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The Meir–Keeler fixed point theorem in incomplete modular spaces with application

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Abstract. In this paper, we present a new generalized modular version of the Meir–Keeler fixed point theorem endowed with an orthogonal relation. Our results improve the results of (Eshaghi Gordji et al., On orthogonal sets and Banach fixed point theorem, Fixed Point Theory, 2017). Finally, this result is applied to the existence and uniqueness of solutions to perturbed integral equations in modular function spaces.

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1. Introduction and preliminaries

A problem that mathematicians have dealt with for almost 50 years is how to generalize the classical function space L^p. A first attempt was made by Birnhaum and Orlicz in 1931 [1]. This generalization found many applications in differential and integral equations with kernels of nonpower types. The more abstract generalization was given by Nakano [2] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called modular. This idea, which was the basis of the theory of modular spaces and initiated by Nakano in connection with the theory of the order space, was refined and generalized by Musielak and Orlicz [3] in 1959. Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis. For more details about modular spaces, we refer the reader to the books edited by Musielak [4] and by Kozlowski [5].

It is well known that fixed point theory is one of the powerful tools in solving integral and differential equations. The Banach contraction mapping principle is one of the pivotal results in fixed point theory and it has a board set of applications. Khamsi et al. [6] investigated the fixed point results in modular function spaces. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see for instance [7–10]).

In 1969, Meir and Keeler [11] presented a generalization of the Banach contraction mapping principle as the following.

Theorem 1.1. [11] *Let (X, d) be a complete metric space and $T : X \to X$ be an operator. Suppose that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $x, y \in X$,*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \qquad \Rightarrow \qquad d(Tx, Ty) < \epsilon.$$

Then, T admits a unique fixed point $z \in X$ and for any $x \in X$, the sequence $\{T^n(x)\}$ converges to z.

Recently, Eshaghi et al. [12] introduced the notion of orthogonal set and then gave an extension of Banach's fixed point theorem. They proved, by means of an example, that their main theorem is a real generalization of Banach's fixed point theorem. The main result of [12] is the following theorem.

Theorem 1.2. [12] *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then, f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

For more details about orthogonal space, we refer the reader to [12–14]. The paper is organized as follows. In Sect. 2, we begin by recalling some basic concepts of modular spaces and definition of orthogonal sets in [12,13] that are reviewed. Strongly orthogonal sequences and their relation to orthogonal sequences by means of some examples are explained. In Sect. 3, we present our main theorem and construct an example which shows that the main theorem of this paper is a real extension of modular version of the Meir–Keeler fixed point theorem. In Sect. 4, as an application, we find the existence and uniqueness of solution for a perturbed integral equations in Musielak–Orlicz space.

2. Preliminaries

We recall some definitions of modular spaces. Let X be an arbitrary vector space over $K(= \mathbb{R} \text{ or } \mathbb{C})$,

(a) A function $\rho : X \to [0, +\infty]$ is called a modular if

(i) $\rho(x) = 0$ *if and only if* $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$ *for every scaler* α *with* $|\alpha| = 1$;

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *if* $\alpha + \beta = 1$ *and* $\alpha \geq 0,\ \beta \geq 0$

for all $x, y \in X$. If (iii) is replaced by

(iii) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *if* $\alpha + \beta = 1$ *and* $\alpha \geq 0,\ \beta \geq 0$,

we say that ρ is convex modular.

Note that for all $x \in X$, the function $\rho(\alpha x)$ is an increasing function of $\alpha \geq 0$, that is, if $\alpha < \beta$ and $\alpha, \beta \geq 0$ then $\rho(\alpha x) \leq \rho(\beta x)$.

(b) A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ given by $X_\rho = \{x \in X : \ \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}$.

(c) The modular space X_ρ can be equipped with the F-norm defined by $|x|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}$. If ρ be convex, then the functional $\|x\|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}$ is a norm called the Luxemburg norm in X_ρ which is equivalent to the F-norm $|.|_\rho$.

Definition 2.1. Let X_ρ be a modular space.

(a) A sequence $\{x_n\}$ in X_ρ is said to be:
   (i) ρ-convergent to x, denoted by $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.
   (ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.

(c) A subset $B \subseteq X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\} \subset B$ with $x_n \xrightarrow{\rho} x$, then $x \in B$.

(d) A subset $B \subseteq X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup\{\rho(x - y) : \ x, y \in B\} < \infty$, where $\delta_\rho(B)$ is called the ρ-diameter of B.

(e) ρ is said to satisfy the $\triangle_2$-condition if $2x_n \xrightarrow{\rho} 0$ whenever $x_n \xrightarrow{\rho} 0$.

Now, we recall the main definition of [12,13].

Definition 2.2. [12,13] Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 \in X : (\forall y, y \perp x_0) \ \ or \ \ (\forall y, x_0 \perp y),$$

then $\perp$ is called an orthogonality relation and the pair $(X, \perp)$ an orthogonal set (briefly *O-set*).

