



Fixed point theorems in b -metric spaces with applications to differential equations

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Abstract. In this paper, we present some fixed point theorems for a class of contractive mappings in b -metric spaces. We verify the T -stability of Picard's iteration and the P property for such mappings. We also give an example to support our assertions. In addition, by using our results, we obtain the existence and uniqueness of solution to some ordinary differential equations with initial value conditions. Further, we provide the precise mathematical expressions of solutions to such equations.

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1. Introduction

In the last decades, fixed point theory has been extended to various abstract spaces and has also been used extensively in the study of all kinds of scientific problems successfully, establishing a connection between pure and applied approaches and even including very relevant computational issues. In particular, several applications of fixed point theory have been introduced for the study and calculation of solutions to differential equations, integral equations, dynamical systems, models in economy and related areas, game theory, physics, engineering, computer science, or neural networks, among many others. Besides, they are basic tools for the study of nonlinear systems, by setting a framework which helps to elevate some basic properties of the solutions to linear models in order to deduce (or approximate) the behavior of the nonlinear ones, whose solutions can be found as the fixed points of a certain operator. Among them, the most influential and celebrating fixed point theorem, known as the Banach contraction principle (see [3]), was proved by the Polish mathematician Banach in 1922. Since then, fixed point theory has had a rapid development. In [2] or [5], the author introduced b -metric space as a sharp generalization of metric space, and proved fixed point theorems in b -metric

spaces, which generalized the famous Banach contraction principle. Subsequently, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b -metric spaces (see [4, 6, 9, 10, 18–21]). In recent years, stability results for fixed point iteration procedures become the center of strong research activity in applications of many branches of mathematics. There are numerous works about stabilities for iteration procedures in various spaces (see, for example, [1, 7, 13, 14]). But the most important iteration procedure among them, is Picard’s iteration, whose stability occupies a prominent place in many areas. On the other hand, quite a few authors are interested in the P properties of fixed points for some mappings (see [7, 8]). In this article, we obtain some fixed point theorems for a class of contractive mappings in b -metric spaces. Moreover, we consider the T -stability of Picard’s iteration and the P property for such mappings. The results greatly improve and generalize the previous results from [6, 8, 9, 14]. Besides this, we illustrate our assertions with an example. In addition, we give some applications to two classes of ordinary differential equations with initial value conditions. We verify the existence and uniqueness of solution to such equations. Further, we give the concrete mathematical expressions of solutions to such equations. To the best of our knowledge, by using different methods of fixed point theory, many authors usually merely deal with the existence and uniqueness of solution for various differential or integral equations. However, they seldom consider the expression of solution. On this basis alone our results are quite valuable.

In the sequel, we always denote by $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ the sets of positive integers, real numbers and nonnegative real numbers, respectively.

First of all, let us recall the concept of b -metric space.

Definition 1.1. [2, 5, 9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d: X \times X \rightarrow \mathbb{R}^+$ is called a b -metric on X if, for all $x, y, z \in X$, the following conditions hold:

- (b1) $d(x, y) = 0$ if and only if $x = y$;
- (b2) $d(x, y) = d(y, x)$;
- (b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b -metric space or metric type space.

For some examples of b -metric spaces, the reader may refer to [2, 4, 5, 10, 18–21] and the references therein. Motivated by Example 1.2 of [6], we give an example of unusual b -metric space as follows.

Example 1.2. Let $H^p(U) = \{f \in H(U) : \|f\|_{H^p} < \infty\}$ ($0 < p < 1$) be H^p space defined on the unit disk U , where $H(U)$ is the set of all holomorphic functions on U and

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Denote $X = H^p(U)$. Define a mapping $d: X \times X \rightarrow \mathbb{R}^+$ by

$$d(f, g) = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \tag{1.1}$$

for all $f, g \in X$. Then (X, d) is a b -metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

Indeed, we only prove that condition (b3) in Definition 1.1 is satisfied. To this end, letting $f, g, h \in X$, by (1.1), we need to show that

$$\begin{aligned} & \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}) - h(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\ & \leq 2^{\frac{1}{p}-1} \left[\sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right. \\ & \quad \left. + \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta}) - h(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right]. \end{aligned} \tag{1.2}$$

Denote $u(re^{i\theta}) = f(re^{i\theta}) - g(re^{i\theta})$, $v(re^{i\theta}) = g(re^{i\theta}) - h(re^{i\theta})$, it follows immediately from (1.2) that

$$\begin{aligned} & \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta}) + v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\ & \leq 2^{\frac{1}{p}-1} \left[\sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right. \\ & \quad \left. + \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right]. \end{aligned} \tag{1.3}$$

To prove (1.3), notice that the following inequalities:

$$\begin{aligned} (a + b)^p & \leq a^p + b^p \quad (a, b \geq 0, 0 < p \leq 1), \\ (a + b)^p & \leq 2^{p-1}(a^p + b^p) \quad (a, b \geq 0, p \geq 1), \end{aligned} \tag{1.4}$$

then

$$\begin{aligned} & \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta}) + v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\ & \leq \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (|u(re^{i\theta})| + |v(re^{i\theta})|)^p d\theta \right)^{\frac{1}{p}} \\ & \leq \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (|u(re^{i\theta})|^p + |v(re^{i\theta})|^p) d\theta \right)^{\frac{1}{p}} \\ & = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} |v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\ & \leq 2^{\frac{1}{p}-1} \sup_{0 < r < 1} \left[\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right] \\ & \leq 2^{\frac{1}{p}-1} \left[\sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} + \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |v(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right]. \end{aligned}$$

Definition 1.3. [18, 19] Let (X, d) be a b -metric space and $\{x_n\}$ a sequence in X . We say that

- (1) $\{x_n\}$ b -converges to $x \in X$ if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (2) $\{x_n\}$ is a b -Cauchy sequence if $d(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$;
- (3) (X, d) is b -complete if every b -Cauchy sequence in X is b -convergent.

Each b -convergent sequence in a b -metric space has a unique limit and it is also a b -Cauchy sequence. Moreover, in general, a b -metric is not necessarily continuous. The following example illustrates this claim.

Example 1.4. [10] Let $X = \mathbb{N} \cup \{\infty\}$. Define a mapping $d: X \times X \rightarrow \mathbb{R}^+$ as follows:

$$d(m, n) = \begin{cases} 0, & \text{if } m = n; \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ (} m \neq n \text{) is even and the other is even or } \infty; \\ 5, & \text{if one of } m, n \text{ (} m \neq n \text{) is odd and the other is odd or } \infty; \\ 2, & \text{others.} \end{cases}$$

It is not hard to verify that

$$d(m, p) \leq \frac{5}{2}[d(m, n) + d(n, p)] \quad (m, n, p \in X).$$

Then (X, d) is a b -metric space with coefficient $s = \frac{5}{2}$. Choose $x_n = 2n$ ($n \in \mathbb{N}$), then

$$d(x_n, \infty) = \frac{1}{2n} \rightarrow 0 \quad (n \rightarrow \infty),$$

that is, $x_n \rightarrow \infty$ ($n \rightarrow \infty$). However, $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ ($n \rightarrow \infty$).

Recently, Qing and Rhoades [14] established the notion of T -stability of Picard’s iteration in metric space as follows.

Definition 1.5. [14] Let (X, d) be a metric space and T a self-map on X . Let x_0 be a point of X , and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T , which yields a sequence $\{x_n\}$ of points from X . Then the iteration procedure $x_{n+1} = f(T, x_n)$, is said to be T -stable with respect to T if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n \rightarrow \infty} d(y_{n+1}, f(T, y_n)) = 0$, we have $\lim_{n \rightarrow \infty} y_n = q$. Particularly, if these conditions hold for Picard’s iteration procedure $x_{n+1} = Tx_n$, then we will say that Picard’s iteration is T -stable.

In the following, we simplify Definition 1.5 and introduce the concept of T -stability of Picard’s iteration in b -metric space.

