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Relation-theoretic metrical fixed-point results via w-distance with applications

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Abstract. In this article, utilizing the concept of w-distance, we prove the celebrated Banach's fixed-point theorem in metric spaces equipped with an arbitrary binary relation. Necessarily, our findings unveil another direction of relation-theoretic metrical fixed-point theory. In addition, our paper consists of several non-trivial examples which signify the motivation of such investigations. Finally, our obtained results enable us to explore the existence of solutions of nonlinear fractional differential equations and fractional thermostat model involving the Caputo fractional derivative.

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Keywords. Complete metric space, binary relation, w-distance, fixed-point, nonlinear fractional differential equation, fractional thermostat model.

1. Introduction

On account of the fact that the metric fixed-point theory imparts a sound basis for exploring many problems in pure and applied sciences, many authors went into the possibility of altering the concepts of metric and metric spaces. One such interesting and important motivation is to establish fixed-point results in metric space endowing with an arbitrary binary relation. Exploiting the concepts of different kind of binary relations such as partial order, strict order, preorder, tolerance, transitive etc. on metric spaces, many mathematician are doing their research during several years, see [3,4,10,13–15]. Very recently, Alam and Imdad [2] presented relation-theoretic metrical fixed-point results due to famous Banach contraction principle using an amorphous relation. No doubt their results extended and improved several comparable results in existing literature, but still, there are some cases where we cannot explain the existence of fixed point employing their results. In this direction, our main goal is to present some improved and refined version of existing results using the concept of w-distance. Due to reader's advantage, we need to recall some important definitions and useful results relevant to this literature.

Throughout this article, the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ have their usual meanings.

Definition 1.1 [7]. Let X be a non-empty set and $\mathcal{R}$ be a binary relation defined on $X \times X$. Then, x is $\mathcal{R}$-related to y if and only if $(x,y) \in \mathcal{R}$.

Definition 1.2 [8]. A binary relation $\mathcal{R}$ defined on X is said to be complete if for all $x, y \in X$, $[x,y] \in \mathcal{R}$, where $[x,y] \in \mathcal{R}$ stands for either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$.

Definition 1.3 [2]. Suppose $\mathcal{R}$ is a binary relation defined on a non-empty set X. Then a sequence (x_n) in X is said to be $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Definition 1.4 [2]. A metric space (X,d) endowed with a binary relation $\mathcal{R}$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in X.

Definition 1.5 [2]. Let X be a non-empty set and f be a self-map defined on X. Then, a binary relation $\mathcal{R}$ on X is said to be f-closed if $(x,y) \in \mathcal{R}$ $\Rightarrow (fx, fy) \in \mathcal{R}$.

Here, we introduce the notion of weak f-closed binary relation.

Definition 1.6. Let X be a non-empty set and f be a self-map defined on X. Then, a binary relation $\mathcal{R}$ on X is said to be weak f-closed if $(x,y) \in \mathcal{R}$ $\Rightarrow [fx, fy] \in \mathcal{R}$.

It is easy to show that every f-closed binary relation $\mathcal{R}$ is weak f-closed but the converse is not true in general. To show this we present the following example.

Example 1.7. Let $X \neq \phi$ be a finite set and $\mathcal{R}$ be a binary relation defined on $\mathcal{P}(X)$, the power set of X, such that $(A,B) \in \mathcal{R}$ if $A \subseteq B$ for some $A, B \in \mathcal{P}(X)$. Now, we define a function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(A) = A^c$, for all $A \in \mathcal{P}(X)$. Then is easy to check that for all $A, B \in \mathcal{P}(X)$ with $(A,B) \in \mathcal{R}$, $(f(A), f(B)) \notin \mathcal{R}$ but $(f(B), f(A)) \in \mathcal{R}$. Hence, the binary relation $\mathcal{R}$ is not f-closed, but it is weak f-closed.

Definition 1.8 [2]. Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$. Then, $\mathcal{R}$ is said to be d-self-closed; if every $\mathcal{R}$-preserving sequence with $x_n \to x$, there is a subsequence (x_{n_k}) of (x_n), such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N} \cup \{0\}$.

For the sake of reader's perception, we recollect some notations from existing literature:

(A) $F(T) = \{x \in X : Tx = x\}$;
(B) $X(T, \mathcal{R}) = \{x \in X : (x, Tx) \in \mathcal{R}\}$.

Before proceeding further, we record the following results.

Theorem 1.9 (Theorem 3.1, Alam and Imdad [2]). *Let (X, d) be a complete metric space equipped with a binary relation $\mathcal{R}$. Suppose T is a self-mapping on X, such that*

(1) $X(T, \mathcal{R}) \neq \phi$;
(2) $\mathcal{R}$ *is* T*-closed;*
(3) *either* T *is continuous or* $\mathcal{R}$ *is* d*-self-closed;*
(4) *there exists* $k \in [0, 1)$*, such that*

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then, $F(T) \neq \phi$.

Theorem 1.10 (Theorem 2.1, Ahmadullah et al. [1]). *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$. Suppose T is a self-mapping on X with the following conditions:*

(1) *There exists* $Y \subseteq X, TX \subseteq Y \subseteq X$*, such that* (Y, d) *is* $\mathcal{R}$*-complete.*
(2) $X(T, \mathcal{R}) \neq \phi$.
(3) $\mathcal{R}$ *is* T*-closed.*
(4) *Either* T *is* $\mathcal{R}$*-continuous or* $\mathcal{R}|_Y$ *is* d*-self-closed.*
(5) *There exists* $\phi \in \Phi$*, such that*

$$d(Tx, Ty) \leq \phi(M_T(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R},$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Then, $F(T) \neq \phi$.

