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Fixed point method for set-valued functional equations

Choonkil Park

Abstract. In this paper, we introduce a set-valued cubic functional equation and a set-valued quartic functional equation and prove the Hyers-Ulam stability of the set-valued cubic functional equation and the set-valued quartic functional equation by using the fixed point method.

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1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the past decades. The pioneering papers by Aumann [5] and Debreu [12] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [28], the monographs by Hindenbrand [19], Aubin and Frankowska [4], Castaing and Valadier [8], Klein and Thompson [25] and the survey by Hess [18].

The stability problem of functional equations originated from a question of Ulam [48] concerning the stability of group homomorphisms. Hyers [20] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [44] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

In [24], Jun and Kim considered the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation*, and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [27], Lee et al. considered the following quartic functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \quad (1.2)$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1,16,17,21,22,41–43,45–47]).

Let X be a set. A function $d: X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Let (X,d) be a generalized metric space. An operator $T:X\to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L\ge 0$ such that $d(Tx,Ty)\le Ld(x,y)$ for all $x,y\in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz:

Theorem 1.1. [9,13] *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n\ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $S=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\le \frac{1}{1-L}d(y,Jy)$ for all $y\in S$.*

In 1996, Isac and Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10,11,14,15,30,31,36,37,40]).

Let Y be a Banach space. We define the following:

2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \quad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \quad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are, respectively, defined by

$$d(x, C) = \inf\{\|x - y\| : y \in C\}, \quad x \in Y,$$
$$s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}, \quad x^* \in Y^*.$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \quad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance:

Proposition 1.2. *For every $C, C', K, K' \in C_{\text{cb}}(Y)$ and $\lambda > 0$, the following properties hold:*

(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;
(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{\text{cb}}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{\text{cb}}(Y), \oplus, h)$ is a complete metric semigroup (see [8]). Debreu [12] proved that $(C_{\text{cb}}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows;

Lemma 1.3. [12] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{\text{cb}}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*

(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $h(A, B) = \|j(A) - j(B)\|_u$;
(d) $j(C_{\text{cb}}(Y))$ *is closed in* $C(B_{Y^*})$

for all $A, B \in C_{\text{cb}}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{\text{cb}}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{\text{cb}}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [7]). In this case, the Debreu integral of f in Ω is the unique element $(D) \int_\Omega f \mathrm{d}\nu \in C_{\text{cb}}(Y)$ such that $j((D) \int_\Omega f \mathrm{d}\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{\text{cb}}(Y)$ will be denoted by $D(\Omega, C_{\text{cb}}(Y))$. Furthermore, on $D(\Omega, C_{\text{cb}}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{\text{cb}}(Y))$. Then we obtain that $((\Omega, C_{\text{cb}}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6,32–35,38,39]).

Using the fixed point method, we prove the Hyers–Ulam stability of the following additive set-valued functional equations:

$$F(2x+y) \oplus F(2x-y) = 2F(x+y) \oplus 2F(x-y) \oplus 12F(x)$$

and

$$F(2x+y) \oplus F(2x-y) \oplus 6F(y) = 4F(x+y) \oplus 4F(x-y) \oplus 24F(x).$$

Throughout this paper, let X be a real normed space and Y a real Banach space.

2. Stability of the set-valued cubic functional equation

Using the fixed point method, we prove the Hyers–Ulam stability of the set-valued cubic functional equation:

Definition 2.1. [26] Let $F : X \to C_{cb}(Y)$. The set-valued cubic functional equation is defined by

$$F(2x+y) \oplus F(2x-y) = 2F(x+y) \oplus 2F(x-y) \oplus 12F(x)$$

for all $x, y \in X$. Every solution of the set-valued cubic functional equation is called a *set-valued cubic mapping*.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{8}\varphi(2x, 2y) \tag{2.1}$$

for all $x, y \in X$. If $F : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h(F(2x+y) \oplus F(2x-y), 2F(x+y) \oplus 2F(x-y) \oplus 12F(x)) \le \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$, then there exists a unique set-valued cubic mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(F(x), C(x)) \le \frac{L}{16 - 16L}\varphi(x, 0) \tag{2.3}$$

for all $x \in X$.

Moreover, if r and M are positive real numbers with $r > 3$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$, then C is single-valued.

Proof. Letting $x = y = 0$ in (2.1), we get $\varphi(0, 0) \le \frac{L}{8}\varphi(0, 0)$ and so $\varphi(0, 0) = 0$.

Letting $x = y = 0$ in (2.2),

$$h(F(0) \oplus F(0), 2F(0) \oplus 2F(0) \oplus 12F(0)) \le \varphi(0, 0) = 0$$

and so $F(0) = \{0\}$.