Definition 1.6. Let (X, d) be a b -metric space, $x_0 \in X$ and $T : X \rightarrow X$ be a mapping with $F(T) \neq \emptyset$, where $F(T)$ denotes the set of all fixed points of T and similarly hereinafter. Then Picard’s iteration $x_{n+1} = Tx_n$ is said to be T -stable with respect to T if $\lim_{n \rightarrow \infty} x_n = q \in F(T)$ and whenever $\{y_n\}$ is a sequence in X with $\lim_{n \rightarrow \infty} d(y_{n+1}, Ty_n) = 0$, we have $\lim_{n \rightarrow \infty} y_n = q$.

For the convenience of reader, we recall the well-posedness of fixed point problems as defined and studied in [15–17] as follows.

Definition 1.7. [15–17] Let (K, ρ) be a bounded complete metric space. We say that the fixed point problem for a mapping $T: K \rightarrow K$ is well posed if there exists a unique $q \in K$ such that $q \in F(T)$ and whenever $\{y_n\}$ is a sequence in X with $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$ we have $\lim_{n \rightarrow \infty} y_n = q$.

Remark 1.8. Comparing Definitions 1.6 to 1.7, we see that the T -stability is different from the well-posedness. Indeed, on the one hand, their space is different since (X, d) is a b -metric space and (K, ρ) is a bounded complete metric space. On the other hand, Definition 1.6 aims at Picard’s iterative sequence and Definition 1.7 aims at a general sequence. In addition, $\lim_{n \rightarrow \infty} d(y_{n+1}, Ty_n) = 0$ and $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$ differ from each other in many ways. Limited by the length of paper, this article only discusses the T -stability of fixed point problems.

What follows is a useful lemma for the proof of our main results.

Lemma 1.9. [11] Let $\{a_n\}, \{c_n\}$ be nonnegative sequences satisfying $a_{n+1} \leq ha_n + c_n$ for all $n \in \mathbb{N}$, $0 \leq h < 1$, $\lim_{n \rightarrow \infty} c_n = 0$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

The following lemma was frequently utilized by many authors in order to overcome the problem of discontinuity for b -metric. However, throughout this paper, we do not use this lemma because we avoid such problem.

Lemma 1.10. [10] Let (X, d) be a b -metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b -convergent to points $x, y \in X$, respectively. Then we have

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z).$$

2. Main results

In this section, we firstly give a useful lemma, which greatly generalizes and implements the counterpart of the existing literature. Secondly, we give several fixed point theorems for contractive mappings on b -complete b -metric spaces. Thirdly, we deduce the T -stability of Picard’s iteration and the P property for such mappings. Fourthly, we give an example to illustrate our conclusions.

Lemma 2.1. Let (X, d) be a b -metric space with coefficient $s \geq 1$ and $T: X \rightarrow X$ be a mapping. Suppose that $\{x_n\}$ is a sequence in X induced by $x_{n+1} = Tx_n$ such that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \tag{2.1}$$

for all $n \in \mathbb{N}$, where $\lambda \in [0, 1)$ is a constant. Then $\{x_n\}$ is a b -Cauchy sequence.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We divide the proof into three cases.

Case 1 $\lambda \in [0, \frac{1}{s})$ ($s > 1$). By (2.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda d(x_{n-1}, x_n) \\ &\leq \lambda^2 d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \lambda^n d(x_0, x_1). \end{aligned}$$

Thus, for any $n > m$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_m, x_n) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\ &\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_n)] \\ &\leq sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^3[d(x_{m+2}, x_{m+3}) + d(x_{m+3}, x_n)] \\ &\leq sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^3d(x_{m+2}, x_{m+3}) \\ &\quad + \dots + s^{n-m-1}d(x_{n-2}, x_{n-1}) + s^{n-m-1}d(x_{n-1}, x_n) \\ &\leq s\lambda^m d(x_0, x_1) + s^2\lambda^{m+1}d(x_0, x_1) + s^3\lambda^{m+2}d(x_0, x_1) \\ &\quad + \dots + s^{n-m-1}\lambda^{n-2}d(x_0, x_1) + s^{n-m-1}\lambda^{n-1}d(x_0, x_1) \\ &\leq s\lambda^m(1 + s\lambda + s^2\lambda^2 + \dots + s^{n-m-2}\lambda^{n-m-2} + s^{n-m-1}\lambda^{n-m-1})d(x_0, x_1) \\ &\leq s\lambda^m \left[\sum_{i=0}^{\infty} (s\lambda)^i \right] d(x_0, x_1) \\ &= \frac{s\lambda^m}{1 - s\lambda} d(x_0, x_1) \rightarrow 0 \quad (m \rightarrow \infty), \end{aligned}$$

which implies that $\{x_n\}$ is a b -Cauchy sequence. In other words, $\{T^n x_0\}$ is a b -Cauchy sequence.

Case 2 Let $\lambda \in [\frac{1}{s}, 1)$ ($s > 1$). In this case, we have $\lambda^n \rightarrow 0$ as $n \rightarrow \infty$, so there is $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1}{s}$. Thus, by Case 1, we claim that

$$\{(T^{n_0})^n x_0\}_{n=0}^{\infty} := \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots, x_{n_0+n}, \dots\}$$

is a b -Cauchy sequence. Then

$$\{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, \dots, x_{n_0-1}\} \cup \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots, x_{n_0+n}, \dots\}$$

is a b -Cauchy sequence in X .

Case 3 Let $s = 1$. Similar to the process of Case 1, the claim holds. □

Remark 2.2. Lemma 2.1 expands the range of [9, Lemma 3.1] from $\lambda \in [0, \frac{1}{s})$ to $\lambda \in [0, 1)$. Clearly, this is a sharp generalization. Otherwise, though Lemma 2.1 is the special case of [12, Lemma 2.2], our proof of Lemma 2.1 is straightforward without utilizing [12, Lemma 2.1] while the proof of [12, Lemma 2.2] strongly relies on this lemma.

Theorem 2.3. *Let (X, d) be a b -complete b -metric space with coefficient $s \geq 1$ and $T: X \rightarrow X$ be a mapping such that*

$$\begin{aligned}
 d(Tx, Ty) \leq & \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)} \\
 & + \lambda_4 \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ty)d(y, Tx)}{1 + d(x, y)}, \tag{2.2}
 \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$. Then T has a unique fixed point in X . Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ ($n \in \mathbb{N}$) b -converges to the fixed point.

Proof. Choose $x_0 \in X$ and construct a Picard iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ ($n \in \mathbb{N}$). If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, i.e., x_{n_0} is a fixed point of T . Next, without loss of generality, let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}
 & d(x_n, x_{n+1}) \\
 &= (Tx_{n-1}, Tx_n) \\
 &\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)} \\
 &\quad + \lambda_3 \frac{d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})}{1 + d(x_{n-1}, x_n)} \\
 &\quad + \lambda_4 \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n)}{1 + d(x_{n-1}, x_n)} + \lambda_5 \frac{d(x_n, Tx_n)d(x_n, Tx_{n-1})}{1 + d(x_{n-1}, x_n)} \\
 &= \lambda_1 d(x_{n-1}, x_n) + \lambda_2 \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} + \lambda_3 \frac{d(x_{n-1}, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_n)} \\
 &\quad + \lambda_4 \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_n)} + \lambda_5 \frac{d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_n)} \\
 &\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_n, x_{n+1}) + s\lambda_4 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})].
 \end{aligned}$$

It follows that

$$(1 - \lambda_2 - s\lambda_4)d(x_n, x_{n+1}) \leq (\lambda_1 + s\lambda_4)d(x_{n-1}, x_n). \tag{2.3}$$