Next, we would like to draw the reader's attention in another direction of metric fixed-point theory. In 1996, Kada et al. [5] introduced the idea of w-distance in metric spaces and established several well-known results using this concept. They defined the w-distance as follows:

Definition 1.11 [5]. Let (X, d) be a metric space. A function p: $X \times X \to [0, \infty)$ is said to be a w-distance if

(w1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(w2) for any $x \in X, p(x, .) : X \to [0, \infty)$ is lower semi-continuous;
(w3) for any $\epsilon > 0$, there exists $\delta > 0$, such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Remark 1.12. Note that a w-distance function p may not be symmetric and also it is possible that $p(x, x) \neq 0$ for some x, i.e., $p(x, y) = 0$ does not imply $x = y$.

The readers are refereed to [5] for some examples and crucial properties of w-distance.

To establish fixed-point results owing to w-distance in metric spaces equipped with arbitrary binary relation $\mathcal{R}$, we need to define the concept of $\mathcal{R}$-lower semi-continuity (briefly, $\mathcal{R}$-LSC) of a function, and then, we show that notion of $\mathcal{R}$-LSC is weaker than $\mathcal{R}$-continuity as well as lower semi-continuity.

Before defining $\mathcal{R}$-lower semi-continuity, we look back on $\mathcal{R}$-continuity of a function defined on a metric space equipped with an arbitrary binary relation $\mathcal{R}$.

Definition 1.13. [2] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation defined on X. A function $f : X \to X$ is said to be $\mathcal{R}$-continuous at x if for every $\mathcal{R}$-preserving sequence (x_n) converging to x, we get

$$f(x_n) \to f(x) \quad \text{as}\, n \to \infty.$$

The notion of $\mathcal{R}$-lower semi-continuity of a function is defined as follows:

Definition 1.14. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation defined on X. A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is said to be $\mathcal{R}$-LSC at x if for every $\mathcal{R}$-preserving sequence (x_n) converging to x, we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

The following example shows that $\mathcal{R}$-LSC is weaker than $\mathcal{R}$-continuity.

Example 1.15. Let (X, d) be a usual metric space where $X = \mathbb{R}$. Define $(x, y) \in \mathcal{R}$ if $x, y \in [n, n + \frac{1}{3})$ for some $n \in \mathbb{Z}$. For every $x \in X$, we can always find an integer $n \in \mathbb{Z}$, such that $x \in [n, n+1]$. Let us define a function $f : X \to X$ by

$$f(x) = \begin{cases} \lceil x \rceil & x \in [n, n + \frac{1}{3}); \\ x - 1 & \text{otherwise}. \end{cases}$$

We claim that this function is not $\mathcal{R}$-continuous, but it is $\mathcal{R}$-lower semi-continuous. Let (x_n) be a non-constant $\mathcal{R}$-preserving sequence converging to an integer k. Then, there exists some $n_0 \in \mathbb{N}$, such that $x_n \in (k, k + \frac{1}{3})$ for all $n > n_0$. Therefore, we have $\lim_{n\to\infty} f(x_n) = k + 1$ and $f(k) = k$. This implies that f is not $\mathcal{R}$-continuous, but

$$\liminf_{n\to\infty} f(x_n) \geq f(k)$$

for every $\mathcal{R}$-preserving sequence (x_n) converging to k. This shows that f is an $\mathcal{R}$-lower semi-continuous function. Indeed, this function is not also lower semi-continuous function. Let us consider (x_n) be a non constant sequence converging to k from left. Then, we must have some $n_k \in \mathbb{N}$, such that $x_n > (k - 1) + \frac{1}{3}$ for all $n \geq n_k$. This implies $f(x_n) = x_n - 1$ for all $n \geq n_k$ and $\lim_{n\to\infty} f(x_n) = k - 1$. Hence, we cannot obtain

$$\liminf_{n\to\infty} f(x_n) \geq f(k)$$

whenever $x_n \to k$. Therefore, f is not lower semi-continuous function.

Example 1.16. Let $X = [0, \infty)$ and d be the usual metric on X. We define $(x, y) \in \mathcal{R}$ if $xy \geq x$ or y. Let $f : X \to X$ be defined as

$$f(x) = \begin{cases} \frac{1}{2} & x \in [0, 1); \\ \frac{3}{4} & x = 1; \\ x & x > 1. \end{cases}$$

We show that this function is neither lower semi-continuous nor $\mathcal{R}$-continuous, but it is an $\mathcal{R}$-lower semi-continuous function. We consider the point $x = 1$. Let (x_n) be a non-constant sequence converging to 1. If $x_n \to 1$ from left, we have $f(x_n) = \frac{1}{2}$ for all $n \in \mathbb{N}$. Again, if $x_n \to 1$ from right, then we have $f(x_n) = x_n$ for all $n \in \mathbb{N}$ which implies that $\lim_{n\to\infty} f(x_n) = 1$. Hence, we can check that

$$\liminf_{n\to\infty} f(x_n) \geq f(1)$$

does not hold. Hence, it is not a lower semi-continuous function at $x = 1$.

Next, we show that this is an $\mathcal{R}$-lower semi-continuous function. Let us consider (x_n) be an $\mathcal{R}$-preserving sequence converging to 1. Then, for all $n \in \mathbb{N}, (x_n, x_{n+1}) \in \mathcal{R} \Rightarrow x_n x_{n+1} \geq x_n$ or x_{n+1} implies the following two cases:

(1) $x_n = 1$ for all $n \in \mathbb{N}$ and $f(x_n) = 1 = f(1)$.
(2) If (x_n) be a non-constant $\mathcal{R}$-preserving sequence, then for all $n \in \mathbb{N}$, we must have $x_n > 1$ and $f(x_n) = x_n$ which shows that $\lim_{n\to\infty} f(x_n) = 1$. Therefore,

$$\liminf_{n\to\infty} f(x_n) \geq \frac{3}{4} = f(1).$$

This implies that f is an $\mathcal{R}$-lower semi-continuous function.