Note that in above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 2.3. [12,13] Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an *orthogonal sequence* (briefly, *O-sequence*) if

$$(\forall n, \ x_n \perp x_{n+1}) \ \ or \ \ (\forall n, \ x_{n+1} \perp x_n).$$

Next, we introduce the new type of sequences in O-sets.

Definition 2.4. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called a strongly orthogonal sequence (briefly, SO-sequence) if

$$(\forall n, k, \ x_n \perp x_{n+k}) \quad or \quad (\forall n, k, \ x_{n+k} \perp x_n).$$

It is obvious that every SO-sequence is an O-sequence that defined in [12,13]. The following example shows that the converse is not true.

Example 2.5. Let $X = \mathbb{Z}$. Suppose $x \perp y$ iff $xy \in \{x, y\}$. Define the sequence $\{x_n\}$ in X as follows:

$$x_n = \begin{cases} 2 & n = 2k, \ \ for \ some \ k \in \mathbb{Z}, \\ 1 & n = 2k+1, \ \ for \ some \ k \in \mathbb{Z}. \end{cases}$$

We obtain that for all $n \in \mathbb{N}$, $x_n \perp x_{n+1}$, but x_{2n} is not orthogonal to x_{4n}. Therefore, $\{x_n\}$ is an O-sequence that is not SO-sequence.

Definition 2.6. Let $(X, \perp)$ be an O-set and d be a metric on X. The triplet $(X, \perp, d)$ is called an orthogonal metric space.

Definition 2.7. Let $(X, \perp, d)$ be an orthogonal metric space. X is said to be strongly orthogonal complete (briefly, SO-complete) if every Cauchy SO-sequence is convergent.

Clearly, every complete metric space is SO-complete. In the next example, X is SO-complete but it is not complete.

Example 2.8. Consider $X = \{u \in C([a,b], \mathbb{R}); \quad \forall t \in [a,b], u(t) \neq 0\}$. X is an incomplete metric space with supremum norm $\|u\| = \sup_{t\in[a,b]} |u(t)|$. Define the orthogonal relation $\perp$ on X by

$$u \perp v \quad \Leftrightarrow \quad \forall t \in [a,b]; \quad u(t)v(t) \geq \max\{u(t), v(t)\}.$$

X is SO-complete, in fact if $\{u_n\}$ is an arbitrary Cauchy SO-sequence in X, then for all $n \in \mathbb{N}$ and $t \in [a,b]$, $u_n(t) \geq 1$. Since $C([a,b], \mathbb{R})$ with supremum norm is a Banach space, we can find the element $u \in C([a,b], \mathbb{R})$ for which $\|u_n - u\| \to 0$ as $n \to \infty$. The uniformly convergent implies the point-wise convergent. Thus, $u(t) \geq 1$ for all $t \in [a,b]$ and hence $u \in X$.

Notice every O-complete metric space that defined in [12,13] is SO-complete, next by means an example we show that the converse is not true.

Example 2.9. Suppose $X = (0, \infty)$ with the Euclidean metric and the orthogonal relation in Example 2.5. Let $\{x_n\}$ be a Cauchy SO-sequence in X, the definition $\perp$ follows that $x_n = 1$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ converges to constant $1 \in X$. Now, consider the sequence

$$x_n = \begin{cases} 1 & n = 2k, \ for \ some \ k \in \mathbb{Z}, \\ k & n = 2k+1, \ for \ some \ k \in \mathbb{Z}. \end{cases}$$

Clearly, $\{x_n\}$ is an O-sequence that is not convergent to any element of X.

Definition 2.10. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is strongly orthogonal continuous (briefly, SO-continuous) in $a \in X$ if for each SO-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is SO-continuous on X if f is SO-continuous in each $a \in X$.

It is easy to see that every continuous mapping is O-continuous that defined in [12,13] and every O-continuous mapping is SO-continuous. The following example shows that the converse is not true.

Example 2.11. Let $X = \mathbb{R}$ with the Euclidean metric. Assume $\perp$ is orthogonal relation in Example 2.5. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ \dfrac{1}{x} & x \in \mathbb{Q}^c. \end{cases}$$

Notice that f is not continuous but we can see that f is SO-continuous. If $\{x_n\}$ is an SO-sequence in X which converges to $x \in X$. Applying definition $\perp$, we obtain that for enough large n, $x_n \in \mathbb{Q}$. This implies that $f(x_n) = 1 \to x = 1$. To see that f is not O-continuous, consider the sequence

$$x_n = \begin{cases} 0 & n = 2k+1,\ \ for\ some\ k \in \mathbb{Z}, \\ \frac{\sqrt{2}}{k} & n = 2k,\ \ for\ some\ k \in \mathbb{Z}. \end{cases}$$

It is clear that $x_n \to 0$ while the sequence $\{f(x_n)\}$ is not convergent to $f(0)$.