Again by (2.2), we have

$$\begin{aligned}
 & d(x_n, x_{n+1}) \\
 &= d(Tx_n, Tx_{n-1}) \\
 &\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{1 + d(x_n, x_{n-1})} + \lambda_3 \frac{d(x_n, Tx_{n-1})d(x_{n-1}, Tx_n)}{1 + d(x_n, x_{n-1})} \\
 &\quad + \lambda_4 \frac{d(x_n, Tx_n)d(x_n, Tx_{n-1})}{1 + d(x_n, x_{n-1})} + \lambda_5 \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n)}{1 + d(x_n, x_{n-1})} \\
 &= \lambda_1 d(x_n, x_{n-1}) + \lambda_2 \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{1 + d(x_n, x_{n-1})} + \lambda_3 \frac{d(x_n, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_n, x_{n-1})} \\
 &\quad + \lambda_4 \frac{d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_n, x_{n-1})} + \lambda_5 \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_n, x_{n-1})} \\
 &\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_n, x_{n+1}) + s\lambda_5 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})].
 \end{aligned}$$

This establishes that

$$(1 - \lambda_2 - s\lambda_5)d(x_n, x_{n+1}) \leq (\lambda_1 + s\lambda_5)d(x_{n-1}, x_n). \tag{2.4}$$

Adding up (2.3) and (2.4) yields

$$d(x_n, x_{n+1}) \leq \frac{2\lambda_1 + s\lambda_4 + s\lambda_5}{2 - 2\lambda_2 - s\lambda_4 - s\lambda_5}d(x_{n-1}, x_n).$$

Put $\lambda = \frac{2\lambda_1 + s\lambda_4 + s\lambda_5}{2 - 2\lambda_2 - s\lambda_4 - s\lambda_5}$. In view of $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$, then $0 \leq \lambda < 1$. Thus, by Lemma 2.1, $\{x_n\}$ is a b -Cauchy sequence in X . Since (X, d) is b -complete, then there exists some point $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

By (2.2), it is easy to see that

$$\begin{aligned} d(x_{n+1}, Tx^*) & \tag{2.5} \\ &= d(Tx_n, Tx^*) \\ &\leq \lambda_1 d(x_n, x^*) + \lambda_2 \frac{d(x_n, Tx_n)d(x^*, Tx^*)}{1 + d(x_n, x^*)} + \lambda_3 \frac{d(x_n, Tx^*)d(x^*, Tx_n)}{1 + d(x_n, x^*)} \\ &\quad + \lambda_4 \frac{d(x_n, Tx_n)d(x_n, Tx^*)}{1 + d(x_n, x^*)} + \lambda_5 \frac{d(x^*, Tx^*)d(x^*, Tx_n)}{1 + d(x_n, x^*)} \\ &= \lambda_1 d(x_n, x^*) + \lambda_2 \frac{d(x_n, x_{n+1})d(x^*, Tx^*)}{1 + d(x_n, x^*)} + \lambda_3 \frac{d(x_n, Tx^*)d(x^*, x_{n+1})}{1 + d(x_n, x^*)} \\ &\quad + \lambda_4 \frac{d(x_n, x_{n+1})d(x_n, Tx^*)}{1 + d(x_n, x^*)} + \lambda_5 \frac{d(x^*, Tx^*)d(x^*, x_{n+1})}{1 + d(x_n, x^*)}. \tag{2.6} \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ from both sides of (2.6), we get $\lim_{n \rightarrow \infty} d(x_{n+1}, Tx^*) = 0$. That is, $x_n \rightarrow Tx^*$ ($n \rightarrow \infty$). Hence, by the uniqueness of limit of b -convergent sequence, it gives that $Tx^* = x^*$. In other words, x^* is a fixed point of T .

Finally, we show the uniqueness of the fixed point. Indeed, if there is another fixed point y^* , then by (2.2),

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \lambda_1 d(x^*, y^*) + \lambda_2 \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(x^*, y^*)} + \lambda_3 \frac{d(x^*, Ty^*)d(y^*, Tx^*)}{1 + d(x^*, y^*)} \\ &\quad + \lambda_4 \frac{d(x^*, Tx^*)d(x^*, Ty^*)}{1 + d(x^*, y^*)} + \lambda_5 \frac{d(y^*, Ty^*)d(y^*, Tx^*)}{1 + d(x^*, y^*)} \\ &= \lambda_1 d(x^*, y^*) + \lambda_3 \frac{d(x^*, y^*)d(x^*, y^*)}{1 + d(x^*, y^*)} \\ &\leq (\lambda_1 + \lambda_3)d(x^*, y^*). \tag{2.7} \end{aligned}$$

Because $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$ implies $\lambda_1 + \lambda_3 < 1$, we conclude from (2.7) that $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. □

Corollary 2.4. *Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a mapping such that*

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)} + \lambda_4 \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ty)d(y, Tx)}{1 + d(x, y)},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$. Then T has a unique fixed point in X . Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ ($n \in \mathbb{N}$) converges to the fixed point.

Proof. Take $s = 1$ in Theorem 2.3, thus the claim holds. □

Remark 2.5. Take $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$ in Theorem 2.3 or in Corollary 2.4, then Theorem 2.3 and Corollary 2.4 are reduced to [6, Corollary 2.3] and Banach contraction principle, respectively. From this point of view, our results are genuine generalizations of the previous results. Otherwise, by the whole proof of Theorem 2.3, we are able to see that Lemma 1.10 does not be used since we dismiss the problem of whether the b -metric being continuous or discontinuous. However, some previous results strongly lie on the discontinuity of b -metric and hence they often have to make the most of Lemma 1.10 (see [10, 18–21]).

Theorem 2.6. *Under the conditions of Theorem 2.3, if $2s\lambda_1 + 2\lambda_3 + (s + s^2)(\lambda_4 + \lambda_5) < 2$, then Picard’s iteration is T -stable.*

Proof. Taking advantage of Theorem 2.3, we get T has a unique fixed point x^* in X . Assume that $\{y_n\}$ is a sequence in X such that $d(y_{n+1}, Ty_n) \rightarrow 0$ as $n \rightarrow \infty$.

Making full use of (2.2), on the one hand, we have

$$\begin{aligned} d(Ty_n, x^*) &= d(Ty_n, Tx^*) \\ &\leq \lambda_1 d(y_n, x^*) + \lambda_2 \frac{d(y_n, Ty_n)d(x^*, Tx^*)}{1 + d(y_n, x^*)} + \lambda_3 \frac{d(y_n, Tx^*)d(x^*, Ty_n)}{1 + d(y_n, x^*)} \\ &\quad + \lambda_4 \frac{d(y_n, Ty_n)d(y_n, Tx^*)}{1 + d(y_n, x^*)} + \lambda_5 \frac{d(x^*, Tx^*)d(x^*, Ty_n)}{1 + d(y_n, x^*)} \\ &\leq \lambda_1 d(y_n, x^*) + \lambda_3 d(x^*, Ty_n) + \lambda_4 d(y_n, Ty_n) \\ &\leq (\lambda_1 + s\lambda_4)d(y_n, x^*) + (\lambda_3 + s\lambda_4)d(x^*, Ty_n), \end{aligned}$$

which means that

$$(1 - \lambda_3 - s\lambda_4)d(x^*, Ty_n) \leq (\lambda_1 + s\lambda_4)d(y_n, x^*). \tag{2.8}$$

On the other hand, we have

$$\begin{aligned} d(Ty_n, x^*) &= d(Tx^*, Ty_n) \\ &\leq \lambda_1 d(x^*, y_n) + \lambda_2 \frac{d(x^*, Tx^*)d(y_n, Ty_n)}{1 + d(x^*, y_n)} + \lambda_3 \frac{d(x^*, Ty_n)d(y_n, Tx^*)}{1 + d(x^*, y_n)} \\ &\quad + \lambda_4 \frac{d(x^*, Tx^*)d(x^*, Ty_n)}{1 + d(x^*, y_n)} + \lambda_5 \frac{d(y_n, Ty_n)d(y_n, Tx^*)}{1 + d(x^*, y_n)} \\ &\leq \lambda_1 d(x^*, y_n) + \lambda_3 d(x^*, Ty_n) + \lambda_5 d(y_n, Ty_n) \\ &\leq (\lambda_1 + s\lambda_5)d(y_n, x^*) + (\lambda_3 + s\lambda_5)d(x^*, Ty_n), \end{aligned}$$

which means that

$$(1 - \lambda_3 - s\lambda_5)d(x^*, Ty_n) \leq (\lambda_1 + s\lambda_5)d(y_n, x^*). \tag{2.9}$$

