J. Fixed Point Theory Appl. (2018) 20:35
https://doi.org/10.1007/s11784-018-0519-4
Published online February 8, 2018

Journal of Fixed Point Theory and Applications

Fixed point theorems for single- and set-valued F-contractions in b-metric spaces

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Abstract. Very recently, Miculescu and Mihail in (J. Fixed Point Theory Appl 19:2153–2163, 2017) gave a sufficient condition for Cauchyness on sequences in b-metric spaces. In this paper, we give a weaker sufficient condition. Also, to show the new sufficient condition is reasonably weak, we give an example. Using the new sufficient condition, we prove fixed point theorems for set-valued F-contractions in complete b-metric spaces. We also prove a fixed point theorem for single-valued F-contractions in complete b-metric spaces.

Mathematics Subject Classification. Primary 54E99, Secondary 54H25.

Keywords. b-Metric space, Cauchy sequence, fixed point theorem, F-contraction.

1. Introduction

We begin by recalling the concept of a semimetric space.

Definition 1. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *semimetric space* if the following hold:

(D1) $d(x, x) = 0$.
(D2) $d(x, y) = 0 \Rightarrow x = y$.
(D3) $d(x, y) = d(y, x)$. (symmetry)

The following concept is called a b-metric space or a pseudometric space.

Definition 2. ([2,3]) Let (X, d) be a semimetric space and let $K \geq 1$. Then (X, d, K) is said to be a *b-metric space* or *pseudometric space* if the following hold:

(D4) $d(x, z) \leq K\big(d(x, y) + d(y, z)\big)$. ($K$-relaxed triangle inequality)

We note that in the case where $K = 1$, every b-metric space is obviously a metric space. So this concept is a weaker concept than that of metric space. Very recently, Miculescu and Mihail proved the following lemma. The author strongly believes that this lemma will play a very important role in the fixed point theory.

Lemma 3. (Lemma 2.2 in [6]) *Let (X, d, K) be a b-metric space. Let $\{x_n\}$ be a sequence in X. Assume $\{d(x_n, x_{n+1})\} \in O(r^n)$ for some $r \in (0, 1)$. Then $\{x_n\}$ is Cauchy.*

Remark. See Definition 5 for the definition of $O(r^n)$. To speak exactly, the assumption in [6] is a little stronger than Lemma 3. However, from the proof in [6], we can tell that Miculescu and Mihail proved the above lemma in [6].

Wardowski in [11] introduced the concept of F-contraction and proved the following fixed point theorem.

Theorem 4. (Theorem 2.1 in Wardowski [11]) *Let (X, d) be a complete metric space and let T be a F-contraction on X, that is, there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and real numbers $\tau \in (0, \infty)$ and $k \in (0, 1)$ satisfying the following :*

(F1) *F is strictly increasing.*
(F2) *For any sequence $\{\alpha_n\}$ of positive numbers,* $\lim_n \alpha_n = 0 \Leftrightarrow \lim_n F(\alpha_n) = -\infty$.
(F3) $\lim[t^k F(t) : t \to +0] = 0$.
(F4) $\tau + F \circ d(Tx, Ty) \leq F \circ d(x, y)$ *for any $x, y \in X$ with $Tx \neq Ty$.*

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

In this paper, we generalize Lemma 3 (see Lemma 11). To show that the assumption of Lemma 11 is reasonably weak, we give an example (see Example 26). Using Lemma 11 essentially, we prove fixed point theorems (Theorems 13 and 14) for set-valued F-contractions in complete b-metric spaces. We also prove a fixed point theorem (Theorem 23) for single-valued F-contractions in complete b-metric spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For an arbitrary set A, we also denote by $\#A$ the cardinal number of A. For $t \in \mathbb{R}$, we denote by $[t]$ the maximum integer not exceeding t.