From the above explanation, it is clear that T is not $\mathcal{R}$-continuous at $x = 1$, since for any $\mathcal{R}$-preserving sequence converging to 1, one can check that $\lim_{n\to\infty} f(x_n) \neq f(1)$.

The above two examples justify that $\mathcal{R}$-LSC is weaker than $\mathcal{R}$-continuity as well as lower semi-continuity.

Remark 1.17. Every lower semi-continuous function is $\mathcal{R}$-lower semi-continuous, but the converse is not true. If $\mathcal{R}$ is a universal relation, then the notions of lower semi-continuity and $\mathcal{R}$-lower semi-continuity will coincide.

Now, we modify the definition of w-distance (Definition 1.11) and the corresponding Lemma 1 presented in [5] in the context of metric spaces endowed with an arbitrary binary relation $\mathcal{R}$.

Definition 1.18. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. A function $p : X \times X \to [0, \infty)$ is said to be a w-distance on X if

(w1') $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(w2') for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is $\mathcal{R}$-lower semi-continuous;
(w3') for any $\epsilon > 0$, there exists $\delta > 0$, such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

To prove our main results, we need the following lemma.

Lemma 1.19. *Let (X, d) be a metric space endowed with binary relation $\mathcal{R}$ and $p : X \times X \to [0, \infty)$ be a w-distance. Suppose (x_n) and (y_n) are two $\mathcal{R}$-preserving sequences in X and $x, y, z \in X$. Let (u_n) and (v_n) be sequences of positive real numbers converging to 0. Then, we have the followings:*

(L1) *If* $p(x_n, y) \le u_n$ *and* $p(x_n, z) \le v_n$ *for all* $n \in \mathbb{N}$*, then* $y = z$*. Moreover, if* $p(x, y) = 0$ *and* $p(x, z) = 0$*, then* $y = z$*.*
(L2) *If* $p(x_n, y_n) \le u_n$ *and* $p(x_n, z) \le v_n$ *for all* $n \in \mathbb{N}$*, then* $y_n \to z$*.*
(L3) *If* $p(x_n, x_m) \le u_n$ *for all* $m > n$*, then* (x_n) *is an* $\mathcal{R}$*-preserving Cauchy sequence in* X*.*
(L4) *If* $p(x_n, y) \le u_n$ *for all* $n \in \mathbb{N}$*, then* (x_n) *is an* $\mathcal{R}$*-preserving Cauchy sequence in* X*.*

Proof. Proof is omitted as it can be done in the line of [5, Lemma 1]. □

Remark 1.20. Under the universal binary relation $\mathcal{R}$, Definition 1.18 will coincide with Definition 1.11 and the Lemma 1.19 will coincide with [5, Lemma 1].

Now, we are in a position to state our main results. Before starting these, we highlight our main objectives which rest on the following considerations:

- We refine the main result of Alam and Imdad [2, Theorem 3.1] by considering more general distance function (w-distance) instead of the usual distance function on metric spaces endowed with an arbitrary binary relation and correspondingly we use a more general contraction principle.
- We present some non-trivial examples which lead to realize the sharpness of our obtained results.
- Finally, we apply our results to obtain solutions (positive solution) of nonlinear fractional differential equations (fractional thermostat model).

2. Main results

We start this section by extending the relation-theoretic version of Banach contraction principle owing to w-distance.

Theorem 2.1. *Let* (X, d) *be a metric space with a* w*-distance* p *and* $\mathcal{R}$ *be any arbitrary binary relation on* X*. Suppose* T *is a self-map on* X *with following conditions:*

(1) *There exists* $Y \subseteq X$ with $T(X) \subseteq Y$*, such that* (Y, d) *is* $\mathcal{R}$*-complete.*
(2) $X(T, \mathcal{R}) \neq \phi$ *and* $\mathcal{R}$ *is* T*-closed.*
(3) *Either* T *is* $\mathcal{R}$*-continuous or if for every* $\mathcal{R}$*-preserving sequence with* $x_n \to x$*, there exists a subsequence* (x_{n_k}) *of* (x_n)*, such that* $(x_{n_k}, x) \in \mathcal{R}$ *for all* $k \in \mathbb{N} \cup \{0\}$*.*
(4) *There exists* $\lambda \in [0, 1)$*, such that*
$$p(Tx, Ty) \le \lambda p(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R},$$

then $F(T) \neq \phi$*.*

Proof. As $X(T, \mathcal{R}) \neq \phi$, so there exists a point $x_0 \in X(T, \mathcal{R})$, such that $(x_0, Tx_0) \in \mathcal{R}$. Now, we define a sequence (x_n) by $x_n = T(x_{n-1}) = T^n(x_0)$. By the property of T-closedness of $\mathcal{R}$, one can easily check that (x_n) is an $\mathcal{R}$-preserving sequence, that is

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Applying the contraction principle of above theorem, we derive

$$\begin{aligned} p(Tx_{n-1}, Tx_n) &\leq \lambda p(x_{n-1}, x_n) \\ \Rightarrow p(x_n, x_{n+1}) &\leq \lambda p(x_{n-1}, x_n) \\ &\leq \lambda^2 p(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \lambda^n p(x_0, x_1). \end{aligned}$$

Using this for all $m > n$, we have,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\leq p(x_0,, x_1)[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}] \\ &\leq \frac{\lambda^n}{1-\lambda} p(x_0, x_1). \end{aligned} \tag{2.1}$$

Let us define $u_n = \frac{\lambda^n}{1-\lambda} p(x_0, x_1)$. Clearly $u_n \to 0$ as $n \to \infty$. Therefore, by $(L3)$, we must have that (x_n) is an $\mathcal{R}$-preserving Cauchy sequence in Y. Being (Y, d) $\mathcal{R}$-complete, we must have $x_n \to \tilde{x}$ as $n \to \infty$ for some $\tilde{x} \in Y$.