Combining (2.8) and (2.9) yields

$$(2 - 2\lambda_3 - s\lambda_4 - s\lambda_5)d(x^*, Ty_n) \leq (2\lambda_1 + s\lambda_4 + s\lambda_5)d(y_n, x^*). \tag{2.10}$$

As a result, we have

$$d(x^*, Ty_n) \leq \frac{2\lambda_1 + s\lambda_4 + s\lambda_5}{2 - 2\lambda_3 - s\lambda_4 - s\lambda_5}d(y_n, x^*).$$

Denote $h = \frac{s(2\lambda_1 + s\lambda_4 + s\lambda_5)}{2 - 2\lambda_3 - s\lambda_4 - s\lambda_5}$. It follows immediately from $2s\lambda_1 + 2\lambda_3 + (s + s^2)(\lambda_4 + \lambda_5) < 2$ that $0 \leq h < 1$. Let $a_n = d(y_n, x^*)$, $c_n = sd(y_{n+1}, Ty_n)$, by (2.10), then

$$a_{n+1} = d(y_{n+1}, x^*) \leq s[d(y_{n+1}, Ty_n) + d(Ty_n, x^*)] \leq ha_n + c_n.$$

Thus, by Lemma 1.9, it leads to $a_n = d(y_n, x^*) \rightarrow 0$ ($n \rightarrow \infty$), that is, $y_n \rightarrow x^*$ ($n \rightarrow \infty$). As a consequence, Picard’s iteration is T -stable. \square

Corollary 2.7. *Under the conditions of Corollary 2.4, Picard’s iteration is T -stable.*

Proof. Let $s = 1$ in Theorem 2.6, then $2s\lambda_1 + 2\lambda_3 + (s + s^2)(\lambda_4 + \lambda_5) < 2$ becomes $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 < 1$. Noticing that Corollary 2.4 is the special case of Theorem 2.3, therefore, by Theorem 2.6, we complete the proof.

It is clear that if T is a map which has a fixed point x^* , then x^* is also a fixed point of T^n for each $n \in \mathbb{N}$. It is well known that the converse is not true. If a map T satisfies $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, then it is said to have the P property (see [7, 8]). The following results are generalizations of the corresponding results in metric spaces.

Theorem 2.8. *Let (X, d) be a b -metric space with coefficient $s \geq 1$. Let $T: X \rightarrow X$ be a mapping such that $F(T) \neq \emptyset$ and that*

$$d(Tx, T^2x) \leq \lambda d(x, Tx) \tag{2.11}$$

for all $x \in X$, where $0 \leq \lambda < 1$ is a constant. Then T has the P property.

Proof. We always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the hypotheses, it is clear that

$$\begin{aligned} d(z, Tz) &= d(TT^{n-1}z, T^2T^{n-1}z) \leq \lambda d(T^{n-1}z, T^n z) = \lambda d(TT^{n-2}z, T^2T^{n-2}z) \\ &\leq \lambda^2 d(T^{n-2}z, T^{n-1}z) \leq \dots \leq \lambda^n d(z, Tz) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Hence, $d(z, Tz) = 0$, that is., $Tz = z$. \square

Theorem 2.9. *Under the conditions of Theorem 2.3, T has the P property.*

Proof. We have to prove that the mapping T satisfies (2.11). In fact, for any $x \in X$, for one thing, we have

$$\begin{aligned}
 d(Tx, T^2x) &= d(Tx, TTx) \\
 &\leq \lambda_1 d(x, Tx) + \lambda_2 \frac{d(x, Tx)d(Tx, TTx)}{1 + d(x, Tx)} + \lambda_3 \frac{d(x, TTx)d(Tx, Tx)}{1 + d(x, Tx)} \\
 &\quad + \lambda_4 \frac{d(x, Tx)d(x, TTx)}{1 + d(x, Tx)} + \lambda_5 \frac{d(Tx, TTx)d(Tx, Tx)}{1 + d(x, Tx)} \\
 &\leq \lambda_1 d(x, Tx) + \lambda_2 d(Tx, T^2x) + \lambda_4 d(x, T^2x) \\
 &\leq (\lambda_1 + s\lambda_4)d(x, Tx) + (\lambda_2 + s\lambda_4)d(Tx, T^2x),
 \end{aligned}$$

which implies that

$$(1 - \lambda_2 - s\lambda_4)d(Tx, T^2x) \leq (\lambda_1 + s\lambda_4)d(x, Tx). \tag{2.12}$$

For another thing, we have

$$\begin{aligned}
 d(Tx, T^2x) &= d(TTx, Tx) \\
 &\leq \lambda_1 d(Tx, x) + \lambda_2 \frac{d(Tx, TTx)d(x, Tx)}{1 + d(Tx, x)} + \lambda_3 \frac{d(Tx, Tx)d(x, TTx)}{1 + d(Tx, x)} \\
 &\quad + \lambda_4 \frac{d(Tx, TTx)d(Tx, Tx)}{1 + d(Tx, x)} + \lambda_5 \frac{d(x, Tx)d(x, TTx)}{1 + d(Tx, x)} \\
 &\leq \lambda_1 d(Tx, x) + \lambda_2 d(Tx, T^2x) + \lambda_5 d(x, T^2x) \\
 &\leq (\lambda_1 + s\lambda_5)d(Tx, x) + (\lambda_2 + s\lambda_5)d(Tx, T^2x),
 \end{aligned}$$

which establishes that

$$(1 - \lambda_2 - s\lambda_5)d(Tx, T^2x) \leq (\lambda_1 + s\lambda_5)d(x, Tx). \tag{2.13}$$

On adding up (2.12) and (2.13), it follows that

$$(2 - 2\lambda_2 - s\lambda_4 - s\lambda_5)d(Tx, T^2x) \leq (2\lambda_1 + s\lambda_4 + s\lambda_5)d(x, Tx).$$

This implies that

$$d(Tx, T^2x) \leq \frac{2\lambda_1 + s\lambda_4 + s\lambda_5}{2 - 2\lambda_2 - s\lambda_4 - s\lambda_5} d(x, Tx).$$

Denote that $\lambda = \frac{2\lambda_1 + s\lambda_4 + s\lambda_5}{2 - 2\lambda_2 - s\lambda_4 - s\lambda_5}$. Note that $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$, then $\lambda < 1$. Accordingly, (2.11) is satisfied. Consequently, by Theorem 2.8, T has the P property. \square

Corollary 2.10. *Under the conditions of Corollary 2.4, T has the P property.*

Proof. Since Corollary 2.4 is the special case of Theorem 2.3, then by Theorem 2.9, we obtain the desired result. \square

Example 2.11. Let $X = [0, 1]$ and define a mapping $d: X \times X \rightarrow \mathbb{R}^+$ by $d(x, y) = |x - y|^p$ ($p \geq 1$). Taking account of (1.4), we claim that (X, d) is a b -complete b -metric space with coefficient $s = 2^{p-1}$. Define a mapping $T: X \rightarrow X$ by $Tx = e^{x-\lambda}$, where $\lambda > 1 + \ln 2$ is a constant. Then by mean value theorem of differentials, for any $x, y \in X$ and $x \neq y$, there exists some real number ξ belonging to between x and y such that

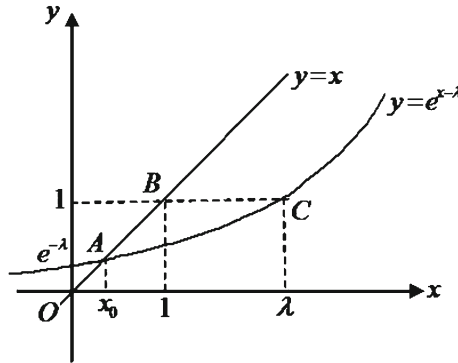


FIGURE 1. The formation of fixed point of the given mapping

$$\begin{aligned}
 d(Tx, Ty) &= |e^{x-\lambda} - e^{y-\lambda}|^p = (e^{\xi-\lambda})^p |x - y|^p \leq (e^{1-\lambda})^p d(x, y) \\
 &\leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)} \\
 &\quad + \lambda_4 \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ty)d(y, Tx)}{1 + d(x, y)},
 \end{aligned}$$

where $\lambda_1 = (e^{1-\lambda})^p$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Obviously, $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$. Hence, all the conditions of Theorem 2.3 are satisfied and T has a unique fixed point in X . See the following Fig. 1. The abscissa of point A , i.e., x_0 , is the fixed point.

