J. Fixed Point Theory Appl. 19 (2017) 2185–2208
DOI 10.1007/s11784-017-0412-6
Published online February 11, 2017

Journal of Fixed Point Theory and Applications

Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects

Tran Dinh Ke and Do Lan

Abstract. We prove the global solvability and weakly asymptotic stability for a semilinear fractional differential inclusion subject to impulsive effects by analyzing behavior of its solutions on the half-line. Our analysis is based on a fixed point principle for condensing multi-valued maps, which is employed for solution operator acting on the space of piecewise continuous functions. The obtained results will be applied to a lattice fractional differential system.

Mathematics Subject Classification. 34A60, 35R11, 34A08, 35B35, 35R12, 47H08, 47H10.

Keywords. Weakly asymptotic stability, fractional differential inclusion, impulsive effect, condensing map, fixed point, measure of non-compactness, MNC estimate.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. Consider the following problem:

$$D_0^\alpha u(t) \in Au(t) + F(t, u(t), u_t), \qquad t > 0, t \neq t_k, k \in \Lambda, \tag{1.1}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{1.2}$$

$$u(s) = \varphi(s), \qquad s \in [-h, 0], \tag{1.3}$$

where $D_0^\alpha, \alpha \in (0,1)$, is the fractional derivative in the Caputo sense, A is a closed linear operator in X which generates a strongly continuous semigroup $W(\cdot)$, $F : \mathbb{R}^+ \times X \times C([-h, 0]; X) \to \mathcal{P}(X)$ is a multi-valued map, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k \in \Lambda \subset \mathbb{N}$, I_k and g are the functions which will be specified in Sect. 3. Here, u_t stands for the history of the state function up to the time t, i.e., $u_t(s) = u(t + s), s \in [-h, 0]$.

The system (1.1)–(1.3) is a generalized Cauchy problem which involves impulsive effect expressed by (1.2). It should be mentioned that the multi-valued nonlinearity in (1.1) appears frequently in control theory, where control factor is taken in the form of multi-valued feedback (see, e.g., [9]) and the delay term is concerned as an inherence in control problems. On the other hand, the impulsive condition (1.3) is employed to describe processes subject to abrupt changes happening in biology, mechanics, electrical engineering, etc. A systematic study on impulsive differential equations can be found in [18,20].

In the last decade, there have been extensive studies devoted to particular cases of our problem in the literature. We refer to some typical results on existence and properties of solution set presented in [4,21–23], in which the solvability on compact intervals and the structure of solution set like R_δ-set were proved. Regarding related control problems, it should be mentioned the results on controllability given in [13–17,24,25,29], where the fixed point approach was employed as a fruitful method. More results on solvability and controllability can be found in the reference quoted in these works.

One of the most important questions associated with problem (1.1)–(1.3) is to analyze the stability of its solutions. Unfortunately, the results on this direction are less known. In [11], we proved a stability result for (1.1)–(1.3) in the case when F is single valued and Lipschitzian. However, the technique used in [11] does not work when F is a multi-valued map. Moreover, the classical concept of stability due to Lyapunov is inappropriate for applying to multi-valued cases. Therefore, we adopt the following concept of weakly asymptotic stability of zero solution to inclusion (1.1): Let $\Sigma(\varphi)$ be the solution set of (1.1)–(1.3) with respect to the initial datum φ such that $0 \in \Sigma(0)$. The zero solution of (1.1) is said to be weakly asymptotically stable if it is

(1) *stable*: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\varphi\|_h < \delta$ then $\|u_t\|_h < \epsilon$ for any $u \in \Sigma(\varphi)$, here $\|\cdot\|_h$ denotes the norm in $C([-h,0];X)$;
(2) *weakly attractive*: for any $\varphi \in \mathcal{B}$, there exists $u \in \Sigma(\varphi)$ satisfying $\|u_t\|_h \to 0$ as $t \to +\infty$.

It should be noted that this concept will coincide with the classical one in the Lyapunov sense if the solution to the Cauchy problem is unique, since the only difference is the weak attractivity. It requires that at least one trajectory (rather than all trajectories) at each starting point approaches the equilibrium point. So this concept is useful in the control theory and has a relation to the viability theory (see, e.g., [1]).