Next, we show that $\tilde{x}$ is a fixed point of T. To prove this, at first we consider that T is $\mathcal{R}$-continuous.

Using $\mathcal{R}$-continuity of T, we obtain

$$d(\tilde{x}, T\tilde{x}) = \lim_{n\to\infty} d(x_{n+1}, T\tilde{x}) = \lim_{n\to\infty} d(T(x_n), T\tilde{x}) = d(T\tilde{x}, T\tilde{x}) = 0.$$

This shows that $\tilde{x}$ is a fixed point of T.

Alternatively, let for every $\mathcal{R}$-preserving sequence with $x_n \to x$, there exists a subsequence (x_{n_k}) of (x_n), such that $(x_{n_k}, x) \in \mathcal{R}$ for all $k \in \mathbb{N} \cup \{0\}$. Combining the Eq. 2.1 with $\mathcal{R}$-lower semi-continuity of p, we get

$$p(x_{n_k+1}, \tilde{x}) \leq \liminf_{k\to\infty} p(x_{n_k+1}, x_{n_k+m}) \leq \liminf_{k\to\infty} \frac{\lambda^{n_k-1}}{1-\lambda} p(x_0, x_1) = 0.$$

Since $\mathcal{R}$ is T-closed and $(x_{n_k}, \tilde{x}) \in \mathcal{R}$, we derive

$$\begin{aligned} p(Tx_{n_k}, T\tilde{x}) &\leq \lambda p(x_{n_k}, \tilde{x}) \\ &\leq \lambda \liminf_{k\to\infty} p(x_{n_k}, x_{n_k+m}) \\ &\leq \liminf_{k\to\infty} \frac{\lambda^{n_k+1}}{1-\lambda} p(x_0, x_1) = 0. \end{aligned}$$

By (L1) of Lemma 1.19, we must have $T\tilde{x} = \tilde{x}$, i.e., $\tilde{x}$ is a fixed point of T. □

The following theorem ensures the uniqueness of the fixed point of T. We like to provide an additional condition to the hypotheses of Theorem 2.1 to ensure that the fixed point in Theorem 2.1 is in fact unique if any of the following conditions holds.

$$\text{For every } x, y \in T(X),\ \exists z \in T(X) \text{ such that } (z, x),\ (z, y) \in \mathcal{R}. \tag{2.2}$$

$$\mathcal{R}|_{TX} \text{ is complete.} \tag{2.3}$$

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, suppose that any of the condition (2.2) or condition (2.3) holds. Then, we obtain the uniqueness of the fixed point of T.*

Proof. We prove the theorem by considering following two possible cases.

Case I Let in addition to the hypotheses of Theorem 2.1, condition (2.2) holds. Then, for any two fixed points $\tilde{x}, \tilde{y}$ of T, there must be an element $z \in T(X)$, such that

$$(z, \tilde{x}) \in \mathcal{R} \quad \text{and} \quad (z, \tilde{y}) \in \mathcal{R}.$$

As $\mathcal{R}$ is T-closed, so for all $n \in \mathbb{N} \cup \{0\}$,

$$(T^n(z), \tilde{x}) \in \mathcal{R} \quad \text{and} \quad (T^n(z), \tilde{y}) \in \mathcal{R}.$$

Using contractivity condition of T, we get

$$p(T^n(z), \tilde{x}) = p(T^n(z), T^n\tilde{x}) \leq \lambda^n p(z, \tilde{x})$$

and

$$p(T^n(z), \tilde{y}) = p(T^n(z), T^n\tilde{y}) \leq \lambda^n p(x_0, \tilde{y}).$$

Let us consider $u_n = \lambda^{n+1}p(z, \tilde{x})$ and $v_n = \lambda^{n+1}p(z, \tilde{y})$. Clearly, (u_n) and (v_n) are two sequences of real numbers converging to 0. Hence, by (L1) of Lemma 1.19, we obtain $\tilde{x} = \tilde{y}$, i.e., T has a unique fixed point.

Case II Let in addition to the hypotheses of Theorem 2.1, condition (2.3) holds. Suppose $\tilde{x}, \tilde{y}$ are two fixed points of T. Then, we must have $(\tilde{x}, \tilde{y}) \in \mathcal{R}$ or $(\tilde{y}, \tilde{x}) \in \mathcal{R}$. For $(\tilde{x}, \tilde{y}) \in \mathcal{R}$, we obtain

$$p(\tilde{x}, \tilde{y}) = p(T(\tilde{x}), T(\tilde{y})) \leq \lambda p(\tilde{x}, \tilde{y}) < p(\tilde{x}, \tilde{y})$$

which leads to a contradiction. Hence, we must have $\tilde{x} = \tilde{y}$.

In a similar way, if $(\tilde{y}, \tilde{x}) \in \mathcal{R}$, we have $\tilde{x} = \tilde{y}$. □

To signify the motivations of our investigation, we present following examples.

Example 2.3. Let (X, d) be a metric space where $X = [1, 3)$ and d is the usual metric defined on X. We define a binary relation $\mathcal{R} = \{(x, y) \in X^2 : x \geq y\}$. Let T be a self-map on X defined by

$$T(x) = \begin{cases} \frac{x}{2}, & x \in [1, 2); \\ 2, & x \in [2, 3). \end{cases}$$

Now, we check the hypotheses of Theorem 3.1 given in Alam and Imdad [2].