Otherwise, by virtue of $\lambda > 1 + \ln 2$, then $\lambda_1 = (e^{1-\lambda})^p < 2^{1-p} = \frac{1}{s}$, so $2s\lambda_1 + 2\lambda_3 + (s + s^2)(\lambda_4 + \lambda_5) < 2$. Accordingly, all conditions of Theorem 2.6 are satisfied. By Theorem 2.6, Picard’s iteration is T -stable. Indeed, take $y_n = \frac{n}{n+1}x_0 \in X$, it follows that

$$d(y_{n+1}, Ty_n) = \left| \frac{n+1}{n+2}x_0 - e^{\frac{n}{n+1}x_0-\lambda} \right|^2 \rightarrow |x_0 - e^{x_0-\lambda}| = 0 \quad (n \rightarrow \infty).$$

Note that $y_n = \frac{n}{n+1}x_0 \rightarrow x_0$ ($n \rightarrow \infty$), the validity of Theorem 2.6 is beyond all doubt.

3. Applications

In this section, firstly, we apply Theorem 2.3 to the first-order initial value problem

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $f: [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}] \times [x_0 - \frac{k}{2}, x_0 + \frac{k}{2}] \rightarrow \mathbb{R}$ is a continuous function and $k > 1$, $r > 2$, t_0, x_0 are four real constants.

Theorem 3.1. Consider the initial value problem (3.1) and suppose that

(i) f satisfies the Lipschitz condition, i.e.,

$$|f(t, x(t)) - f(t, y(t))| \leq k|x(t) - y(t)| \tag{3.2}$$

for all $(t, x), (t, y) \in R$, where $R = \{(t, x) : |t - t_0| \leq (\frac{1}{k})^{r-1}, |x - x_0| \leq \frac{k}{2}\}$;

(ii) f is bound on R , i.e.,

$$|f(t, x)| \leq \frac{k^r}{2}. \tag{3.3}$$

Then the initial value problem (3.1) has a unique solution on the interval $I = [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}]$. Further, the solution is exhibited as follows:

$$x(t) = x_0 + \lim_{n \rightarrow \infty} \int_{t_0}^t f(\tau, x_n(\tau))d\tau,$$

where

$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^t f(\tau, x_{n-1}(\tau))d\tau \quad (n = 1, 2, \dots).$$

Proof. Let $C(I)$ be the set of all continuous functions on I . Let $X = \{x \in C(I) : |x(t) - x_0| \leq \frac{k}{2}\}$. Define a mapping $d: X \times X \rightarrow \mathbb{R}^+$ by

$$d(x, y) = \max_{t \in I} |x(t) - y(t)|^2. \tag{3.4}$$

Clearly, $(C(I), d)$ is a b -complete b -metric space with coefficient $s = 2$. Since X is a closed subspace of $C(I)$, then (X, d) is a b -complete b -metric space with coefficient $s = 2$.

Integrating (3.1), we have

$$x(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau))d\tau. \tag{3.5}$$

As a consequence, finding the solution of (3.1) is equivalent to finding the fixed point of mapping $T: X \rightarrow X$ defined by

$$Tx(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau))d\tau. \tag{3.6}$$

Note that if $\tau \in I$ then $|\tau - t_0| \leq (\frac{1}{k})^{r-1}$, and $x \in X$ means $|x(\tau) - x_0| \leq \frac{k}{2}$, so $(\tau, x(\tau)) \in R$. Since f is continuous on R , then the integral (3.6) exists and T is well-defined for all $x \in X$.

We make a conclusion that T is a self-mapping on X . Indeed, making full use of (3.3) and (3.6), it follows that

$$\begin{aligned} |Tx(t) - x_0| &= \left| \int_{t_0}^t f(\tau, x(\tau))d\tau \right| \\ &\leq \int_{t_0}^t |f(\tau, x(\tau))|d\tau \\ &\leq \frac{k^r}{2}|t - t_0| \end{aligned}$$

$$\begin{aligned} &\leq \frac{k^r}{2} \left(\frac{1}{k}\right)^{r-1} \\ &= \frac{k}{2}. \end{aligned}$$

Next, by using (3.2), (3.4) and (3.6), we get

$$\begin{aligned} |Tx(t) - Ty(t)|^2 &= \left| \int_{t_0}^t [f(\tau, x(\tau)) - f(\tau, y(\tau))]d\tau \right|^2 \\ &\leq \left[\int_{t_0}^t |f(\tau, x(\tau)) - f(\tau, y(\tau))|d\tau \right]^2 \\ &\leq \left[\int_{t_0}^t k|x(\tau) - y(\tau)|d\tau \right]^2 \\ &\leq k^2 \left[\int_{t_0}^t \max_{\tau \in I} |x(\tau) - y(\tau)|d\tau \right]^2 \\ &= k^2 \max_{\tau \in I} |x(\tau) - y(\tau)|^2 |t - t_0|^2 \\ &\leq k^2 \max_{\tau \in I} |x(\tau) - y(\tau)|^2 \left(\frac{1}{k}\right)^{2r-2} \\ &= \left(\frac{1}{k}\right)^{2r-4} \max_{\tau \in I} |x(\tau) - y(\tau)|^2 \\ &= \left(\frac{1}{k}\right)^{2r-4} d(x, y), \end{aligned}$$

which establishes that

$$\begin{aligned} d(Tx, Ty) &\leq \left(\frac{1}{k}\right)^{2r-4} d(x, y) \\ &\leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)} \\ &\quad + \lambda_4 \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ty)d(y, Tx)}{1 + d(x, y)}, \end{aligned}$$

where $\lambda_1 = \left(\frac{1}{k}\right)^{2r-4}$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Because $k > 1$ and $r > 2$, it means that $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$.

Owing to the above statement, all conditions of Theorem 2.3 are satisfied. Hence, T has a unique fixed point. That is to say, there exists a unique solution to (3.1).

Now, by utilizing successive approximation method, we find the unique solution of (3.1). For this purpose, put $x_0(t) = x_0$ and

$$x_n(t) = x_0 + \int_{t_0}^t f(\tau, x_{n-1}(\tau))d\tau. \tag{3.7}$$

It is easy to see that

$$x_{n-1}(t) = x_0 + \int_{t_0}^t f(\tau, x_{n-2}(\tau))d\tau. \tag{3.8}$$

Combining (3.7) and (3.8), we deduce

$$x_n(t) - x_{n-1}(t) = \int_{t_0}^t [f(\tau, x_{n-1}(\tau)) - f(\tau, x_{n-2}(\tau))]d\tau. \tag{3.9}$$

Letting

$$y_n(t) = x_n(t) - x_{n-1}(t), \quad y_0(t) = x_0, \tag{3.10}$$

we get

$$x_n(t) = \sum_{i=0}^n y_i(t).$$

Using (3.2), (3.9) and (3.10), we have

$$|y_n(t)| \leq k \int_{t_0}^t |y_{n-1}(\tau)|d\tau.$$

Further, we have

$$|y_n(t)| \leq \frac{|x_0|k^n}{n!} |t - t_0|^n.$$

Since the series $\sum_{n=1}^{\infty} \frac{|x_0|k^n}{n!} |t - t_0|^n$ is convergent in I , then the series $\sum_{n=1}^{\infty} y_n(t)$ is convergent to some function $x(t)$, that is., $x_n(t) = \sum_{i=0}^n y_i(t)$ converges to $x(t)$ as $n \rightarrow \infty$.