Definition 5. Let $\{a_n\}$ be a sequence in $[0, \infty)$ and let $\{b_n\}$ be a sequence in $(0, \infty)$. Then we write $\{a_n\} \in O(b_n)$ if there exists $C > 0$ satisfying $a_n \leq C\, b_n$ for all $n \in \mathbb{N}$.

The following proposition is well known and is easily proved.

Proposition 6. *Let $r, s \in (0, 1)$ satisfy $r < s$ and let $\alpha, \beta \in (1, \infty)$ satisfy $\alpha < \beta$. Then the following hold :*

(i) $O(r^n) \subset O(n^{-\alpha})$.
(ii) $\{n^{-\alpha}\} \notin O(r^n)$.

(iii) $O(r^n) \subset O(s^n)$.
(iv) $\{s^n\} \notin O(r^n)$.
(v) $O(n^{-\beta}) \subset O(n^{-\alpha})$.
(vi) $\{n^{-\alpha}\} \notin O(n^{-\beta})$.

Definition 7. Let (X, d) be a semimetric space, let $\{x_n\}$ be a sequence in X and let $x \in X$.

- $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$.
- $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup\{d(x_n, x_m) : m > n\} = 0$.
- X is said to be *Hausdorff* if $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$ imply $x = y$.
- X is said to be *complete* if every Cauchy sequence converges.

In general, not every semimetric space is metrizable. However, it is well known that every b-metric space is metrizable. So we can define the closedness. See Proposition 14.5 in [3] and others.

Definition 8. Let (X, d, K) be a b-metric space. Let A be a subset of X.

- A is said to be *closed* if for any convergent sequence in A, its limit belongs to A.
- A is said to be *bounded* if $\sup\{d(x, y) : x, y \in A\} < \infty$.

Let (X, d, K) be a b-metric space and let $\mathrm{CB}(X)$ be the set of all nonempty, bounded and closed subsets of X. For $x \in X$ and any subset A of X, we define $d(x, A) = \inf\{d(x, y) : y \in A\}$. Then the *Hausdorff–Pompeiu metric* H with respect to d is defined by

$$H(A, B) = \max\big\{\sup\{d(u, B) : u \in A\},\ \sup\{d(v, A) : v \in B\}\big\}$$

for all $A, B \in \mathrm{CB}(X)$.

Lemma 9. *Define a function f from $\mathbb{N}$ into $\mathbb{N} \cup \{0\}$ by*

$$f(n) = -[-\log_2 n]. \tag{1}$$

Then the following hold:

(i) $2^{k-1} < n \le 2^k \;\Leftrightarrow\; f(n) = k$ *for any* $n \in \mathbb{N}$ *and* $k \in \mathbb{N} \cup \{0\}$.
(ii) (Lemma 7 in [8]) f *is nondecreasing.*

Proof. Obvious. □

Lemma 10. (Lemma 5 in [8]) *Let (X, d, K) be a b-metric space. Define f by (1). For $n \in \mathbb{N}$ and $(x_0, \ldots, x_n) \in X^{n+1}$,*

$$d(x_0, x_n) \le K^{f(n)} \sum_{j=0}^{n-1} d(x_j, x_{j+1})$$

holds.

3. Lemmas

In this section, we prove two lemmas, which are used in Sect. 4. We first generalize Lemma 3.

Lemma 11. *Let (X, d, K) be a b-metric space. Let $\{x_n\}$ be a sequence in X. Assume*

$$\{d(x_n, x_{n+1})\} \in \bigcup \{O(n^{-\beta}) : \beta > 1 + \log_2 K\}.$$

Then $\{x_n\}$ is Cauchy.

Remark. We do not know that the above assumption is best possible. However, at least, we can tell that the number $1 + \log_2 K$ is best possible. See Example 26 below. There exists a sequence $\{x_n\}$ in a b-metric space (X, d, K) such that $\{x_n\}$ is not Cauchy and $\{d(x_n, x_{n+1})\} \in O(n^{-(1+\log_2 K)})$ holds.