3. Main results

Let X be an arbitrary vector space over $K(=\mathbb{R}$ or $\mathbb{C})$. We start our work with the following definitions.

Definition 3.1. Let ρ be a modular function on X and $\perp$ be an orthogonal relation on X_ρ. The triplet $(X, \perp, \rho)$ is called an orthogonal modular space.

Definition 3.2. Let $(X, \perp, \rho)$ be an orthogonal modular space.

(a) A subset B of X_ρ is called SO-ρ-closed, if for any SO-sequence $\{x_n\} \subset B$ with $x_n \xrightarrow{\rho} x$, then $x \in B$.
(b) $(X, \perp, \rho)$ is said to be strongly orthogonal ρ-complete (briefly, SO-ρ-complete) if every ρ-Cauchy SO-sequence in X_ρ is ρ-convergent.
(c) Let B be a subset of X_ρ. A mapping $f : B \to B$ is called:
   (i) $\perp$-preserving if $f(x)\perp f(y)$ whenever $x\perp y$ and $x, y \in B$.
   (ii) Strongly orthogonal ρ-continuous (briefly, SO-ρ-continuous) in $a \in B$ if for each SO-sequence $\{a_n\}$ in B then $a_n \xrightarrow{\rho} a$ implies $f(a_n) \xrightarrow{\rho} f(a)$. Also, f is SO-ρ-continuous on B if f is SO-ρ-continuous in each $a \in B$.

In below, we show that SO-ρ-closedness does not imply ρ-closedness.

Example 3.3. Let $(X, \|.\|)$ be a Banach space and $\rho = \|.\|$. Let T be a Picard operator on X, it means that there exists x^* in X for which for every $y \in X$, $\lim_{n\to\infty} T^n(y) = x^*$. Consider the non-closed subspace B of X such that $x^* \in B$. Define $x\perp y$ iff $\lim_{n\to\infty} T^n(x) = y$. It is clear that for every $y \in X$, $y\perp\ x^*$. B is a SO-ρ-closed subset of X. In fact, if $\{x_n\}$ is a ρ-Cauchy SO-sequence in B that converges to $z \in X$. Then, the definition of orthogonality implies that for $n \geq 2$, $x_n = x^*$. Therefore, $z = x^* \in B$.

Definition 3.4. Let $(X, \perp, \rho)$ be an orthogonal modular space. Let B be an SO-ρ-closed subset of X_ρ and $c, l \in (0, \infty)$ with $c > l$. We say that a mapping $T : B \to B$ satisfies the Meir–Keeler condition whenever For every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that:

$$x \neq y,\ x\perp y \quad and \quad \epsilon \leq \rho(l(x-y)) < \epsilon + \delta(\epsilon) \quad \Rightarrow \quad \rho(c(Tx - Ty)) < \epsilon. \tag{3.1}$$

Theorem 3.5. *Let $(X, \perp, \rho)$ be an SO-ρ-complete orthogonal modular space (not necessarily ρ-complete) with an orthogonal element x_0 and ρ satisfies the Δ_2-condition. Suppose that B is an SO-ρ-closed subset of X_ρ such that $x_0 \in B$ and there exist $c, l \in (0, \infty)$ with $c > l$. Assume that $T : B \to B$ is $\perp$-preserving, SO-ρ-continuous such that satisfying the Meir–Keeler condition. Then, T has a unique fixed point $z \in B$. Also, T is a Picard operator, that is, for all $x \in B$, the sequence $\{T^n(x)\}$ is ρ-convergent to z.*

Proof. Let $\alpha \in (1, \infty)$ be the conjugate of c/l ; i. e., $\frac{l}{c} + \frac{1}{\alpha} = 1$. By definition of orthogonality, we have

$$(\forall y \in X_\rho,\ x_0 \perp y) \ \ or \ \ (\forall y \in X_\rho,\ y \perp x_0).$$

It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Put

$$x_1 = Tx_0,\ x_2 = T(x_1) = T^2(x_0), \ldots, x_{n+1} = T(x_n) = T^{n+1}(x_0)$$

for all $n \in \mathbb{N}$. It is clear that

$$(\forall n \in \mathbb{N},\ x_0 \perp x_n) \ \ or \ \ (\forall n \in \mathbb{N},\ x_n \perp x_0).$$

Since T is $\perp$-preserving, we see that

$$(\forall n, k \in \mathbb{N},\ x_k = T^k(x_0) \perp T^k(x_n) = x_{n+k}) \ \ or$$
$$(\forall n, k \in \mathbb{N}, x_{n+k} = T^k(x_n) \perp T^k(x_0) = x_k).$$

This implies that $\{x_n\}$ is an SO-sequence. For better readability, we divide the proof into several steps.

Step 1: $\lim_{n\to\infty} \rho(l(x_{n+1} - x_n)) = 0$.