In the following, we show that $x(t) = \sum_{n=0}^{\infty} y_n(t)$ is the solution of (3.5). This implies that $x(t) = \sum_{n=0}^{\infty} y_n(t)$ is also the solution of (3.1). For this reason, assume that

$$x(t) = x_n(t) + \Delta_n(t). \tag{3.11}$$

It is valid that $\lim_{n \rightarrow \infty} |\Delta_n(t)| = 0$. Combining (3.7) and (3.11), we get

$$x(t) - \Delta_n(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau) - \Delta_{n-1}(\tau))d\tau.$$

Thus, we arrive at

$$\begin{aligned} x(t) - x_0 - \int_{t_0}^t f(\tau, x(\tau))d\tau \\ = \Delta_n(t) + \int_{t_0}^t [f(\tau, x(\tau) - \Delta_{n-1}(\tau)) - f(\tau, x(\tau))]d\tau. \end{aligned} \tag{3.12}$$

By (3.2) and (3.12) it yields that

$$\begin{aligned} \left| x(t) - x_0 - \int_{t_0}^t f(\tau, x(\tau))d\tau \right| \\ = \left| \Delta_n(t) + \int_{t_0}^t [f(\tau, x(\tau) - \Delta_{n-1}(\tau)) - f(\tau, x(\tau))]d\tau \right| \end{aligned}$$

$$\begin{aligned}
 &\leq |\Delta_n(t)| + \int_{t_0}^t |f(\tau, x(\tau) - \Delta_{n-1}(\tau)) - f(\tau, x(\tau))|d\tau \\
 &\leq |\Delta_n(t)| + \int_{t_0}^t k|\Delta_{n-1}(\tau)|d\tau \\
 &\leq |\Delta_n(t)| + k \left(\frac{1}{k}\right)^{r-1} \max_{\tau \in I} |\Delta_{n-1}(\tau)|.
 \end{aligned} \tag{3.13}$$

Taking the limit as $n \rightarrow \infty$ from both sides of (3.13) and noting $\lim_{n \rightarrow \infty} |\Delta_n(t)| = 0$, it is evident that

$$x(t) - x_0 - \int_{t_0}^t f(\tau, x(\tau))d\tau = 0.$$

Hence, we deduce

$$x(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau))d\tau.$$

Then $x(t) = \sum_{n=0}^{\infty} y_n(t)$ is the solution of (3.5). In other words, $x(t) = \sum_{n=0}^{\infty} y_n(t)$ is the solution of (3.1).

Finally, we look for the mathematical expression of the solution of (3.1). To this end, taking advantage of (3.9) and (3.10), we obtain

$$\begin{aligned}
 x(t) &= \sum_{n=0}^{\infty} y_n(t) \\
 &= y_0(t) + y_1(t) + \sum_{n=2}^{\infty} y_n(t) \\
 &= y_0(t) + y_1(t) + \sum_{n=2}^{\infty} [x_n(t) - x_{n-1}(t)] \\
 &= x_0 + x_1(t) - x_0 + \sum_{n=2}^{\infty} \int_{t_0}^t [f(\tau, x_{n-1}(\tau)) - f(\tau, x_{n-2}(\tau))]d\tau \\
 &= x_1(t) + \sum_{n=2}^{\infty} \int_{t_0}^t [f(\tau, x_{n-1}(\tau)) - f(\tau, x_{n-2}(\tau))]d\tau \\
 &= x_0 + \int_{t_0}^t f(\tau, x_0)d\tau + \int_{t_0}^t \sum_{n=2}^{\infty} [f(\tau, x_{n-1}(\tau)) - f(\tau, x_{n-2}(\tau))]d\tau \\
 &= x_0 + \int_{t_0}^t f(\tau, x_0)d\tau + \int_{t_0}^t \lim_{n \rightarrow \infty} f(\tau, x_{n-1}(\tau))d\tau - \int_{t_0}^t f(\tau, x_0)d\tau \\
 &= x_0 + \lim_{n \rightarrow \infty} \int_{t_0}^t f(\tau, x_n(\tau))d\tau,
 \end{aligned} \tag{3.14}$$

where $x_n(t)$ ($n = 1, 2, \dots$) is written by (3.7). □

Remark 3.2. In the proof of Theorem 3.1, (3.14) can also be obtained by the continuity of f . Actually, since f is continuous and $x(t) = \lim_{n \rightarrow \infty} x_n(t)$ is the solution of (3.1), then by (3.5), we have

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t f(\tau, x(\tau))d\tau \\ &= x_0 + \int_{t_0}^t f\left(\tau, \lim_{n \rightarrow \infty} x_n(\tau)\right) d\tau \\ &= x_0 + \lim_{n \rightarrow \infty} \int_{t_0}^t f(\tau, x_n(\tau))d\tau. \end{aligned}$$

Secondly, we look for the general solution of n th-order nonhomogeneous differential equation. In order to start this purpose, we give the following n th-order initial value problem:

$$\begin{cases} \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots \\ \quad + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x), \quad x \in [0, x_0], \\ y(0) = C_0, y'(0) = C_1, y''(0) = C_2, \dots, y^{(n-1)}(0) = C_{n-1}, \end{cases} \tag{3.15}$$

where $a_{n-1}(x), a_{n-2}(x), \dots, a_1(x), a_0(x) \in C([0, x_0])$ (the set of all continuous real functions defined on $[0, x_0]$) are given, and $C_0, C_1, C_2, \dots, C_{n-1}$ are constants.

Theorem 3.3. Consider initial value problem (3.15), and set

$$M = \max_{0 \leq t, x \leq x_0} \left| \sum_{k=0}^{n-1} \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} \right|.$$

If $x_0 M < 1$, then (3.15) has a unique solution in $C([0, x_0])$. Further, the solution is exhibited as follows:

$$y = \frac{1}{(n-1)!} \sum_{i=0}^{\infty} \int_0^x (x-t)^{n-1} u_i(t) dt + \sum_{k=0}^{n-1} \frac{C_k}{k!} x^k,$$

where

$$\begin{aligned} u_0(x) &= f(x) - \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_j(x) \frac{C_{k+j}}{k!} x^k, \\ u_i(x) &= - \sum_{k=0}^{n-1} \int_0^x \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} u_{i-1}(t) dt, \quad i = 1, 2, \dots \end{aligned}$$