Our main aim in this work is to prove, for the first time, the weakly asymptotic stability of the zero solution to (1.1). To this end, we make use of fixed point approach. More precisely, by constructing a new measure of noncompactness (MNC), denoted by χ^*, on weighted spaces of piecewise continuous functions on the half-line, denoted by PC_ϱ, we are able to show that the solution operator is χ^*-condensing, which implies the existence of exponentially bounded solutions. Consequently, when the semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is exponentially stable, we obtain the weakly asymptotic stability as aforementioned. The features of our work include

- Constructing a new MNC on the space PC_ϱ in order to present a compactness condition on this space.
- Analyzing the fixed point set of the solution operator associated with (1.1)–(1.3) on PC_ϱ to derive an asymptotic estimate of solutions to

(1.1)–(1.3). Based on this estimate, the weakly asymptotic stability will be proved.

The rest of our work is as follows. In the next section, we introduce the MNC χ^* on PC_ϱ which can be used to characterize the compactness on this space. We also give some estimates via MNCs which will be used in Sect. 3. Section 3 is devoted to proving the solvability of (1.1)–(1.3) on PC_ϱ. In Sect. 4, the result on weakly asymptotic stability of the zero solution is derived when the semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is exponentially stable. The last section shows an application of the obtained results to a lattice differential system.

2. Preliminaries

2.1. Measure of noncompactness and condensing operators

Let $\mathcal{E}$ be a Banach space. Denote

$$\mathcal{P}(\mathcal{E}) = \{Y \subset \mathcal{E} : Y \neq \emptyset\},$$
$$\mathcal{P}_b(\mathcal{E}) = \{Y \in \mathcal{P}(\mathcal{E}) : Y \text{ is bounded}\},$$
$$K_v(\mathcal{E}) = \{Y \in \mathcal{P}(\mathcal{E}) : Y \text{ is compact and convex}\}.$$

We will use the following definition of measure of noncompactness (see [9]).

Definition 2.1. A function $\beta : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^+$ is called *a measure of noncompactness* (MNC) in $\mathcal{E}$ if

$$\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega) \qquad \text{for every } \ \Omega \in \mathcal{P}_b(\mathcal{E}),$$

where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

- (i) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}_b(\mathcal{E}),\ \Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
- (ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}, \Omega \in \mathcal{P}_b(\mathcal{E})$;
- (iii) invariant with respect to union with compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}_b(\mathcal{E})$;
- (iv) algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}_b(\mathcal{E})$;
- (v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as follows, for $\Omega \in \mathcal{P}_b(\mathcal{E})$,

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \quad \text{has a finite } \varepsilon\text{-net}\}.$$

The Hausdorff MNC satisfies all properties stated in Definition 2.1. If $\mathcal{T} \in L(\mathcal{E})$, the space of bounded linear operators on X, we define the χ-norm of $\mathcal{T}$ as follows:

$$\|\mathcal{T}\|_\chi = \inf\{\eta > 0 : \chi(\mathcal{T}(B)) \leq \eta{\cdot}\chi(B), \quad \text{for all bounded set } B \subset \mathcal{E}\}. \tag{2.1}$$

It is clear that

$$\chi\left(\mathcal{T}(B)\right) \leq \|\mathcal{T}\|_\chi \cdot \chi(B), \quad \forall B \subset \mathcal{E}.$$

In addition, $\|\mathcal{T}\|_\chi \leq \|\mathcal{T}\|$ and $\mathcal{T}$ is a compact operator iff $\|\mathcal{T}\|_\chi = 0$.

Let $\mathcal{E} = \mathrm{PC}(J; X)$, the space of X-valued functions defined on $J \subset \mathbb{R}$, satisfying that, for every $u \in \mathrm{PC}(J; X)$

- u is continuous on $J \setminus \{t_k : k \in \Lambda\}$;
- there exist $u(t_k^+) = \lim_{t\to t_k^+} u(t)$ and $u(t_k^-) = \lim_{t\to t_k^-} u(t)$ such that $u(t_k^-) = u(t_k)$.

If J is a compact interval, $\mathrm{PC}(J;X)$ with the norm

$$\|u\|_{\mathrm{PC}} = \sup_{t\in J} \|u(t)\|,$$

becomes a Banach space. Let χ_{PC} be the Hausdorff MNC in $\mathrm{PC}(J;X)$, then one knows that (see [8]), for any bounded set $D \subset \mathrm{PC}(J;X)$,

- $\sup_{t\in J} \chi(D(t)) \le \chi_{\mathrm{PC}}(D)$, where $D(t) = \{u(t) : d \in D\}$;
- If D is equicontinuous on each interval $(t_{k-1}, t_k] \subset J$ then $\chi_{\mathrm{PC}}(D) = \sup_{t\in J} \chi(D(t))$.