(1) Let $Y = [1, 2]$. Then, it is clear that $T(X) \subseteq Y$ and (Y, d) is $\mathcal{R}$-complete.
(2) For $x = 1, Tx = \frac{1}{2}$, such that $(x, Tx) \in \mathcal{R}$, i.e., $X(T, \mathcal{R}) \neq \phi$.
(3) Let (x_n) be an $\mathcal{R}$-preserving sequence converging to x. Therefore for all $n \in \mathbb{N}, (x_n, x_{n+1}) \in \mathcal{R}$, i.e., $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ and this implies that (x_n) is a decreasing sequence converging to x. Therefore, we must have $(x_n, x) \in \mathcal{R}$ for all $n \in \mathbb{N}$.
(4) Now, we show that we cannot employ the contraction principle given in Theorem 3.1 of Alam and Imdad [2].

For example, we consider $x = 2, y = 1$. Then, clearly $(x, y) \in \mathcal{R}$ and $Tx = 2,\ Ty = \frac{1}{2}$. Then, $d(Tx, Ty) = d(2, \frac{1}{2}) = \frac{3}{2}$ and $d(x, y) = 1$. Therefore, we cannot find any $k \in [0, 1)$, such that

$$d(Tx, Ty) \leq kd(x, y)$$

holds. However, if we choose a w-distance function p as $p(x, y) = |x| + |y|$, then for all $x, y \in X$, we have

$$p(Tx, Ty) \leq \lambda p(x, y)$$

with $(x, y) \in \mathcal{R}$ and for some $\lambda \in [0, 1)$.

Hence, all the hypotheses of our theorem satisfy and note that $x = 2$ is a fixed point of T and it is the unique fixed point.

Note: It is worth mentioning that the results of Ahmadullah et al. [1] are more generalized and improved version than that of Alam and Imdad [2], but still in that example, we cannot employ the main result (Theorem 2.1) of Ahmadullah et al. [1]. For $x = 2, y = 1$, we obtain

$$M_T(2, 1) = \max\left\{d(2, 1), d\left(1, \frac{1}{2}\right), d(2, 2), \frac{d(1, 2) + d(2, \frac{1}{2})}{2}\right\} = \frac{5}{4}.$$

In Theorem 2.1 given in [1], as ϕ is a function with $\phi(t) < t, t > 0$, we cannot find any function ϕ with that property, so that

$$d(Tx, Ty) \leq \phi(M_T(x, y))$$

holds. Hence, we cannot employ the results of Ahmadullah et al. [1] in that example.

Next, we furnish another important example.

Example 2.4. Let us consider the metric space (X, d), where $X = [0, 2]$, d is the usual metric on X and $(x, y) \in \mathcal{R}$ if $xy \leq x$ or y. We define a w-distance $p : X \times X \to X$ by $p(x, y) = y$. Let us define a function $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{3}, & 0 \leq x \leq \frac{2}{3}; \\ 1 - x, & \frac{2}{3} < x < 1; \\ \frac{3}{4}, & x = 1; \\ x - \frac{1}{2}, & x > 1. \end{cases}$$

Now, if $(x, y) \in \mathcal{R}$, then $xy \leq x$ *or* y. Let us consider $xy \leq x$. Therefore, we have the following cases:

Case 1 Let $x = 0$. Then, for any $y \in [0, 2],\ (x, y) \in \mathcal{R}$. Therefore, we get:

(i) for $0 \leq y \leq \frac{2}{3}$, $Tx = 0$ and $Ty = \frac{y}{3}$. Therefore, $p(Tx, Ty) = Ty = \frac{y}{3}$ and $p(Tx, Ty) = \frac{y}{3} \leq \frac{1}{3}p(x, y)$;

(ii) if $\frac{2}{3} < y < 1$, then $Ty \in (0, \frac{1}{3})$ and $p(Tx, Ty) = 1 - y < y = p(x, y)$. In particular, $p(Tx, Ty) \leq kp(x, y)$, where $k \in [\frac{1}{2}, 1)$;

(iii) let $y = 1$. Then, $p(T0, T1) = \frac{3}{4} \leq \frac{3}{4}p(0, 1)$;

(iv) for $y > 1$, we have $p(Tx, Ty) = y - \frac{1}{2} \leq ky = kp(x, y)$, where $k \in [\frac{3}{4}, 1)$.

Case 2 For all $y \in [0, 2]$ and $x = 0$, we have $p(Ty, Tx) = 0 = kp(y, x)$ for all $k \in [0, 1)$.

Case 3 Let $x \neq 0$. Then, $y \leq 1$. Therefore, we have,

(i) for $0 \leq y \leq \frac{2}{3}$, $p(Tx, Ty) \leq \frac{1}{3}p(x, y)$;
(ii) for $\frac{2}{3} < y < 1$, $p(Tx, Ty) \leq kp(x, y)$, where $k \in [\frac{1}{2}, 1)$;
(iii) for $y = 1$, $p(Tx, T1) = \frac{3}{4} \leq \frac{3}{4}p(x, 1)$ for all $x \in X$;
(iv) for $y \leq 1$ and $x > 1$, we have $p(Ty, Tx) \leq kp(y, x)$, where $k \in [\frac{3}{4}, 1)$.

The above three cases show that T satisfies the condition (5) of Theorem 2.1. Next, we check the remaining hypotheses of our theorem.

(1) Let us consider $Y = [0, \frac{3}{2}]$. Then we must have $TX \subseteq Y$ and $\mathcal{R}|_Y$ is $\mathcal{R}$-complete.
(2) Clearly, $X(T, \mathcal{R}) \neq \phi$.
(3) $\mathcal{R}$ is T-closed.
(4) Note that T is not $\mathcal{R}$-continuous at $x = 1$ and $x = \frac{2}{3}$. However, for every $\mathcal{R}$-preserving sequence (x_n) with $x_n \to x$, we can always find a subsequence (x_{n_k}) of (x_n), such that $(x_{n_k}, x) \in \mathcal{R}$ for all $k \in \mathbb{N} \cup \{0\}$.
(5) For any $x, y \in Y$, one can always find $z \in Y$, such that $(z, x), (z, y) \in \mathcal{R}$.