If there exists $n_0 \in \mathbb{N}$, $x_{n_0} = x_{n_0+1}$ and easily the result follows. Now, let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then using the Meir–Keeler condition, for any $n \in \mathbb{N}$ we have

$$\rho(l(x_{n+1} - x_n)) < \rho(l(x_n - x_{n-1})).$$

This means that the sequence $\{\rho(l(x_{n+1} - x_n))\}$ is strictly decreasing and hence it converges. Put $\lim_{n\to\infty} \rho(l(x_{n+1} - x_n)) = r$. Now we prove $r = 0$. Suppose that $r > 0$. Applying the Meir–Keeler condition to $r > 0$, we can find $\delta(r) > 0$ such that

$$x \neq y,\ x \perp y \ \ and \qquad r \leq \rho(l(x - y)) < r + \delta(r) \qquad \Rightarrow \qquad \rho(c(Tx - Ty)) < r.$$

Since $\lim_{n\to\infty} \rho(l(x_{n+1} - x_n)) = r$, then there exists $n_0 \in \mathbb{N}$ such that

$$r \leq \rho(l(x_{n_0} - x_{n_0-1})) < r + \delta(r) \qquad \Rightarrow \ \rho(c(Tx_{n_0} - Tx_{n_0-1})) < r.$$

Since $\rho(\alpha x)$ is an increasing function of $\alpha \geq 0$, then $\rho(l(x_{n_0+1} - x_{n_0})) < r$ and this is a contradiction because $r = \inf\{\rho(l(x_n - x_{n-1}))\ ;\ n \in \mathbb{N}\}$. Therefore, $r = 0$.

Step 2: $\{x_n\}$ is a ρ-Cauchy SO-sequence .

Suppose $\{x_n\}$ is not a ρ-Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k \geq K$,

$$\rho(c(x_{m_k} - x_{n_k})) \geq \epsilon \quad \ and \qquad \rho(c(x_{m_k-1} - x_{n_k})) < \epsilon. \tag{3.2}$$

To prove (3.2), suppose

$$\sum_k = \{m \in \mathbb{N};\ \exists n_k \geq k, \quad \rho(c(x_m - x_{n_k})) \geq \epsilon\ ,\ m > n_k \geq k\}.$$

Obviously, $\sum_k \neq \emptyset$ and $\sum_k \subseteq \mathbb{N}$, then by the well ordering principle, the minimum element of $\sum_k$ is denoted m_k, and clearly (3.2) holds. There exists $\delta(\epsilon) > 0$ (which can be chosen $\delta(\epsilon) \leq \epsilon$) satisfying (3.1). The Δ_2-condition and Step 1 show that there exists $n_0 \in \mathbb{N}$ for which $\rho(M(x_{n_0} - x_{n_0-1})) < \delta(\epsilon)$, where $M \geq \max\{\alpha l, c\}$. Fix $k \geq n_0$, we have

$$\begin{aligned}\rho(l(x_{m_k-1} - x_{n_k-1})) &= \rho\Big(l(x_{m_k-1} - x_{n_k}) + l(x_{n_k} - x_{n_k-1})\Big)\\ &\leq \rho(c(x_{m_k-1} - x_{n_k})) + \rho(\alpha l(x_{n_k-1} - x_{n_k}))\\ &\leq \rho(c(x_{m_k-1} - x_{n_k})) + \rho(M(x_{n_k-1} - x_{n_k}))\\ &< \epsilon + \delta(\epsilon).\end{aligned}$$

Now, we consider two cases:

Case 1 $\rho(l(x_{m_k-1} - x_{n_k-1})) \geq \epsilon$.

Since x_{n_k-1} and x_{m_k-1} are $\perp$-comparable, applying the condition (3.1) we get

$$\epsilon \leq \rho(l(x_{m_k-1} - x_{n_k-1})) < \epsilon + \delta(\epsilon) \ \Rightarrow\ \rho(c(x_{m_k} - x_{n_k})) < \epsilon.$$

Case 2 $\rho(l(x_{m_k-1} - x_{n_k-1})) < \epsilon$.

Since x_{m_k-1} and x_{n_k-1} are $\perp$-comparable, then using (3.1) we get

$$\rho(c(x_{m_k} - x_{n_k})) < \rho(l(x_{m_k-1} - x_{n_k-1})) < \epsilon.$$

Hence in each cases $\rho(c(x_{m_k} - x_{n_k})) < \epsilon$ and this is a contradiction with (3.2). Therefore, $\{x_n\}$ is a ρ-Cauchy SO-sequence. Since X_ρ is an SO-ρ-complete orthogonal modular space and B is an SO-ρ-closed subset of X_ρ, then there exists $z \in B$ such that $x_n \xrightarrow{\rho} z$. The Δ_2-condition of ρ implies that $\rho(c(x_n - z)) \to 0$ as $n \to \infty$. On the other hand, T is an SO-ρ-continuous function, then for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(c(x_{n_0+1} - z)) < \frac{\epsilon}{2} \quad \text{and} \quad \rho(c(Tx_{n_0} - Tz)) < \frac{\epsilon}{2}.$$

Now,

$$\begin{aligned}\rho(\frac{c}{2}(Tz - z)) &= \rho\left(\frac{c}{2}(Tz - Tx_{n_0}) + \frac{c}{2}(Tx_{n_0} - z)\right)\\ &\leq \rho(c(Tz - Tx_{n_0}) + \rho(c(x_{n_0+1} - z))\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

It follows that $\rho(\frac{c}{2}(Tz - z)) = 0$ and so $Tz = z$.