Proof. From the assumption, there exist $\beta > 1+\log_2 K$ and $C > 0$ satisfying

$$d(x_n, x_{n+1}) \le C\, n^{-\beta}$$

for any $n \in \mathbb{N}$. We note $2\,K < 2^\beta$. Choose $\mu \in \mathbb{N}$ satisfying $2\,K^{1+1/\mu} < 2^\beta$. Then we have

$$2^\mu\, K^{\mu+1} < 2^{\mu\,\beta} \quad \text{and hence} \quad K^{\mu+1}\, 2^{\mu\,(1-\beta)} < 1. \tag{2}$$

Define a function h from $\mathbb{N}$ into itself by

$$h(n) := 2 + \frac{2^{\mu\, n} - 1}{2^\mu - 1} = 2 + \sum_{j=0}^{n-1} 2^{\mu\, j}.$$

For $k, m, n \in \mathbb{N}$ with $h(k) \le m < n \le h(k+1)$, we have by Lemmas 9 and 10

$$\begin{aligned} d(x_m, x_n) &\le K^{f(n-m)} \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \qquad (3)\\ &\le K^{f\left(h(k+1)-h(k)\right)} \sum_{j=h(k)}^{h(k+1)-1} d(x_j, x_{j+1})\\ &= K^{f(2^{\mu\, k})} \sum_{j=h(k)}^{h(k+1)-1} d(x_j, x_{j+1})\\ &= K^{\mu\, k} \sum_{j=h(k)}^{h(k+1)-1} d(x_j, x_{j+1})\\ &\le C\, K^{\mu\, k} \sum_{j=h(k)}^{h(k+1)-1} j^{-\beta}\\ &\le C\, K^{\mu\, k} \int_{h(k)-1}^{h(k+1)-1} t^{-\beta}\, dt\\ &\le C\, K^{\mu\, k} \int_{h(k)-2+1/(2^\mu-1)}^{h(k+1)-2+1/(2^\mu-1)} t^{-\beta}\, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{C\,K^{\mu\,k}}{1-\beta}\left[t^{1-\beta}\right]_{2^{\mu\,k}/(2^{\mu}-1)}^{2^{\mu\,(k+1)}/(2^{\mu}-1)} \\
&= C\,K^{\mu\,k}\,\frac{1-2^{\mu\,(1-\beta)}}{\beta-1}\,(2^{\mu}-1)^{\beta-1}\,2^{\mu\,k\,(1-\beta)} \\
&= M\,(K\,2^{1-\beta})^{\mu\,k},
\end{aligned}
$$

where we put $M{:=}C\,(1-2^{\mu\,(1-\beta)})\,(2^{\mu}-1)^{\beta-1}/(\beta-1)>0$. Since

$$(K\,2^{1-\beta})^{\mu} = K^{\mu+1}\,2^{\mu\,(1-\beta)}/K < 1/K \le 1$$

holds by (2), we note

$$\lim_{k\to\infty} M\,(K\,2^{1-\beta})^{\mu\,k} = 0. \tag{4}$$

For $k,\ell,m,n\in\mathbb{N}$ with $k<\ell$, $h(k)\le m<h(k+1)$ and $h(\ell)<n\le h(\ell+1)$, we have by (2) and (3)

