)\Big),\ (k-1)\prod_{i=1}^{k}\lambda_i(m^3)\right\} \\
&= \alpha_m^{r+1}\prod_{i=1}^{k} h_i(x^3).
\end{aligned}
$$

Thus, by induction, we have shown that (4.7) holds for every $x_i \in \mathbb{R}_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (4.7), we obtain the equality

$$
Q_m\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) = \sum_{i=1}^{k} Q_m(x_i), \quad x_i \in \mathbb{R}_0, m \in \mathcal{U}. \tag{4.8}
$$

In this way, for each $m \in \mathcal{U}$, we obtain a function Q_m such that (4.8) holds for $x_1, x_2, \ldots, x_k \in \mathbb{R}$ and

$$
\big\|f(x) - Q_m(x)\big\|_* \leq \sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x) \leq \prod_{i=1}^{k-1}\lambda_i(m^3)\alpha_m^n\prod_{i=1}^{k} h_i(x^3),
$$

for all $x \in X$ and all $m \in \mathcal{U}$. Since

$$
\lim_{m\to\infty}\prod_{i=1}^{k-1}\lambda_i(m) = 0,
$$

it follows, with $m \to \infty$, that f fulfils (1.2). □

According to Theorem 4.1, we derive the following particular case.

Corollary 4.2. *Let Y be a non-Archimedean space, $\theta \geq 0$ and $q_i \in \mathbb{R}$ with $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^{k} q_i < 0$. Suppose that $f : \mathbb{R} \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\left\| f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k} f(x_i) \right\|_* \leq \theta \prod_{i=1}^{k} \left|x_i^3\right|^{q_i}, \quad x_i \in \mathbb{R}_0. \tag{4.9}$$

Then, f satisfies (1.2) on $\mathbb{R}$.

Proof. The proof follows from Theorem 4.1 by defining $h_i : \mathbb{R}\backslash\{0\} \to \mathbb{R}_+$ by $h_i(x_i) = \theta_i \left|x_i\right|^{q_i}$ with $\theta_i \in \mathbb{R}_+$ and $q_i \in \mathbb{R}$ such that $\prod_{i=1}^{k} \theta_i = \theta$ and $\sum_{i=1}^{k} q_i < 0$.

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \lambda_i(n) &= \inf \left\{t \in \mathbb{R}_+ : h_i(nx^3) \leq t\, h_i(x^3), \quad x \in \mathbb{R}_0\right\} \\ &= \inf \left\{t \in \mathbb{R}_+ : \theta_i \left|nx^3\right|^{q_i} \leq t\, \theta_i \left|x^3\right|^{q_i}, \quad x \in \mathbb{R}_0\right\} \\ &= n^{q_i}. \end{aligned}$$

Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\prod_{i=1}^{k} \lambda_i\big((k-1)n^3 + 1\big) \ + \ (k-1) \prod_{i=1}^{k} \lambda_i(n^3) = \Big((k-1)n^3 + 1\Big)^{\sum_{i=1}^{k} q_i} \\ &\quad + (k-1)\big(n^3\big)^{\sum_{i=1}^{k} q_i} < 1 \end{aligned}$$

for all $n \geq n_0$. Since $\sum_{i=1}^{k} q_i < 0$, one of q_i must be positive. Assume that $q_k > 0$. Then,

$$\begin{cases} \lim_{n\to\infty} \prod_{i=1}^{k-1} \lambda_i(n) = \lim_{n\to\infty} n^{\sum_{i=1}^{k-1} q_i} = 0, \\ \lim_{n\to\infty} \prod_{i=1}^{k} \lambda_i(n) = \lim_{n\to\infty} n^{\sum_{i=1}^{k} q_i} = 0. \end{cases}$$

Thus, according to Theorem 4.1, we get the desired result. □

The next corollary proves the hyperstability results for the inhomogeneous radical functional equation.

Corollary 4.3. *Let $\theta, q_i \in \mathbb{R}$ such that $\theta \geq 0$ and $\sum_{i=1}^{k} q_i < 0$. Assume that $G : \mathbb{R}^k \to Y$ and $f : \mathbb{R} \to Y$ satisfy $f(0) = 0$ and the inequality*

$$\left\| f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k} f(x_i) - G(x_1, x_2, \ldots, x_k) \right\|_* \leq \theta \prod_{i=1}^{k} \left|x^3\right|^{q_i} \tag{4.10}$$

for all $x_i \in \mathbb{R}_0$. If the functional equation

$$f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) = \sum_{i=1}^{k} f(x_i) + G(x_1, x_2, \ldots, x_k), \quad x_1, x_2, \ldots, x_k \in \mathbb{R}_0, \tag{4.11}$$

has a solution $f_0 : \mathbb{R} \to Y$, then f is a solution of (4.11).

Proof. From (4.10), we get that the function $K : \mathbb{R} \to Y$ defined by $K := f - f_0$ satisfies (4.9). Consequently, Corollary 4.2 implies that K is a solution of the radical functional Eq. (1.2). Therefore,

$$
\begin{aligned}
& f\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) - \sum_{i=1}^{k} f(x_i) - G(x_1, x_2, \ldots, x_k) \\
& = K\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) + f_0\left(\sqrt[3]{\sum_{i=1}^{k} x_i^3}\right) \\
& \quad - \sum_{i=1}^{k} K(x_i) - \sum_{i=1}^{k} f_0(x_i) - G(x_1, x_2, ..., x_k) \\
& = 0
\end{aligned}
$$

for all $x_1, x_2, \ldots, x_k \in \mathbb{R}_0$, which means that f is a solution of (4.11). □

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