Now, we show that T is a Picard operator. Let $x \in B$ be arbitrary. By our choice of x_0, we have

$$[x_0 \perp z \text{ and } x_0 \perp x] \text{ or } [z \perp x_0 \text{ and } x \perp x_0].$$

$\perp$-preserving of T implies that

$$[x_n \perp z \text{ and } x_n \perp T^n(x)] \text{ or } [z \perp x_n \text{ and } T^n(x) \perp x_n]$$

for all $n \in \mathbb{N}$. Now, we show that the sequence $\{\rho(c(T^n(x) - x_n))\}$ converges to zero. If for some m_0, $T^{m_0}(x) = x_{m_0}$, then $\rho(c(T^n(x) - x_n)) = 0$ for all $n \geq m_0$. Now, let $T^n(x) \neq x_n$ for all $n \in \mathbb{N}$. The Meir–Keeler condition implies that the sequence $\{\rho(l(T^n(x) - x_n))\}$ is strictly decreasing. Using the same argument of Step 1, we can get that $\lim_{n\to\infty} \rho(c(T^n(x) - x_n)) = 0$. Now, for all $n \in \mathbb{N}$ we obtain that

$$\rho\left(\frac{c}{2}(T^n(x) - z)\right) \leq \rho(c(T^n(x) - x_n)) + \rho(c(x_n - z)).$$

As $n \to \infty$, since $\rho(\alpha x)$ is an increasing function of α and also ρ satisfies the Δ_2-condition then $T^n(x) \xrightarrow{\rho} z$.

Finally, to prove the uniqueness of fixed point, let $x^* \in B$ be a fixed point of T. Then, $T^n(x^*) = x^*$ for all $n \in \mathbb{N}$. It follows from T is a Picard operator that $x^* = z$. □

Corollary 3.6. *Let (X, ρ) be a ρ-complete modular space and ρ-satisfies Δ_2-condition. Suppose that B is a ρ-closed subset of X_ρ and there exist $c, l \in (0, \infty)$ with $c > l$. Assume that $T : B \to B$ be an operator such that:*

For every $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in B$,

$$\epsilon \leq \rho(l(x - y)) < \epsilon + \delta(\epsilon) \quad \Rightarrow \quad \rho(c(Tx - Ty)) < \epsilon. \tag{3.3}$$

Then, T admits a unique fixed point $z \in B$ and for any $x \in B$, the sequence $\{T^n(x)\}$ is ρ-convergent to z.

Proof. For all $x, y \in B$ define $x \perp y$ iff $\rho(l(Tx - Ty)) \leq \rho(l(x - y))$. It is clear that for all $x, y \in B$, $x \perp y$. So $(B, \perp)$ is an O-set. Since X_ρ is ρ-complete and B is ρ-closed, then B is an SO-ρ-closed subset of X_ρ. Also, the definition $\perp$ and condition (3.3) imply that T is $\perp$-preserving, SO-ρ-continuous, and the Meir–Keeler condition holds. Therefore by applying Theorem 3.5, we can see the results.

Now, we show that our main theorem is a real generalization of Corollary 3.6.

Example 3.7. Let

$$X = \{\{x_n\} \subset \mathbb{R};\ \exists n_1, n_2, \ldots, n_k;\ \forall n \neq n_1, n_2, \ldots, n_k, \quad x_n = 0\}$$

and ρ is the norm $\rho(x) = \sum_{n=1}^{\infty} |x_n|$, where $x = \{x_n\} \in X$. Note that (X, ρ) is not ρ-complete because, $A_n = \{1, \frac{1}{2}, \ldots, \frac{1}{2^n}, 0, 0, 0, \ldots\}$, $n \in \mathbb{N}$, is a sequence in X where limit of $\{A_n\}$ is not belong to X. For $x, y \in X$, define

$$x \perp y \iff \exists \alpha \in \{0, 1\} \quad \text{such that } x = \alpha y \text{ or } y = \alpha x.$$