$$
\begin{aligned}
&d(x_m,x_n) \\
&\le K\,d(x_m,x_{h(k+1)}) + K\,d(x_{h(k+1)},x_n) \\
&\le K\,M\,(K\,2^{1-\beta})^{\mu\,k} + K^2\,d(x_{h(k+1)},x_{h(k+2)}) + K^2\,d(x_{h(k+2)},x_n) \\
&\le \sum_{i=0}^{2} K^{i+1}\,M\,(K\,2^{1-\beta})^{\mu\,(k+i)} + K^3\,d(x_{h(k+3)},x_n) \\
&\le \cdots \le \sum_{i=0}^{\ell-k-1} K^{i+1}\,M\,(K\,2^{1-\beta})^{\mu\,(k+i)} + K^{\ell-k}\,d(x_{h(\ell)},x_n) \\
&\le \sum_{i=0}^{\ell-k-1} K^{i+1}\,M\,(K\,2^{1-\beta})^{\mu\,(k+i)} + K^{\ell-k}\,M\,(K\,2^{1-\beta})^{\mu\,\ell} \\
&\le \sum_{i=0}^{\ell-k} K^{i+1}\,M\,(K\,2^{1-\beta})^{\mu\,(k+i)} \\
&\le \sum_{i=0}^{\infty} K^{i+1}\,M\,(K\,2^{1-\beta})^{\mu\,(k+i)} \\
&= K\,M\,(K\,2^{1-\beta})^{\mu\,k}\,\frac{1}{1-K^{\mu+1}\,2^{\mu\,(1-\beta)}}.
\end{aligned}
$$

Noting (4), we obtain that $\{x_n\}$ is Cauchy. □

The following lemma is essentially proved in the proof of Theorem 2.1 in [11].

Lemma 12. ([11]) *Let* $\{t_n\}$ *be a sequence in* $(0,\infty)$*. Assume that there exist a function* F *from* $(0,\infty)$ *into* $\mathbb{R}$ *and real numbers* $\tau\in(0,\infty)$ *and* $k\in(0,1)$ *satisfying (F2),(F3) and the following:*

- $n\,\tau + F(t_{n+1}) \le F(t_1)$.

Then $\{t_n\}\in O(n^{-1/k})$ *holds.*

Proof. From the assumption, we have $\lim_n F(t_n) = -\infty$. By (F2), we have $\lim_n t_n = 0$. So, using (F3), we obtain $\lim_n {t_n}^k F(t_n) = 0$. We have

$$\begin{aligned} 0 &= \lim_{n\to\infty} {t_n}^k \big(F(t_n) - F(t_1) - \tau\big) \\ &\le \liminf_{n\to\infty} (-{t_n}^k\, n\,\tau) \le \limsup_{n\to\infty} (-{t_n}^k\, n\,\tau) \\ &\le 0, \end{aligned}$$

which implies $\lim_n n\,{t_n}^k = 0$. Choose $\nu \in \mathbb{N}$ satisfying $n\,{t_n}^k \le 1$ for any $n \ge \nu$. Then we have $t_n \le n^{-1/k}$ for any $n \ge \nu$. We obtain the desired result. □

4. Set-valued mappings

We prove fixed point theorems for set-valued F-contractions in complete b-metric spaces.

Theorem 13. *Let (X, d, K) be a complete b-metric space. Let T be a mapping from X into $CB(X)$. Assume that there exist a function F from $(0,\infty)$ into $\mathbb{R}$,*

$$k \in \big(0, 1/(1 + \log_2 K)\big)$$

and $\tau \in (0,\infty)$ satisfying (F2),(F3) and the following:

(F5) *For any $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that either $v = u$ or*

$$\tau + F \circ d(u, v) \le F \circ d(x, y)$$

holds.

Then T has a fixed point.

Remark. We do not need (F1). On the other hand, we do need (F3). Compare Theorem 13 with Theorem 23.

Proof. Arguing by contradiction, we assume that T does not have a fixed point. Fix $u_1 \in X$ and $u_2 \in Tu_1$. We note $u_2 \ne u_1$. Since $u_2 \notin Tu_2$, from the assumption, we can choose $u_3 \in Tu_2$ satisfying

$$\tau + F \circ d(u_2, u_3) \le F \circ d(u_1, u_2).$$

Continuing this argument, we can choose a sequence $\{u_n\}$ in X satisfying

$$u_{n+1} \in Tu_n \quad \text{and} \quad \tau + F \circ d(u_{n+1}, u_{n+2}) \le F \circ d(u_n, u_{n+1})$$