We have already checked that T satisfies the contractivity condition. Therefore, all the hypotheses of our theorem hold. Note that $x = 0$ is a fixed point of T and it is the unique fixed point of T.

Remark 2.5. (1) It is notable that the binary relation $\mathcal{R}$ considered in our example is not reflexive, irreflexive and transitive. Here, $\mathcal{R}$ satisfies only symmetrical condition.

(2) It is interesting to note that the mapping T in above example neither satisfies the contractive condition of Theorem 3.1 in Alam and Imdad [2] nor the contractive condition of Theorem 2.1 in Ahmadullah et al. [1].

For example, we choose $x = 1$ and $y = \frac{3}{4}$. Clearly, $(x, y), (y, x) \in \mathcal{R}$. Therefore

$$\begin{aligned} M_T(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &= \max\left\{d(1,\frac{3}{4}), d(1,\frac{3}{4}), d(\frac{3}{4},\frac{1}{4}), \frac{d(1,\frac{1}{4})+d(\frac{3}{4},\frac{3}{4})}{2}\right\} \\ &= \max\left\{\frac{1}{4},\frac{1}{2},\frac{3}{8}\right\} \\ &= \frac{1}{2}, \end{aligned} \tag{2.4}$$

and $d(Tx, Ty) = d(\frac{3}{4}, \frac{1}{4}) = \frac{1}{2}$.

In Theorem 2.1 of Ahmadullah et al. [1], ϕ being an increasing function with $\phi(t) < t$, for $t > 0$, the mapping T does not satisfy the contractive condition of this theorem, and hence, we cannot exploit this theorem to obtain any fixed point. Again, since the Theorem 2.1 of Ahmadullah et al. [1] is improved version over Theorem 3.1 of Alam and Imdad [2] and also Theorem 2.1 of Samet and Turinici [11] (for symmetric binary relation), we cannot employ these results also in that example.

Analysing above two examples, it is transparent that our findings unveil another direction of relation-theoretic metrical fixed-point results where the main result given in Alam and Imdad [2] (Theorem 3.1) does not work (even the main result of Ahmadullah et al.[1] (Theorem 2.1) does not work here).

Remark 2.6. If we set $p(x, y) = d(x, y)$, in Theorem 2.1, then we obtain the Theorem 3.1 of Alam and Imdad [2]. Hence our Theorem 2.1 is an improved and generalized version of relation-theoretic metrical fixed-point theorem due to Banach contraction given in Alam and Imdad [2].

3. Applications

In this section, we employ our main result to obtain a solution of a nonlinear fractional differential equation. Moreover, we apply our main result to find a positive solution of a fractional thermostat model.

3.1. Application to fractional boundary value problem

We consider the following nonlinear fractional differential equation given by

$$ {}^{C}D^{\beta}x(t) = f(t, x(t)) \quad (0 < t < 1, 1 < \beta \leq 2), \tag{3.1} $$

with boundary conditions

$$ x(0) = 0,\ Ix(1) = x'(0), \tag{3.2} $$

where ${}^{C}D^{\beta}$ stands for the Caputo fractional derivative of order β defined by

$$ {}^{C}D^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^n(s)\mathrm{d}s \ (n-1 < \beta < n; n = [\beta]+1), $$

and $I^{\beta}f(t)$ denotes the Riemann–Liouville fractional integral of a continuous function $f(t)$ of order β (for detail, see [12]) given by

$$ I^{\beta}f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\mathrm{d}s,\ \beta > 0. $$

We consider $X = C[0, 1]$, the set of all real valued continuous functions defined on $[0, 1]$ with supremum norm $||x||_{\infty} = \sup_{t\in[0,1]}|x(t)|$. Therefore, $(X, ||.||_{\infty})$ is a Banach space.

At first, we find out the solution of Eq. 3.1 using the boundary conditions 3.2. For this purpose, we need to recall the following important lemma.

Lemma 3.1. [6] *For* $\beta > 0$,

$$ I^{\beta C}D^{\beta}x(t) = x(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, $$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$ *and* n *is the smallest integer greater than or equal to* β.

Lemma 3.2. *Suppose* $f \in C[0, 1]$. *Then, for any function* $x \in C[0, 1]$

$$ {}^{C}D^{\beta}x(t) = f(t) \quad (0 < t < 1, 1 < \beta \leq 2), $$

with

$$ x(0) = 0, \quad Ix(1) = x'(0), $$

has unique solution

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\mathrm{d}s + \frac{2t}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r\mathrm{d}s.$$

Proof. Using Lemma 3.1, there exist some $c_0, c_1 \in \mathbb{R}$, such that

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\mathrm{d}s - c_0 - c_1 t. \tag{3.3}$$

Using $x(0) = 0$, it implies that $c_0 = 0$. Now, the Riemann–Liouville integral of order one is given by

$$\begin{aligned} Ix(t) &= \int_0^t \left[\frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r - c_1 s \right] \mathrm{d}s \\ &= \frac{1}{\Gamma(\beta)} \int_0^t \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r\mathrm{d}s - c_1 \frac{t^2}{2}. \end{aligned}$$

Utilizing second condition $Ix(1) = x'(0)$, we have

$$\begin{aligned} -c_1 &= \frac{1}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r\mathrm{d}s - \frac{c_1}{2} \\ \Rightarrow c_1 &= -\frac{2}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r\mathrm{d}s \end{aligned}$$

Substituting the values of c_0 and c_1 in 3.3, we obtain the solution:

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\mathrm{d}s + \frac{2t}{\Gamma(\beta)} \int_0^1 \int_0^s (s-r)^{\beta-1} f(r)\mathrm{d}r\mathrm{d}s.$$