We claim that X is SO-ρ-complete. Take a ρ-Cauchy SO-sequence $\{A_n\}$ in X. If for all $n \in \mathbb{N}$, $A_n = 0$, then $\{A_n\}$ ρ-converges to $A = 0 \in X$. Assume there exists $n_0 \in \mathbb{N}$ which $A_{n_0} \neq 0$. Without loss of generality, $A_1 \neq 0$. For every $n \in \mathbb{N}$, the definition of $\perp$ follows that $A_n = \alpha_n A_1$, where $\alpha_n = 0$ *or* 1. Since $\{A_n\}$ is ρ-Cauchy sequence. Hence,

$$|\alpha_n - \alpha_m|\rho(A_1) = \rho(\alpha_n A_1 - \alpha_m A_1) = \rho(A_n - A_m) \to 0 \ as \ n \to \infty.$$

This shows that $\{\alpha_n\}$ is Cauchy in $\mathbb{R}$. Let $\lim_{n\to\infty} \alpha_n = \alpha$. Obviously, $\alpha = 0$ *or* 1. Let $A = \alpha A_1$. It is clear that $A \in X$. Also

$$\rho(A_n - A) = \rho(\alpha_n A_1 - \alpha A_1) = (\alpha_n - \alpha)\rho(A_1) \to 0 \ as \ n \to \infty.$$

Thus, $\{A_n\}$ is ρ-convergent to $A \in X$ and, hence, X is SO-ρ-complete.

Define a mapping $T : X \to X$ by the formulate:

$$Tx = \begin{cases} \{\frac{x_i}{4}\}, & \text{if } 0 \le \rho(x) \le \frac{1}{4}, \\ \{\frac{1}{12}, 0, 0, \ldots\}, & \text{if } \rho(x) > \frac{1}{4}. \end{cases}$$

First, observe that if $x \perp y$, then $x = 0$ or $y = 0$ or $x = y$, and so $Tx = 0$ or $Ty = 0$ or $Tx = Ty$. In each case, $Tx \perp Ty$. Thus, T is $\perp$-preserving. By the first of the example, we can prove that T is SO-ρ-continuous. Below we show that T satisfies the Meir–Keeler condition for $c = 2$ and $l = 1$. Define $\phi : [0, \infty) \to [0, \infty)$ as:

$$\phi(t) = \begin{cases} \frac{2}{3}t, & if \ 0 \le t \le \frac{3}{4}, \\ 2t - 1, & if \ \frac{3}{4} < t \le 1, \\ 1, & if \ 1 \le t < \infty. \end{cases}$$

Indeed, ϕ is a L-function, that is, $\phi(0) = 0$, $\phi(s) > 0$ for $s > 0$ and for every $s > 0$ there exists $u > s$ such that $\phi(t) < s$ for $t \in [s, u]$. Claim for all $x, y \in X$ with $x \neq y$ and $x \perp y$,

$$\rho\big(2(Tx - Ty)\big) < \phi(\rho(x - y)). \tag{3.4}$$

For any $x, y \in X$ with $x \neq y$ and $x \perp y$, the following cases are hold:

Case 1 $0 \le \rho(x - y) \le \frac{3}{4}$, $x = 0$, $\rho(y) \le \frac{1}{4}$, (or $y = 0, \rho(x) \le \frac{1}{4}$). Then

$$\rho\big(2(Tx - Ty)\big) = 2\rho(Ty) = \frac{\rho(y)}{2} = \frac{\rho(x - y)}{2} < \frac{2}{3}\rho(x - y) = \phi(\rho(x - y)).$$

Case 2 $0 \le \rho(x - y) \le \frac{3}{4}$, $\rho(x) \le \frac{1}{4}$, $\rho(y) > \frac{1}{4}$. The definition of $\perp$ implies that $x = 0$. Hence

$$\rho\big(2(Tx - Ty)\big) = \frac{1}{6} < \frac{2}{3}\rho(y) = \frac{2}{3}\rho(x - y) = \phi(\rho(x - y)).$$

Case 3 $0 \le \rho(x - y) \le \frac{3}{4}$, $\rho(x), \rho(y) > \frac{1}{4}$. In this case

$$\rho\big(2(Tx - Ty)\big) = 0 < \frac{2}{3}\rho(x - y) = \phi(\rho(x - y)).$$

Case 4 $\frac{3}{4} < \rho(x - y) \le 1$, $\rho(x) \le \frac{1}{4}$, $\rho(y) > \frac{1}{4}$. Then, $x = 0$ and

$$\rho\big(2(Tx - Ty)\big) = \frac{1}{6} < 2\rho(x - y) - 1 = \phi(\rho(x - y)).$$