for any $n \in \mathbb{N}$. So it is obvious that

$$n\,\tau + F \circ d(u_{n+1}, u_{n+2}) \le F \circ d(u_1, u_2)$$

holds for any $n \in \mathbb{N}$. By Lemma 12, $\{d(u_n, u_{n+1})\} \in O(n^{-1/k})$ holds. Since

$$1/k \in (1 + \log_2 K, \infty)$$

holds, by Lemma 11, we obtain that $\{u_n\}$ is Cauchy. Since X is complete, $\{u_n\}$ converges to some $z \in X$. For $n \in \mathbb{N}$, there exists $z_n \in Tz$ such that either

$$z_n = u_{n+1} \quad \text{or} \quad \tau + F \circ d(u_{n+1}, z_n) \le F \circ d(u_n, z)$$

holds. Let $\{f(n)\}$ be an arbitrary subsequence of the sequence $\{n\}$ in $\mathbb{N}$. We consider the following two cases:

- $\#\{n \in \mathbb{N} : z_{f(n)} = u_{f(n)+1}\} = \infty$.
- $\#\{n \in \mathbb{N} : z_{f(n)} = u_{f(n)+1}\} < \infty$.

In the first case, there exists a subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying $z_{f\circ g(n)} = u_{f\circ g(n)+1}$. It is obvious that $\lim_n d(z_{f\circ g(n)}, z) = 0$ holds. In the second case, there exists a subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying

$$\tau + F \circ d(u_{f\circ g(n)+1}, z_{f\circ g(n)}) \le F \circ d(u_{f\circ g(n)}, z).$$

Since $\lim_n d(u_n, z) = 0$ holds, we have $\lim_n d(u_{f\circ g(n)+1}, z_{f\circ g(n)}) = 0$ by (F2). We have

$$\lim_{n\to\infty} d(z_{f\circ g(n)}, z) \le \lim_{n\to\infty} K \left(d(z_{f\circ g(n)}, u_{f\circ g(n)+1}) + d(u_{f\circ g(n)+1}, z)\right) = 0.$$

Therefore, we have proved $\lim_n d(z_{f\circ g(n)}, z) = 0$ in both cases. Since f is arbitrary, we obtain $\lim_n d(z_n, z) = 0$. Since Tz is closed, we have $z \in Tz$. This is a contradiction. Therefore, we have shown that T has a fixed point. □

Altun, Mınak and Dăg in [1] proved a fixed point theorem for set-valued F-contractions in complete metric spaces under the following assumption:

(F6) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A \in (0, \infty)$.

It is obvious that under (F1),(F6) and the following are equivalent.

(F7) F is upper semicontinuous.

We extend Theorem 2.5 in [1] to b-metric spaces.

Theorem 14. *Let (X, d, K) be a complete b-metric space. Let T be a mapping from X into $CB(X)$. Assume that there exist a function F from $(0, \infty)$ into $\mathbb{R}$, $k \in \left(0, 1/(1 + \log_2 K)\right)$ and $\tau \in (0, \infty)$ satisfying (F1)–(F3),(F7) and the following:*

(F8) *For any $x, y \in X$ with $Tx \ne Ty$,*

$$\tau + F \circ H(Tx, Ty) \le F \circ d(x, y)$$

holds.

Then T has a fixed point.

Remark. Considering the continuous function $t \mapsto \ln(t)$, we find that Theorem 14 is a generalization of Corollary 14 in [8]. See also Example 2.1 in [11].

Proof. Fix $x, y \in X$ and $u \in Tx$. We consider the following two cases:

- $d(u, Ty) = 0$.
- $d(u, Ty) > 0$.