Proof. Put

$$u(x) = \frac{d^n y}{dx^n}, \quad p(x) = \frac{d^{n-1} y}{dx^{n-1}}, \quad q(x) = \frac{d^{n-2} y}{dx^{n-2}}, \quad r(x) = \frac{d^{n-3} y}{dx^{n-3}},$$

then $u(x), p(x), q(x), r(x) \in C([0, x_0])$. Considering the initial conditions, we get that

$$\frac{d^{n-1} y}{dx^{n-1}} = \int_0^x u(t) dt + C_{n-1}, \tag{3.16}$$

and

$$\begin{aligned}
 \frac{d^{n-2}y}{dx^{n-2}} &= \int_0^x p(s)ds + C_{n-2} \\
 &= \int_0^x \left[\int_0^s u(t)dt + C_{n-1} \right] ds + C_{n-2} \\
 &= \int_0^x \int_0^s u(t)dt ds + C_{n-1}x + C_{n-2} \\
 &= \int_0^x dt \int_t^x u(t)ds + C_{n-1}x + C_{n-2} \\
 &= \int_0^x (x-t)u(t)dt + C_{n-1}x + C_{n-2}, \tag{3.17}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d^{n-3}y}{dx^{n-3}} &= \int_0^x q(s)ds + C_{n-3} \\
 &= \int_0^x \left[\int_0^s (s-t)u(t)dt + C_{n-1}s + C_{n-2} \right] ds + C_{n-3} \\
 &= \int_0^x \int_0^s (s-t)u(t)dt ds + \int_0^x (C_{n-1}s + C_{n-2})ds + C_{n-3} \\
 &= \int_0^x dt \int_t^x (s-t)u(t)ds + \frac{1}{2}C_{n-1}x^2 + C_{n-2}x + C_{n-3} \\
 &= \frac{1}{2} \int_0^x (x-t)^2u(t)dt + \frac{1}{2}C_{n-1}x^2 + C_{n-2}x + C_{n-3}, \tag{3.18}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d^{n-4}y}{dx^{n-4}} &= \int_0^x r(s)ds + C_{n-4} \\
 &= \int_0^x \left[\frac{1}{2} \int_0^s (s-t)^2u(t)dt + \frac{1}{2}C_{n-1}s^2 + C_{n-2}s + C_{n-3} \right] ds \\
 &\quad + C_{n-4} \\
 &= \frac{1}{2} \int_0^x \int_0^s (s-t)^2u(t)dt ds + \int_0^x \left(\frac{1}{2}C_{n-1}s^2 + C_{n-2}s + C_{n-3} \right) ds \\
 &\quad + C_{n-4} \\
 &= \frac{1}{2} \int_0^x dt \int_t^x (s-t)^2u(t)ds + \frac{1}{6}C_{n-1}x^3 + \frac{1}{2}C_{n-2}x^2 + C_{n-3}x + C_{n-4} \\
 &= \frac{1}{6} \int_0^x (x-t)^3u(t)dt + \frac{1}{6}C_{n-1}x^3 + \frac{1}{2}C_{n-2}x^2 + C_{n-3}x + C_{n-4}. \tag{3.19}
 \end{aligned}$$

By mathematical induction, we arrive at

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{(n-2)!} \int_0^x (x-t)^{n-2}u(t)dt + \frac{1}{(n-2)!}C_{n-1}x^{n-2} + \frac{1}{(n-3)!}C_{n-2}x^{n-3} \\
 &\quad + \dots + \frac{1}{6}C_4x^3 + \frac{1}{2}C_3x^2 + C_2x + C_1, \tag{3.20}
 \end{aligned}$$

and

$$y = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} u(t) dt + \frac{1}{(n-1)!} C_{n-1} x^{n-1} + \frac{1}{(n-2)!} C_{n-2} x^{n-2} + \dots + \frac{1}{6} C_3 x^3 + \frac{1}{2} C_2 x^2 + C_1 x + C_0. \tag{3.21}$$

Substituting (3.16)–(3.21) into (3.15), we have

$$\begin{aligned} & u(x) + a_{n-1}(x) \left[\int_0^x u(t) dt + C_{n-1} \right] \\ & + a_{n-2}(x) \left[\int_0^x (x-t)u(t) dt + C_{n-1}x + C_{n-2} \right] \\ & + a_{n-3}(x) \left[\frac{1}{2} \int_0^x (x-t)^2 u(t) dt + \frac{1}{2} C_{n-1}x^2 + C_{n-2}x + C_{n-3} \right] \\ & + a_{n-4}(x) \left[\frac{1}{6} \int_0^x (x-t)^3 u(t) dt + \frac{1}{6} C_{n-1}x^3 + \frac{1}{2} C_{n-2}x^2 \right. \\ & \left. + C_{n-3}x + C_{n-4} \right] \\ & + \dots \\ & + a_1(x) \left[\frac{1}{(n-2)!} \int_0^x (x-t)^{n-2} u(t) dt + \frac{1}{(n-2)!} C_{n-1}x^{n-2} \right. \\ & \left. + \frac{1}{(n-3)!} C_{n-2}x^{n-3} + \dots + \frac{1}{6} C_4 x^3 + \frac{1}{2} C_3 x^2 + C_2 x + C_1 \right] \\ & + a_0(x) \left[\frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} u(t) dt + \frac{1}{(n-1)!} C_{n-1}x^{n-1} \right. \\ & \left. + \frac{1}{(n-2)!} C_{n-2}x^{n-2} + \dots + \frac{1}{6} C_3 x^3 + \frac{1}{2} C_2 x^2 + C_1 x + C_0 \right] \\ & = f(x). \end{aligned}$$

As a consequence, we have

$$\begin{aligned} f(x) &= u(x) + \int_0^x \left[a_{n-1}(x) + a_{n-2}(x)(x-t) + \frac{a_{n-3}(x)}{2}(x-t)^2 \right. \\ & \left. + \frac{a_{n-4}(x)}{6}(x-t)^3 + \dots + \frac{a_1(x)}{(n-2)!}(x-t)^{n-2} \right. \\ & \left. + \frac{a_0(x)}{(n-1)!}(x-t)^{n-1} \right] u(t) dt \\ & + a_{n-1}(x)C_{n-1} + a_{n-2}(x)(C_{n-1}x + C_{n-2}) + a_{n-3}(x) \\ & \times \left(\frac{1}{2} C_{n-1}x^2 + C_{n-2}x + C_{n-3} \right) \\ & + a_{n-4}(x) \left(\frac{1}{6} C_{n-1}x^3 + \frac{1}{2} C_{n-2}x^2 + C_{n-3}x + C_{n-4} \right) \\ & + \dots \\ & + a_1(x) \left(\frac{1}{(n-2)!} C_{n-1}x^{n-2} + \frac{1}{(n-3)!} C_{n-2}x^{n-3} + \dots \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{6}C_4x^3 + \frac{1}{2}C_3x^2 + C_2x + C_1 \Big) \\
 & + a_0(x) \left(\frac{1}{(n-1)!}C_{n-1}x^{n-1} + \frac{1}{(n-2)!}C_{n-2}x^{n-2} + \dots \right. \\
 & \left. + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_1x + C_0 \right) \\
 = & u(x) + \int_0^x \sum_{k=0}^{n-1} \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} u(t) dt + a_0(x) \sum_{k=0}^{n-1} \frac{C_k}{k!} x^k \\
 & + a_1(x) \sum_{k=0}^{n-2} \frac{C_{k+1}}{k!} x^k + a_2(x) \sum_{k=0}^{n-3} \frac{C_{k+2}}{k!} x^k \\
 & + \dots + a_{n-4}(x) \sum_{k=0}^3 \frac{C_{n-4+k}}{k!} x^k \\
 & + a_{n-3}(x) \sum_{k=0}^2 \frac{C_{n-3+k}}{k!} x^k + a_{n-2}(x) \sum_{k=0}^1 \frac{C_{n-2+k}}{k!} x^k \\
 & + a_{n-1}(x) \sum_{k=0}^0 \frac{C_{n-1+k}}{k!} x^k \\
 = & u(x) + \sum_{k=0}^{n-1} \int_0^x \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} u(t) dt \\
 & + \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_j(x) \frac{C_{k+j}}{k!} x^k.
 \end{aligned}$$

Then we speculate that

$$\begin{aligned}
 u(x) & = f(x) - \sum_{k=0}^{n-1} \int_0^x \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} u(t) dt \\
 & \quad - \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_j(x) \frac{C_{k+j}}{k!} x^k \\
 & = - \int_0^x \sum_{k=0}^{n-1} \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k} u(t) dt \\
 & \quad + f(x) - \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_j(x) \frac{C_{k+j}}{k!} x^k. \tag{3.22}
 \end{aligned}$$

Take

$$K(x, t) = - \sum_{k=0}^{n-1} \frac{a_k(x)}{(n-1-k)!} (x-t)^{n-1-k},$$

$$F(x) = f(x) - \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_j(x) \frac{C_{k+j}}{k!} x^k. \tag{3.23}$$

Now by (3.22) and (3.23), we claim that (3.15) is equivalent to the following Volterra-type integral equation:

$$u(x) = \int_0^x K(x, t)u(t)dt + F(x).$$

Let $X = C([0, x_0])$. Put $d: X \times X \rightarrow \mathbb{R}^+$ as $d(u, v) = \max_{0 \leq x \leq x_0} |u(x) - v(x)|^2$. It is valid that (X, d) is a b -complete b -metric space with coefficient $s = 2$.