Case 5 $\frac{3}{4} < \rho(x - y) \le 1$, $\rho(x), \rho(y) > \frac{1}{4}$. Obviously,

$$\rho\big(2(Tx - Ty)\big) = 0 < 2\rho(x - y) - 1 = \phi(\rho(x - y)).$$

Case 6 $\rho(x-y)\geq 1,\ \rho(x)\leq\frac{1}{4},\ \rho(y)>\frac{1}{4}$. Then

$$\rho\big(2(Tx-Ty)\big)=\frac{1}{6}<1=\phi(\rho(x-y)).$$

Case 7 $\rho(x-y)>1$ and $\rho(x),\ \rho(y)>\frac{1}{4}$. Obviously,

$$\rho\big(2(Tx-Ty)\big)=0<\phi(\rho(x-y)).$$

In each above case, $\rho\big(2(Tx-Ty)\big)<\phi(\rho(x-y))$. Therefore, (3.4) holds. Let $\epsilon>0$ be given. Since ϕ is L-function, there exists $\delta(\epsilon)>0$ such that for all $t\in[\epsilon,\ \epsilon+\delta(\epsilon)], \phi(t)\leq\epsilon$. So if $x\perp y$, $x\neq y$ and $\epsilon\leq\rho(x-y)<\epsilon+\delta(\epsilon)$, then $\rho\big(2(Tx-Ty)\big)<\phi(\rho(x-y))\leq\epsilon$. This implies Meir–Keeler condition. The existence of unique fixed point of T implies Theorem 3.5.

Notice that the Meir–Keeler condition is not hold for all $x,y\in X$. For example, let $x=\{\frac{1}{4},0,0,\ldots\}$ and $y=\{\frac{1}{4}+\frac{1}{100},0,0,\ldots\}$. We have $\rho(x-y)=\frac{1}{100}\leq\frac{2}{3}$ and $\rho\big(2(Tx-Ty)\big)=\frac{1}{12}>\frac{1}{150}=\frac{2}{3}\rho(x-y)=\phi(\rho(x-y))$.

4. Application

In this section, the existence and uniqueness of a solution to the integral equations in Musielak–Orlicz spaces are studied. Consider the following integral equation:

$$u(t)=\int_o^t e^{s-t}\left(\int_0^b e^{-\xi}g(s,\xi,u(\xi))\ \mathrm{d}\xi\right)\mathrm{d}s, \tag{4.1}$$

where ρ is a convex modular on L^{φ}, satisfying the Δ_2-condition and B is a convex, ρ-closed, ρ-bounded subset of L^{φ} and $0\in B$. Let $b>0$, g be a function from $[0,b]\times[0,b]\times B$ into B, $\gamma:[0,b]\times[0,b]\times[0,b]\to\mathbb{R}^+$ be measurable functions for which:

(H_1) (i): $g(t,.,x):s\to g(t,s,x)$ is a measurable function for every $x\in B$ and for almost all $t\in[0,b]$. (ii): $g(t,s,.):x\to g(t,s,x)$ is ρ-continuous on B for almost $t,s\in[0,b]$.

(H_2) (i): $g(t,s,x)\geq 0$ for all $x\geq 0$ and for almost $t,s\in[0,b]$. (ii) $g(t,s,x)g(t',r,y)\geq g(t,t',xy)$ for each $x,y\in B$ with $xy\geq 0$ and for almost $t,t',s,r\in[0,b]$.

(H_3) There exists $\lambda>0$ such that $\rho(g(t,s,x)-g(t,s,y))\leq\lambda\rho(x-y)$ for all $(t,s,x),(t,s,y)\in[0,b]\times[0,b]\times B$ with $xy\geq 0$.

(H_4) $\rho(g(t,s,x)-g(\tau,s,x))\leq\gamma(t,\tau,s)$ for all $(t,s,x),(\tau,s,x)\in[0,b]\times[0,b]\times B$ and
$\lim_{t\to\infty}\int_0^b\gamma(t,\tau,s)\ \mathrm{d}s=0$ uniformly for $\tau\in[0,b]$.

We denote by $D=C([0,b],B)$ the space of all ρ-continuous function from $[0,b]$ into B, endowed with the modular ρ_a defined by $\rho_a(u)=$

$\sup_{t\in[0,b]} e^{-at}\rho(u(t))$, where $a \geq 0$. By ([7], Prop. 2.1), D is a convex, ρ_a-bounded, ρ_a-closed subset of ρ_a-complete space $C([0,b], L^{\varphi})$. Define the operators T and S on D by

$$Tu(t) = \int_0^b e^{-s} g(t,s,u(s))\ \mathrm{d}s$$
$$Su(t) = \int_0^t e^{s-t} Tu(s)\ \mathrm{d}s.$$

Note that the fixed points of S are the solutions of (4.1). By ([15], Prop. 3.3), D is invariant under the operators T and S.