In the first case, since Ty is closed, we have $u \in Ty$. In the second case, we have $Tx \neq Ty$. So,

$$\tau + F \circ H(Tx, Ty) \leq F \circ d(x, y)$$

holds. From (F1) and $d(u, Ty) \leq H(Tx, Ty)$, we have

$$\tau + F \circ d(u, Ty) \leq F \circ d(x, y).$$

From (F6), we have

$$\begin{aligned} \inf\{F \circ d(u, w) : w \in Ty\} = F \circ d(u, Ty) &\leq F \circ d(x, y) - \tau \\ &< F \circ d(x, y) - \tau/2. \end{aligned}$$

So we can choose $v \in Ty$ satisfying

$$\tau/2 + F \circ d(u, v) \leq F \circ d(x, y).$$

Thus, we have shown (F5) with $\tau := \tau/2$. By Theorem 13, we obtain the desired result. □

5. Preliminaries, part 2

Throughout this section, we let η be a function from $[0, \infty)$ into itself. We define the following condition:

(H1) For any sequence $\{a_n\}$ in $[0, \infty)$, $\lim_n \eta(a_n) = 0 \Leftrightarrow \lim_n a_n = 0$. (See Lemma 6 in Jachymski [4])

The proofs of the following lemmas are obvious.

Lemma 15. ([9]) *The following are equivalent:*

- (i) *η satisfies (H1).*
- (ii) *The conjunction of the following holds:*
  - (a) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $t < \delta$ implies $\eta(t) < \varepsilon$.*
  - (b) *For any $\delta > 0$, there exists $\varepsilon > 0$ such that $\eta(t) < \varepsilon$ implies $t < \delta$.*

Lemma 16. (Lemma 2.2 in [10]) *Let η satisfy (H1). Then $\eta^{-1}(0) = \{0\}$ holds, that is, $\eta(\alpha) = 0 \Leftrightarrow \alpha = 0$.*

Lemma 17. ([9]) *Let (X, d) be a semimetric space and let η satisfy (H1). Define a function p from $X \times X$ into $[0, \infty)$ by $p = \eta \circ d$. Let $\{x_n\}$ be a sequence in X and let $x \in X$. Then the following hold:*

- (i) *(X, p) is a semimetric space.*
- (ii) *$\{x_n\}$ converges to x in (X, d) iff $\{x_n\}$ converges to x in (X, p).*
- (iii) *$\{x_n\}$ is Cauchy in (X, d) iff $\{x_n\}$ is Cauchy in (X, p).*
- (iv) *(X, d) is complete iff (X, p) is complete.*
- (v) *(X, d) is Hausdorff iff (X, p) is Hausdorff.*

We will show that (D4) implies the following (D5).

Lemma 18. *Let (X, d, K) be a b-metric space. Then the following holds:*

(D5) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ and $d(y, z) < \delta$ imply $d(x, z) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Put $\delta := \varepsilon/(2\,K) > 0$. Let $x, y, z \in X$ satisfy $d(x, y) < \delta$ and $d(y, z) < \delta$. Then we have by (D4)

$$d(x, z) \le K\,\big(d(x, y) + d(y, z)\big) < 2\,K\,\delta = \varepsilon.$$

Therefore, we obtain (D5). □

Lemma 19. *Let (X, d) be a semimetric space. Assume (D5). Let η satisfy (H1). Then (D5) with $d := \eta \circ d$ holds.*

Proof. Define p by $p = \eta \circ d$. Fix $\varepsilon' > 0$. Then by Lemma 15, there exists $\varepsilon > 0$ such that $t < \varepsilon$ implies $\eta(t) < \varepsilon'$. By (D5), there exists $\delta > 0$ such that $d(x, y) < \delta$ and $d(y, z) < \delta$ imply $d(x, z) < \varepsilon$. By Lemma 15 again, there exists $\delta' > 0$ such that $\eta(t) < \delta'$ implies $t < \delta$. We let $x, y, z \in X$ satisfy $p(x, y) < \delta'$ and $p(y, z) < \delta'$. Then we have $d(x, y) < \delta$ and $d(y, z) < \delta$. From the above, $d(x, z) < \varepsilon$ holds. Hence, $p(x, z) < \varepsilon'$ holds. We have shown (D5) with $d := p$. □