Let $y = 0$ in (2.2). Since $F(x)$ is convex, we get

$$h(2F(2x), 16F(x)) \le \varphi(x, 0) \tag{2.4}$$

and so

$$h\left(F(x), 8F\left(\frac{x}{2}\right)\right) \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{16}\varphi\left(x, 0\right) \tag{2.5}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{\mathrm{cb}}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on S,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ h(g(x), f(x)) \leq \mu\varphi(x, 0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [15, Theorem 2.4] and [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g, f) = \varepsilon$. Then

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} h(Jg(x), Jf(x)) &= h\left(8g\left(\frac{x}{2}\right), 8f\left(\frac{x}{2}\right)\right) = 8h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \\ &\leq \varepsilon L\varphi(x, 0) \end{aligned}$$

for all $x \in X$. So $d(g, f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d(Jg, Jf) \leq Ld(g, f)$$

for all $g, f \in S$.

It follows from (2.5) that $d(F, JF) \leq \frac{L}{16}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.6}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$K = \{g \in S : d(F, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$h(F(x), C(x)) \leq \mu\varphi(x, 0)$$

for all $x \in X$;

(2) $d(J^nF, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n F\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(F,C) \le \frac{1}{1-L} d(F,JF)$, which implies the inequality

$$d(F,C) \le \frac{L}{16-16L}.$$

This implies that the inequality (2.3) holds.

It follows from (2.2) that

$$\begin{aligned} &h(C(2x+y) \oplus C(2x-y), 2C(x+y) \oplus 2C(x-y) \oplus 12C(x)) \\ &= \lim_{n\to\infty} \left[8^n h \left(F\left(\frac{2x+y}{2^n}\right) \oplus F\left(\frac{2x-y}{2^n}\right), \right.\right. \\ &\quad \left.\left. \times\ F\left(\frac{x+y}{2^n}\right) \oplus F\left(\frac{x-y}{2^n}\right) \oplus 12F\left(\frac{x}{2^n}\right)\right)\right] \\ &\le \lim_{n\to\infty} \left[8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right] = 0 \end{aligned}$$

for all $x, y \in X$. Thus

$$C(2x+y) \oplus C(2x-y) = 2C(x+y) \oplus 2C(x-y) \oplus 12C(x)$$

for all $x, y \in X$.

If r and M are positive real numbers with $r > 3$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$, then $\operatorname{diam}\left(8^n F\left(\frac{x}{2^n}\right)\right) \le \frac{8^n}{2^{rn}} M\|x\|^r$ for all $x \in X$ and so $C(x) = \lim_{n\to\infty}\left[8^n F\left(\frac{x}{2^n}\right)\right]$ is a singleton set. □

Corollary 2.3. *Let $p > 3$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. If $F : X \to (C_{\mathrm{cb}}(Y), h)$ is a mapping satisfying*

$$\begin{aligned} &h(F(2x+y) \oplus F(2x-y), 2F(x+y) \oplus 2F(x-y) \oplus 12F(x)) \\ &\le \theta(||x||^p + ||y||^p) \end{aligned} \tag{2.7}$$

for all $x, y \in X$, then there exists a unique set-valued cubic mapping $C : X \to Y$ satisfying

$$h(F(x), C(x)) \le \frac{\theta}{2(2^p - 8)} ||x||^p$$

for all $x \in X$.

Moreover, if r and M are positive real numbers with $r > 3$ and $\operatorname{diam} F(x) \le M\|x\|^r$ for all $x \in X$, then C is single-valued.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired results. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. *If* $F : X \to (C_{\mathrm{cb}}(Y), h)$ *is a mapping satisfying* $F(0) = \{0\}$ *and* (2.2), *then there exists a unique set-valued cubic mapping* $C : X \to (C_{\mathrm{cb}}(Y), h)$ *such that*

$$h(F(x), C(x)) \leq \frac{1}{16 - 16L}\varphi(x, 0)$$

for all $x \in X$.

Moreover, if r *and* M *are positive real numbers with* $r < 3$ *and* $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all* $x \in X$, *then* C *is single-valued.*

Proof. It follows from (2.4) that

$$h\left(F(x), \frac{1}{8}F(2x)\right) \leq \frac{1}{16}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let* $3 > p > 0$ *and* $\theta \geq 0$ *be real numbers, and let* X *be a real normed space. If* $F : X \to (C_{\mathrm{cb}}(Y), h)$ *is a mapping satisfying* $F(0) = \{0\}$ *and* (2.7), *then there exists a unique set-valued cubic mapping* $C : X \to Y$ *satisfying*

$$h(F(x), C(x)) \leq \frac{\theta}{2(8 - 2^p)}||x||^p$$

for all $x \in X$.