Theorem 4.1. *Under mention conditions, for all $b > 0$ the integral equation (4.1) has a unique solution in D.*

Proof. We consider the following orthogonal relation in D:

$$u \perp v \Leftrightarrow \forall t,s \in [0,b],\ \ u(t)\ v(s) \geq 0.$$

Since D is a ρ_a-closed subset of ρ_a-complete modular space $C([0,b], L^{\varphi})$, then D is an SO-ρ-closed subset of SO-ρ-complete orthogonal modular space $C([0,b], L^{\varphi})$. To complete the proof, we need the following steps:

Step 1: S is $\perp$-preserving. In fact, for each $u, v \in D$ with $u \perp v$, by hypothesis $(H_2)(i)$ and $(H_2)(ii)$, we obtain

$$\begin{aligned} Tu(t)\ Tv(t') &= \int_0^b e^{-s} g(t,s,u(s))\ \mathrm{d}s \int_0^b e^{-r} g(t',r,v(r))\ \mathrm{d}r \\ &= \int_0^b \int_0^b e^{-(s+r)} g(t,s,u(s)) g(t',r,v(r))\ \mathrm{d}r\ \mathrm{d}s \\ &\geq \int_0^b \int_0^b e^{-(s+r)} g(t,t',u(s)v(r))\ \mathrm{d}r\ \mathrm{d}s \qquad u(s)v(r) \geq 0 \\ &\geq 0 \end{aligned}$$

for each $t, t' \in [0,b]$. So $Tu \perp Tv$. Definition of S implies that $Su \perp Sv$.

Step 2: We show that S is ρ_a-Lipschitz on $\perp$-comparable elements. Let $K = \{s_0, s_1, \ldots, s_m\}$ be a subdivision of $[0,b]$. Then, $\sum_{i=0}^{m-1}(s_{i+1} - s_i)e^{-s_i}x(s_i)$ is $\|.\|$-convergent and, consequently, ρ-convergent to $\int_0^b e^{-s}x(s)\mathrm{d}s$ in L^{φ} when $|K| = \sup\{|s_{i+1} - s_i|;\ i = 0,1,\ldots,m-1\} \to 0$ as $m \to \infty$. Let $u \perp v$, then

$$\int_0^b e^{-s} g\big(t,s,u(s)) - g(t,s,v(s)\big)\ \mathrm{d}s = \lim_{m\to\infty} \sum_{i=0}^{m-1} (s_{i+1} - s_i) e^{-s_i} \big(g(t,s_i,u(s_i) - g(t,s_i,v(s_i))\big)$$

and $\sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{-s_i} \leq \int_0^b e^{-s}\mathrm{d}s = 1-e^{-b} < 1$, by Fatou property and condition (H_3)

$$\begin{aligned}\rho(Tu(t)-Tv(t)) &\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{-s_i}\rho\big(g(t,s_i,u(s_i))-g(t,s_i,v(s_i))\big)\\ &\leq \lambda \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{-s_i}\rho(u(s_i)-v(s_i))\\ &\leq \lambda \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{as_i}\rho_a(u-v).\end{aligned}$$

Therefore,

$$\begin{aligned}e^{-at}\rho(Tu(t)-Tv(t)) &\leq \lambda e^{-at}\left(\int_0^b e^{as}\mathrm{d}s\right)\rho_a(u-v)\\ &\leq \lambda\frac{e^{ab}-1}{a}\ \rho_a(u-v).\end{aligned}$$

Hence,

$$\rho_a(Tu-Tv) \leq \lambda\frac{e^{ab}-1}{a}\ \rho_a(u-v).$$

Definition of S implies that

$$\rho_a(Su-Sv) \leq R\ \rho_a(u-v).$$

where $R = \frac{\lambda}{a(a+1)}(1-e^{-(a+1)b})(e^{ab}-1)$.

Step 3: S is SO-ρ-continuous. To see this, let $\{u_n\} \subset D$ be an SO-sequence in D converging to $u \in D$. By the definition of orthogonality, $u_k(t)u_{n+k}(t') \geq 0$ for all $n, k \in \mathbb{N}$ and $t, t' \in [0, b]$. By Δ_2-condition of ρ we have $u_k(t)u(t') \geq 0$ for all $k \in \mathbb{N}$ and $t, t' \in [0, b]$. This implies that $u_k \perp u$ for all $k \in \mathbb{N}$. By applying Step 2, we have

$$\rho_a(Su_n-Su) \leq R\ \rho_a(u_n-u).$$

This implies the SO-ρ-continuity of S.

Step 4: S satisfies the Meir–Keeler condition for $c = 2$ and $l = 1$. Indeed, define

$$\delta(\epsilon) = \inf\{\rho_a(u-v);\ \rho_a(2(Su-Sv)) \geq \epsilon \ \ \textit{and}\ u\perp v\}.$$

Let $0 < R < 1$ and $\epsilon > 0$ be given. If $u \perp v$ and $\rho_a(2(Su-Sv)) \geq \epsilon$, by Step 2, we get $\rho_a(u-v) \geq R^{-1}\epsilon$. So $\delta(\epsilon) \geq R^{-1}\epsilon > \epsilon$. Using the same argument of ([16], Theorem 1), we get that S satisfies Meir–Keeler condition.

In the end, by Theorem 3.5, S has a unique fixed point which is a solution of the integral equation (4.1). □

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