Lemma 20. *Let (X, d) be a semimetric space. Assume (D5). Then (X, d) is Hausdorff.*

Proof. Assume $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $d(u, v) < \delta$ and $d(v, w) < \delta$ imply $d(u, w) < \varepsilon$. For sufficiently large $n \in \mathbb{N}$, we have

$$d(x, x_n) = d(x_n, x) < \delta \quad \text{and} \quad d(x_n, y) < \delta.$$

Hence, $d(x, y) < \varepsilon$ holds. Since $\varepsilon > 0$ is arbitrary, we obtain $x = y$. □

6. Single-valued mappings

We prove a fixed point theorem for single F-contractions in b-metric spaces, using the following splendid fixed point theorem proved by Jachymski, Matkowski and Świątkowski in [5].

Theorem 21. (a corollary of Theorem 1 in [5]) *Let (X, d) be a Hausdorff, complete semimetric space. Assume the following:*

(D6) *There exist $\delta > 0$ and $\varepsilon > 0$ such that $d(x, y) < \delta$ and $d(y, z) < \delta$ imply $d(x, z) < \varepsilon$.*

Let T be a contraction on X, that is, there exists $r \in [0, 1)$ satisfying

$$d(Tx, Ty) \le r\,d(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

Theorem 22. *Let (X, d) be a complete semimetric space. Assume (D5). Let T be a mapping on X. Assume that there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and a real number $\tau \in (0, \infty)$ satisfying (F2) and (F4). Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Proof. Define a function η from $[0,\infty)$ into itself by

$$\eta(t) = \begin{cases} 0 & \text{if } t = 0 \\ \exp \circ F(t) & \text{if } t > 0. \end{cases}$$

From (F2), it is obvious that η satisfies (H1). By (F4), we also have

$$\eta \circ d(Tx, Ty) \le \exp(-\tau)\, \eta \circ d(x, y)$$

for any $x, y \in X$ with $Tx \ne Ty$. It is obvious that this inequality holds even when $Tx = Ty$. We note $\exp(-\tau) < 1$. Define a function p by $p = \eta \circ d$. Then by Lemmas 17, 19 and 20, (X, p) is a Hausdorff, complete semimetric space satisfying (D5). We note that (D6) is weaker than (D5). Thus, we have shown all the assumptions of Theorem 21. So by Theorem 21, we obtain the desired result. □

Theorem 23. *Let (X, d, K) be a complete b-metric space. Let T be a mapping on X. Assume that there exist a function F from $(0,\infty)$ into $\mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying (F2) and (F4). Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Remark. We need neither (F1) nor (F3).

Proof. By Lemma 18, (D5) holds. So by Theorem 22, we obtain the desired result. □

Using Theorem 23, we obtain the following corollary.

Corollary 24. *Let (X, d) be a complete metric space. Let T be a mapping on X. Assume that there exist a function F from $(0,\infty)$ into $\mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying (F2) and (F4). Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Remark. Considering the proof of Theorem 22, we can tell that Corollary 24 is a corollary of Theorem 9 in Jachymski [4]. See also Remark below the proof of Theorem 17 in [7].

7. Counterexample

We give a counterexample concerning Lemma 11.

Lemma 25. *Let $\{t_n\}$ be a sequence in $(0,\infty)$. Assume that there exist $\mu \in (1,\infty)$, $\beta, C \in (0,\infty)$ and $\nu \in \mathbb{R}$ such that*

$$\mu^k + \nu \le n \quad \textit{implies} \quad t_n \le \frac{C}{\mu^{\beta k}}$$

for any $k, n \in \mathbb{N}$. Then $\{t_n\} \in O(n^{-\beta})$ holds.