Define a mapping $T: X \rightarrow X$ by

$$Tu(x) = \int_0^x K(x, t)u(t)dt + F(x).$$

For any $u, v \in X$, we have

$$\begin{aligned} d(Tu, Tv) &= \max_{0 \leq x \leq x_0} \left| \int_0^x K(x, t)u(t)dt - \int_0^x K(x, t)v(t)dt \right|^2 \\ &= \max_{0 \leq x \leq x_0} \left| \int_0^x K(x, t)[u(t) - v(t)]dt \right|^2 \\ &\leq x_0^2 M^2 \max_{0 \leq t \leq x_0} |u(t) - v(t)|^2 = x_0^2 M^2 d(u, v) \\ &\leq \lambda_1 d(u, v) + \lambda_2 \frac{d(u, Tu)d(v, Tv)}{1 + d(u, v)} + \lambda_3 \frac{d(u, Tv)d(v, Tu)}{1 + d(u, v)} \\ &\quad + \lambda_4 \frac{d(u, Tu)d(u, Tv)}{1 + d(u, v)} + \lambda_5 \frac{d(v, Tv)d(v, Tu)}{1 + d(u, v)}, \end{aligned}$$

where $\lambda_1 = x_0^2 M^2$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

By virtue of $x_0 M < 1$, then $\lambda_1 < 1$, so $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$. Owing to the above statement, all conditions of Theorem 2.3 are satisfied, then by Theorem 2.3, T has a unique fixed point in X . That is to say, the initial value problem (3.15) has a unique solution in $C([0, x_0])$.

Eventually, we look for the expression of solution. For this purpose, take

$$y_n(x) = \sum_{i=0}^n u_i(x), \quad x \in [0, x_0],$$

where

$$u_0(x) = F(x), \quad u_i(x) = \int_0^x K(x, t)u_{i-1}(t)dt, \quad i = 1, 2, \dots$$

Note that

$$\begin{aligned} y_{n+1}(x) &= u_0(x) + \sum_{i=1}^{n+1} u_i(x) \\ &= F(x) + \sum_{i=1}^{n+1} \int_0^x K(x, t)u_{i-1}(t)dt \end{aligned}$$

$$\begin{aligned}
&= F(x) + \int_0^x \left[K(x, t) \sum_{i=1}^{n+1} u_{i-1}(t) \right] dt \\
&= F(x) + \int_0^x \left[K(x, t) \sum_{i=0}^n u_i(t) \right] dt \\
&= F(x) + \int_0^x K(x, t) y_n(t) dt \\
&= T y_n(x)
\end{aligned}$$

for any $n \in \mathbb{N}$, so $y_{n+1} = T y_n$ is a Picard's iteration. Based on the proof of Theorem 2.3, it is not hard to verify that $\{y_n\}$ b -converges to the fixed point $u(x)$ of T . In other words,

$$u(x) = \lim_{n \rightarrow \infty} y_n(x) = \sum_{i=0}^{\infty} u_i(x).$$

Substituting $u(x) = \sum_{i=0}^{\infty} u_i(x)$ into (3.21), we easily claim that the solution of (3.15) is the following form:

$$y = \frac{1}{(n-1)!} \sum_{i=0}^{\infty} \int_0^x (x-t)^{n-1} u_i(t) dt + \sum_{k=0}^{n-1} \frac{C_k}{k!} x^k.$$

□

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References

- [1] Asadi, M., Soleimani, H., Vaezpour, S.M., Rhoades, B.E.: On T -stability of Picard iteration in cone metric spaces. *Fixed Point Theory Appl.* **2009**, 751090-1–751090-6 (2009)
- [2] Bakhtin, I.A.: The contraction mapping principle in almost metric space (Russian). *Funct. Anal. Ulyanovsk. Gos. Ped. Inst. Ulyanovsk* **30**, 26–37 (1989)
- [3] Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations, intégrales. *Fundam. Math.* **3**, 133–181 (1922)
- [4] Boriceanu, M., Bota, M., Petruşel, A.: Multivalued fractals in b -metric spaces. *Cent. Eur. J. Math.* **8**(2), 367–377 (2010)
- [5] Czerwik, S.: Contraction mappings in b -metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1**, 5–11 (1993)
- [6] Huang, H., Radenović, S., Deng, G.: A sharp generalization on cone b -metric space over Banach algebra. *J. Nonlinear Sci. Appl.* **10**(2), 429–435 (2017)

- [7] Huang, H., Xu, S., Liu, H., Radenović, S.: Fixed point theorems and T -stability of Picard iteration for generalized Lipschitz mappings in cone metric spaces over Banach algebras. *J. Comput. Anal. Appl.* **20**(5), 869–888 (2016)
- [8] Jeong, G.S., Rhoades, B.E.: Maps for which $F(T) = F(T^n)$. *Fixed Point Theory Appl.* **6**, 71–105 (2005)
- [9] Jovanović, M., Kadelburg, Z., Radenović, S.: Common fixed point results in metric-type spaces. *Fixed Point Theory Appl.* **2010**, 978121 (2010)
- [10] Latif, A., Parvaneh, V., Salimi, P., Al-Mazrooei, A.E.: Various Suzuki type theorems in b -metric spaces. *J. Nonlinear Sci. Appl.* **8**(4), 363–377 (2015)
- [11] Liu, Q.: A convergence theorem of the sequence of Ishikawa iterates for quasi-contractive mappings. *J. Math. Anal. Appl.* **146**(2), 301–305 (1990)
- [12] Miculescu, R., Mihail, A.: New fixed point theorems for set-valued contractions in b -metric spaces. *J. Fixed Point Theory Appl.* **19**(3), 2153–2163 (2017)
- [13] Osilike, M.O.: Stability results for fixed point iteration procedures. *J. Niger. Math. Soc.* **14–15**, 17–29 (1995)
- [14] Qing, Y., Rhoades, B.E.: T -Stability of Picard iteration in metric spaces. *Fixed Point Theory Appl.* **2008**, 418971 (2008)
- [15] Reich, S., Zaslavski, A.J.: Generic well-posedness of fixed point problems. *Vietnam J. Math.* **46**(1), 5–13 (2018)
- [16] Reich, S., Zaslavski, A.J.: *Genericity in Nonlinear Analysis*, pp. 157–159. Springer, New York (2014)
- [17] Reich, S., Zaslavski, A.J.: Well-posedness of fixed point problems. *Far East J. Math. Sci. Special Volume, Part III*, 393–401 (2001)
- [18] Sintunavarat, W.: Fixed point results in b -metric spaces approach to the existence of a solution for nonlinear integral equations. *R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **110**, 585–600 (2016)
- [19] Sintunavarat, W.: Nonlinear integral equations with new admissibility types in b -metric spaces. *J. Fixed Point Theory Appl.* **18**(2), 397–416 (2016)
- [20] Yamaod, O., Sintunavarat, W., Cho, Y.J.: Common fixed point theorems for generalized cyclic contraction pairs in b -metric spaces with applications. *Fixed Point Theory Appl.* **2015**, 164 (2015)
- [21] Yamaod, O., Sintunavarat, W., Cho, Y.J.: Existence of a common solution for a system of nonlinear integral equations via fixed point methods in b -metric spaces. *Open Math.* **14**, 128–145 (2016)

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