Moreover, if r *and* M *are positive real numbers with* $r < 3$ *and* $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all* $x \in X$, *then* C *is single-valued.*

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

3. Stability of the set-valued quartic functional equation

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued quartic functional equation.

Definition 3.1. [26] Let $F : X \to C_{\mathrm{cb}}(Y)$. The set-valued quartic functional equation is defined by

$$F(2x + y) \oplus F(2x - y) \oplus 6F(y) = 4F(x + y) \oplus 4F(x - y) \oplus 24F(x)$$

for all $x, y \in X$. Every solution of the set-valued quartic functional equation is called a *set-valued quartic mapping.*

Theorem 3.2. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq \frac{L}{16}\varphi(2x, 2y) \tag{3.1}$$

for all $x, y \in X$. *If* $F : X \to (C_{\text{cb}}(Y), h)$ *is a mapping satisfying*

$$h\left(F(2x+y) \oplus F(2x-y) \oplus 6F(y), 4F(x+y) \oplus 4F(x-y) \oplus 24F(x)\right) \\ \leq \varphi(x,y) \tag{3.2}$$

for all $x, y \in X$, *then there exists a unique set-valued quartic mapping* $Q : X \to (C_{\text{cb}}(Y), h)$ *such that*

$$h(F(x), Q(x)) \leq \frac{L}{32 - 32L}\varphi(x,0)$$

for all $x \in X$. *Moreover, if* r *and* M *are positive real numbers with* $r > 4$ *and* $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all* $x \in X$, *then* Q *is single-valued.*

Proof. Letting $x = y = 0$ in (3.1), we get $\varphi(0,0) \leq \frac{L}{16}\varphi(0,0)$ and so $\varphi(0,0) = 0$.

Letting $x = y = 0$ in (3.2),

$$h(F(0) \oplus F(0) \oplus 6F(0), 4F(0 \oplus 4F(0) \oplus 24F(0)) \leq \varphi(0,0) = 0$$

and so $F(0) = \{0\}$.

Let $y = 0$ in (3.2). Since $F(x)$ is convex, we get

$$h(2F(2x), 32F(x)) \leq \varphi(x,0) \tag{3.3}$$

and so

$$h\left(F(x), 16F\left(\frac{x}{2}\right)\right) \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq \frac{L}{32}\varphi(x,0) \tag{3.4}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{\text{cb}}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on S,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \leq \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [15, Theorem 2.4] and [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.4) that $d(F, JF) \leq \frac{L}{32}$.

The rest of the proof is similar to the Proof of Theorem 2.2. □

Corollary 3.3. *Let* $p > 4$ *and* $\theta \geq 0$ *be real numbers, and let* X *be a real normed space. If* $F : X \to (C_{\text{cb}}(Y), h)$ *is a mapping satisfying*

$$h\left(F(2x+y) \oplus F(2x-y) \oplus 6F(y), 4F(x+y) \oplus 4F(x-y) \oplus 24F(x)\right) \\ \leq \theta(||x||^p + ||y||^p) \tag{3.5}$$

for all $x, y \in X$, *then there exists a unique set-valued quartic mapping* $Q : X \to Y$ *satisfying*

$$h(F(x), Q(x)) \leq \frac{\theta}{2(2^p - 16)}||x||^p$$

for all $x \in X$.

Moreover, if r and M are positive real numbers with $r > 4$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all $x \in X$, then Q is single-valued.*

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$.

Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. If $F : X \to (C_{\mathrm{cb}}(Y), h)$ is a mapping satisfying $F(0) = \{0\}$ and (3.2), then there exists a unique set-valued quartic mapping $Q : X \to (C_{\mathrm{cb}}(Y), h)$ such that

$$h(F(x), Q(x)) \leq \frac{1}{32 - 32L}\varphi(x, 0)$$

for all $x \in X$.

Moreover, if r and M are positive real numbers with $r < 4$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all $x \in X$, then Q is single-valued.*

Proof. It follows from (3.3) that

$$h\left(F(x), \frac{1}{16}F(2x)\right) \leq \frac{1}{32}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Corollary 3.5. *Let $0 < p < 4$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. If $F : X \to (C_{\mathrm{cb}}(Y), h)$ is a mapping satisfying $F(0) = \{0\}$ and (3.5), then there exists a unique set-valued quartic mapping $Q : X \to Y$ satisfying*

$$h(F(x), Q(x)) \leq \frac{\theta}{2(16 - 2^p)}||x||^p$$

for all $x \in X$.

Moreover, if r and M are positive real numbers with $r < 4$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ *for all $x \in X$, then Q is single-valued.*

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

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Choonkil Park
Research Institute for Natural Sciences
Hanyang University
Seoul 04763
Republic of Korea
e-mail: `baak@hanyang.ac.kr`