In the case J is the half-line, i.e., $J = [0, +\infty)$, we consider the following space:

$$\mathrm{PC}_\varrho := \mathrm{PC}_\varrho([0,+\infty);X) = \{u \in \mathrm{PC}([0,+\infty);X) : \lim_{t\to+\infty} \frac{u(t)}{\varrho(t)} = 0\},$$

where $\varrho : \mathbb{R}^+ \to [1, +\infty)$ is a continuous and nondecreasing function. We see that $\mathrm{PC}_\varrho([0,+\infty);X)$ with the norm

$$\|u\|_\varrho = \sup_{t\ge 0} \frac{\|u(t)\|}{\varrho(t)},$$

is a Banach space. In fact, we have no formulation of the Hausdorff MNC in PC_ϱ. We will define a new MNC in this space, which is monotone, nonsingular and regular. For $u \in \mathrm{PC}_\varrho$, we denote by $\pi_T(u)$ the restriction of u to $[0,T]$, i.e., $\pi_T(u) \in \mathrm{PC}([0,T];X)$. For $D \subset \mathrm{PC}_\varrho$, put

$$\chi_\infty(D) = \sup_{T>0} \chi_{\mathrm{PC}}(\pi_T(D)), \tag{2.2}$$

$$d_\infty(D) = \lim_{T\to+\infty} \sup_{u\in D} \sup_{t\ge T} \frac{\|u(t)\|}{\varrho(t)}, \tag{2.3}$$

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.4}$$

It is easily seen that $\chi_\infty(\cdot)$ and $d_\infty(\cdot)$ are monotone and nonsingular MNCs, so is $\chi^*(\cdot)$. We will prove an important property of $\chi^*(\cdot)$ in the next lemma.

Lemma 2.1. *Let $\Omega \subset \mathrm{PC}_\varrho([0,+\infty);X)$ be a bounded set such that $\chi^*(\Omega) = 0$. Then, Ω is relatively compact.*

Proof. Let $\epsilon > 0$. Since $d_\infty(\Omega) = 0$, one can choose $T > 0$ such that

$$\left\| \frac{u(t)}{\varrho(t)} \right\| < \frac{\epsilon}{3}, \quad \forall t \ge T, \quad \forall u \in \Omega. \tag{2.5}$$

Let $\{u_n\}$ be a sequence in Ω. Then, $\chi_\infty(\{u_n\}) = 0$. This implies $\chi_{\mathrm{PC}}(\pi_T(\{u_n\})) = 0$, and hence $\{u_n|_{[0,T]}\}$ has a convergent subsequence in $\mathrm{PC}([0,T];X)$ (still indexed by n). So there exists $N(\epsilon) \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|u_n(t) - u_m(t)\| < \frac{\epsilon}{3}, \quad \forall n, m \ge N(\epsilon).$$

Accordingly,

$$\sup_{t\in[0,T]}\left\|\frac{u_n(t)}{\varrho(t)}-\frac{u_m(t)}{\varrho(t)}\right\|<\frac{\epsilon}{3},\quad \forall n,\ m\geq N(\epsilon). \tag{2.6}$$

Combining (2.5) and (2.6), one gets

$$\begin{aligned}\|u_n-u_m\|_\varrho &= \sup_{t\geq 0}\left\|\frac{u_n(t)}{\varrho(t)}-\frac{u_m(t)}{\varrho(t)}\right\|\\ &\leq \sup_{t\in[0,T]}\left\|\frac{u_n(t)}{\varrho(t)}-\frac{u_m(t)}{\varrho(t)}\right\|+\sup_{t\geq T}\left\|\frac{u_n(t)}{\varrho(t)}-\frac{u_m(t)}{\varrho(t)}\right\|\\ &\leq \sup_{t\in[0,T]}\left\|\frac{u_n(t)}{\varrho(t)}-\frac{u_m(t)}{\varrho(t)}\right\|+\sup_{t\geq T}\left\|\frac{u_n(t)}{\varrho(t)}\right\|+\sup_{t\geq T}\left\|\frac{u_m(t)}{\varrho(t)}\right\|\\ &\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon,\end{aligned}$$

for all $n, m \geq N(\epsilon)$. Therefore, $\{u_n\}$ is a Cauchy sequence in $\mathrm{PC}_\varrho([0,+\infty);X)$. The proof is complete. □

We now recall some notions of set-valued analysis and fixed point theory for condensing multi-valued maps. Let Y be a metric space.