Proof. Choose $\kappa \in \mathbb{N}$ satisfying $\mu^\kappa + \nu < \mu^{\kappa+1}$. It is obvious that

$$\mu^k + \nu < \mu^{k+1}$$

holds for any $k \geq \kappa$ because $\nu < \mu^\kappa (\mu - 1) \leq \mu^k (\mu - 1)$. Fix $n \in \mathbb{N}$ with $n \geq \mu^\kappa + \nu$. We choose $k \in \mathbb{N}$ satisfying

$$\mu^k + \nu \leq n < \mu^{k+1} + \nu.$$

Then we have

$$n^\beta < (\mu^{k+1} + \nu)^\beta < (\mu^{k+2})^\beta = \mu^{\beta\, k}\, \mu^{2\,\beta}.$$

Hence,

$$t_n \leq \frac{C}{\mu^{\beta\, k}} \leq \frac{C\, \mu^{2\,\beta}}{n^\beta}$$

holds. Noting $\#\{n \in \mathbb{N} : n < \mu^\kappa + \nu\} < \infty$, we obtain the desired result. □

Example 26. (Example 11 in [8]) Let $K > 1$. Let X be a subset of $[1, \infty)$ satisfying $\mathbb{N} \subset X$ and $\#\big(X \cap [n, n+1)\big) = 2^n$ for any $n \in \mathbb{N}$. Define a strictly increasing function χ from $\mathbb{N}$ into X satisfying $\chi(\mathbb{N}) = X$. Define a sequence $\{\nu_j\}$ in $\mathbb{N}$ satisfying $\chi(\nu_j) = j$ for any $j \in \mathbb{N}$. Define a sequence $\{\alpha_n\}$ in $(0, \infty)$ by $\alpha_n = 2^{-n}\, K^{-n}$. Define a function f from $\mathbb{N}$ into $\mathbb{N} \cup \{0\}$ by (1). Define a function g from $\mathbb{N} \cup \{0\}$ into $[0, \infty)$ by

$$\begin{aligned} &g(0) = 0, \\ &g(n) = (2\,n - 2^{f(n)})\, K^{f(n)} + (2^{f(n)} - n)\, K^{f(n)-1} \end{aligned}$$

for $n \in \mathbb{N}$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned} &d\big(\chi(m), \chi(n)\big) \\ &= \begin{cases} 0 & \text{if } m = n \\ g(n-m)\, \alpha_k & \text{if } \nu_k \leq m < n \leq \nu_{k+1} \text{ for some } k \in \mathbb{N} \\ d\big(\chi(m), j+1\big) & \text{if } \nu_j \leq m < \nu_{j+1} \leq \nu_k \\ \quad + \displaystyle\sum_{i=j+1}^{k-1} d(i, i+1) + d\big(k, \chi(n)\big) & \qquad < n \leq \nu_{k+1} \text{ for some } j, k \in \mathbb{N} \\ d\big(\chi(n), \chi(m)\big) & \text{if } m > n \end{cases} \end{aligned}$$

for all $m, n \in \mathbb{N}$. Then the following assertions hold:

(i) (X, d, K) is a b-metric space.
(ii) $\{\chi(n)\}$ is not Cauchy.
(iii) $\{d\big(\chi(n), \chi(n+1)\big)\} \in O(n^{-(1+\log_2 K)})$.

Proof. We have proved (i) and (ii) in [8]. Let us prove (iii). We note

$$2^k \leq n + 1 < 2^{k+1} \;\Rightarrow\; d\big(\chi(n), \chi(n+1)\big) = \alpha_k = \frac{1}{(2\,K)^k} = 2^{-(1+\log_2 K)\, k}$$

for any $n, k \in \mathbb{N}$. By Lemma 25, we obtain $\{d\big(\chi(n), \chi(n+1)\big)\} \in O(n^{-(1+\log_2 K)})$. □

Acknowledgements

The author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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