□

Our next aim is to investigate the existence of a solution of a nonlinear fractional differential equation via relation-theoretic metrical fixed-point result. To show this, we consider the following fractional differential equation:

$$^{C}D^{\beta} x(t) = f(t, x(t)) \quad (0 < t < 1, 1 < \beta \leq 2), \tag{3.4}$$

with the boundary conditions

$$x(0) = 0, \quad Ix(1) = x'(0),$$

where

(1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
(2) $x(t) : [0,1] \to \mathbb{R}$ is continuous

satisfying the following conditions:

$$|f(t,x) - f(t,y) \leq L|x-y|$$

for all $t \in [0,1]$ and $\forall x, y \in X$, such that $x(t)y(t) \geq 0$ and L is a constant, such that $L\lambda < 1$ where $\lambda = \frac{1}{\Gamma(\beta+1)} + \frac{2}{\Gamma(\beta+2)}$. Then, the differential Eq. (3.4) has a unique solution.

Proof. We consider the following binary relation on $X(=C[0,1])$:

$$(x,y)\in\mathcal{R}\quad \text{if } x(t)y(t)\geq 0, \forall t\in[0,1].$$

We consider $d(x,y)=\sup_{t\in[0,1]}||x(t)-y(t)||$ for all $x,y\in X$. Therefore, (X,d) is an $\mathcal{R}$-complete metric space.

We define a mapping $T:X\to X$ by

$$\begin{aligned} Tx(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))\mathrm{d}s \\ &\quad + \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,x(r))\mathrm{d}r\mathrm{d}s \end{aligned}$$

for all $t\in[0,1]$.

From Lemma 3.2, it is clear that the fixed points of T are precisely the solutions of Eq. 3.4. To prove the existence of fixed point of T, we show that $\mathcal{R}$ is T-closed and T satisfies the contraction condition.

At first, we show that $\mathcal{R}$ is T-closed. Let, for all $t\in[0,1]$, $(x(t),y(t))\in\mathcal{R}$. Then, we have

$$\begin{aligned} Tx(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))\mathrm{d}s \\ &\quad + \frac{2t}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1} f(r,x(r))\mathrm{d}r\mathrm{d}s > 0 \end{aligned}$$

which implies that $(Tx,Ty)\in\mathcal{R}$, i.e., $\mathcal{R}$ is T-closed. Also, it is clear that for any $x(t)\geq 0, t\in[0,1]$, we have $Tx(t)\geq 0$ for all $t\in[0,1]$, i.e., $(x(t),Tx(t))\in\mathcal{R}$ for all $t\in[0,1]$ which implies that $X(T,\mathcal{R})\neq\phi$.

For all $t\in[0,1]$ and $(x(t),y(t))\in\mathcal{R}$, we obtain

$$\begin{aligned} |Tx-Ty| &= \Bigg|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))\mathrm{d}s \\ &\quad + \frac{2t}{\Gamma(\beta)}\int_0^1\left(\int_0^s (s-r)^{\beta-1} f(r,x(r))\mathrm{d}r\right)\mathrm{d}s \\ &\quad - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))\mathrm{d}s \\ &\quad - \frac{2t}{\Gamma(\beta)}\int_0^1\left(\int_0^s (s-r)^{\beta-1} f(r,y(r))\mathrm{d}r\right)\mathrm{d}s\Bigg| \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\big|f(s,x(s))-f(s,y(s))\big|\mathrm{d}s \\ &\quad + \frac{2}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1}\big|f(r,x(r))-f(r,y(r))\big|\mathrm{d}r\mathrm{d}s \\ &\leq \frac{L||x-y||}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathrm{d}s \\ &\quad + \frac{2L||x-y||}{\Gamma(\beta)}\int_0^1\int_0^s (s-r)^{\beta-1}\mathrm{d}r\mathrm{d}s \end{aligned}$$

$$\leq \frac{L||x-y||}{\Gamma(\beta+1)} + \frac{2L||x-y||B(\beta+1,1)}{\Gamma(\beta+1)}, \quad \text{where } B \text{ is the beta function,}$$

$$\leq L||x-y|| \left(\frac{1}{\Gamma(\beta+1)} + \frac{2}{\Gamma(\beta+2)} \right)$$

$$\Rightarrow ||Tx - Ty|| \leq L\lambda||x-y||.$$

Now, if we set $p(x, y) = d(x, y)$, then we have

$$p(Tx, Ty) \leq L\lambda p(x, y)$$

which shows that T satisfies the contraction condition as $L\lambda < 1$.

Next, we consider that (x_n) is an $\mathcal{R}$-preserving Cauchy sequence converging to x. So, we must have $x_n(t)x_{n+1}(t) \geq 0$ for all $t \in [0, 1]$ and $n \in \mathbb{N}$. This gives us two possibilities: either $x_n(t) \geq 0$ or $x_n(t) \leq 0$ for all $n \in \mathbb{N}$ and each $t \in [0, 1]$. Let us consider the case $x_n(t) \geq 0$ for each $t \in [0, 1]$ and $n \in \mathbb{N}$. Then, for every $t \in [0, 1]$, $x_n(t)$ produces a sequence of non-negetive real numbers which converges to $x(t)$. Hence, we must get $x(t) \geq 0$ for each $t \in [0, 1]$, i.e., $(x_n(t), x(t)) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $t \in [0, 1]$. So, by Theorem 2.1, $x(t)$ is a fixed point of T which is the required solution of Eq. 3.4.

Finally, we show that $x(t)$ is the unique solution of Eq. 3.4. If possible, let $y(t)$ be another solution of Eq. 3.4 which implies that $Ty(t) = y(t)$ for all $t \in [0, 1]$. Now, we consider a constant function $z(t) = 0$ for all $t \in [0, 1]$. Then, it is trivial to show that $(z(t), x(t)) \in \mathcal{R}$ and $(z(t), y(t)) \in \mathcal{R}$ for all $t \in [0, 1]$. Hence, by Theorem 2.2, we claim that $x(t)$ is the unique solution of Eq. 3.4. □

3.2. Application to fractional thermostat model

Now we are interested to find a positive solution of a fractional thermostat model employing our relation-theoretic metrical fixed-point results under a certain condition.

At first, we recall the fractional thermostat model [9] given by

$$^{C}D^{\alpha}x(t) = -f(t, x(t)) \quad (0 \leq t \leq 1, 1 < \alpha \leq 2), \tag{3.5}$$

with boundary conditions

$$x'(0) = 0, \quad \beta^{C}D^{\alpha}x(t) + x(\eta) = x'(0), \tag{3.6}$$

where $\beta > 0, 0 \leq \eta \leq 1$ are given constants. The authors of [9] have already shown that any function $x(t) \in C[0, 1]$ is a solution of Eq. 3.5 if and only if

$$x(t) = \int_0^1 G(t, s)f(s)\mathrm{d}s,$$

where $G(t, s)$ is the Green's function (depending on α) given by

$$G(t, s) = \beta + H_\eta(s) - H_t(s) \tag{3.7}$$

and for $r \in [0, 1], H_r[0, 1] \to \mathbb{R}$ is defined by $H_r(s) = \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \leq r$ and $H_r(s) = 0$ for $s > r$.

To find a positive solution of Eq. 3.5 with boundary condition 3.6 utilizing our relation-theoretic metrical fixed-point result (Theorem 2.1), we present the following theorem.

Theorem 3.3. *Let $x(t) \in C[0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ be a continuous function satisfying Eqs. 3.5 with 3.6, such that*

$$\beta\Gamma(\alpha) \geq (1-\eta)^{\alpha-1},$$

and

$$|f(t,x) - f(t,y) \leq L|x-y|$$

for all $t \in [0,1]$; $\forall x, y \in C[0,1]$, such that $x(t)y(t) \geq 0$ and L is a constant with $L\lambda < 1$, where $\lambda = \beta + \frac{2}{\Gamma(\alpha+1)}$. Then, the Eq. 3.5 has a positive solution.

Proof. Let $X = C[0,1]$. Then, (X,d) is a complete metric space endowed with the metric $d(x,y) = \limsup_{t\in[0,1]} ||x(t) - y(t)||$. We define a binary relation $\mathcal{R}$ on X by $(x,y) \in \mathcal{R}$ if $x(t)y(t) \geq 0$ for all $t \in [0,1]$. Then it is clear that (X,d) is an $\mathcal{R}$-complete metric space. Next, we define a self mapping T on $C[0,1]$ by

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))\mathrm{d}s,$$

where $G(t,s)$ is the Green's function given by 3.7. Since $\beta\Gamma(\alpha) \geq (1-\eta)^{\alpha-1}$ implies $G(t,s) \geq 0$ (see, [9]), one can easily observe that $Tx(t) \geq 0$.

To find the fixed points of T, we only show that T satisfies the contraction condition of our Theorem 2.1, as one can check the other conditions in the line of our previous application.

Let for all $t \in [0,1]$, $(x(t), y(t)) \in \mathcal{R}$, then we have

$$\begin{aligned}
&|Tx - Ty| \\
&\quad= \left| \int_0^1 G(t,s)f(s,x(s))\mathrm{d}s - \int_0^1 G(t,s)f(s,y(s))\mathrm{d}s \right| \\
&\quad= \left| \beta \int_0^1 f(s,x(s))\mathrm{d}s + \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s \right. \\
&\qquad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s - \beta \int_0^1 f(s,y(s))\mathrm{d}s \\
&\qquad \left. - \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\mathrm{d}s \right| \\
&\quad\leq \beta \left| \int_0^1 f(s,x(s))\mathrm{d}s - \int_0^1 f(s,y(s))\mathrm{d}s \right| \\
&\qquad + \left| \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s - \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\mathrm{d}s \right| \\
&\qquad + \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\mathrm{d}s \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta\int_0^1 |f(s,x(s))\mathrm{d}s - f(s,y(s))|\mathrm{d}s + \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) \\
&\quad - f(s,y(s))|\mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|\mathrm{d}s \\
&\leq \beta L||x-y|| + L||x-y|| \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \\
&\quad + L||x-y|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \\
&\leq \beta L||x-y|| + \frac{2L||x-y||}{\Gamma(\alpha+1)} \\
&\leq L||x-y||\left(\beta + \frac{2}{\Gamma(\alpha+1)}\right) \\
&\Rightarrow ||Tx-Ty|| \leq L\lambda||x-y||,
\end{aligned}
$$

where $\lambda = \beta + \frac{2}{\Gamma(\alpha+1)}$. As we consider that $L\lambda < 1$, so T satisfies the contraction condition of Theorem 2.1.

Therefore, (X,d) being an $\mathcal{R}$-complete metric space and T satisfying all the conditions of our Theorem 2.1, there exists $x \in X$ with $x(t) \geq 0$, for all $t \in [0,1]$, such that $x(t) = Tx(t)$ which implies that there exists a positive solution of Eq. 3.5. □

Remark 3.4. We have shown that using relation-theoretic metrical fixed-point result (Theorem 2.1), one can obtain a positive solution of fractional thermostat model whenever $\beta\Gamma(\alpha) \geq (1-\eta)^{\alpha-1}$. However, if $\beta\Gamma(\alpha) < (1-\eta)^{\alpha-1}$, we cannot employ our result to find a positive solution of the model, since $G(t,s)$ fails to be non-negative throughout the domain, as a result $\mathcal{R}$ may not be T-closed.

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