Definition 2.2. [9] A multi-valued map (multimap) $\mathcal{F}: Y \to \mathcal{P}(\mathcal{E})$ is said to be:

- (i) upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) := \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ is a closed subset of Y for every closed set $V \subset \mathcal{E}$;
- (ii) closed if its graph $\Gamma_{\mathcal{F}} := \{(y,z) : z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times \mathcal{E}$.

Definition 2.3. A multimap $\mathcal{F} : Z \subseteq E \to \mathcal{P}(\mathcal{E})$ is said to be condensing with respect to an MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the inequality

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in $\mathcal{E}$. We recall a fixed point principle for condensing multi-valued maps (see, e.g., [9]), which is the main tool for our purpose.

Theorem 2.2. [9, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and let $\mathcal{F} : \mathcal{M} \to K_v(\mathcal{M})$ be a closed and β-condensing multimap. Then,* $\mathrm{Fix}(\mathcal{F}) := \{x \in \mathcal{M} : x \in \mathcal{F}(x)\}$ *is nonempty.*

2.2. Fractional calculus

Let $L^p(0,T;X), p \in (1,+\infty)$ be the space of X-valued functions u defined on $[0,T]$ such that the function $t \mapsto \|u(t)\|^p$ is integrable. The integrals appeared in this work will be understood in the Bochner sense. The notation $L^p(0,T)$ stands for $L^p(0,T;\mathbb{R})$. We now recall some notions in fractional calculus (see, e.g., [10,27]).

Definition 2.4. The fractional integral of order $\beta > 0$ of a function $f \in L^1(0,T;X)$ is defined by

$$I_0^\beta f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\beta-1} f(s)\mathrm{d}s,$$

where Γ is the Gamma function, provided the integral converges.

Definition 2.5. For a function $f \in C^1([0,T];X)$, the Caputo fractional derivative of order $\alpha \in (0,1)$ is defined by

$$D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s.$$

Consider the following problem:

$$D_0^\alpha u(t) = Au(t) + f(t), \quad t > 0,\ t \neq t_k \in (0,+\infty),\ k \in \Lambda, \tag{2.7}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{2.8}$$

$$u(s) = \varphi(s), \quad s \in [-h, 0], \tag{2.9}$$

where $\alpha \in (0,1)$ and $f \in L^p(0,T;X)$. In this note, we assume that the C_0-semigroup $W(\cdot)$ generated by A is globally bounded, i.e.

$$\|W(t)x\| \le M_A\|x\|, \quad \forall t \ge 0,\ x \in X. \tag{2.10}$$

for some $M_A \ge 1$. By the arguments in [11] and [28], we have the following presentation:

$$u(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\mathrm{d}s, \quad t > 0, \tag{2.11}$$

where

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta) W(t^\alpha\theta)x\mathrm{d}\theta, \tag{2.12}$$

$$P_\alpha(t)x = \alpha \int_0^\infty \theta\phi_\alpha(\theta) W(t^\alpha\theta)x\mathrm{d}\theta, \quad x \in X, \tag{2.13}$$

$$\phi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha).$$

Following [28], we have the following estimates:

$$\|S_\alpha(t)x\| \le M_A\|x\|, \tag{2.14}$$

$$\|P_\alpha(t)x\| \le \frac{M_A}{\Gamma(\alpha)}\|x\|, \quad \forall x \in X. \tag{2.15}$$

Let $\Phi(t,s)$ be a family of bounded linear operators on X for $t, s \in [0,T], s \le t$. The following result was proved in [19, Lemma 1].

Proposition 2.3. *Assume that Φ satisfies the following conditions:*

(Φ1) *there exists a function $\rho \in L^q(0,T), q > 1$ such that $\|\Phi(t,s)\| \le \rho(t-s)$ for all $t, s \in [0,T], s \le t$;*

(Φ2) $\|\Phi(t,s)-\Phi(r,s)\| \le \epsilon$ *for* $0 \le s \le r-\epsilon, r < t = r+h \le T$ *with* $\epsilon = \epsilon(h) \to 0$ *as* $h \to 0$.

Then, the operator $\mathbf{S}: L^{q'}(0,T;X) \to C([0,T];X)$ *defined by*

$$(\mathbf{S}g)(t) := \int_0^t \Phi(t,s)g(s)\mathrm{d}s$$

sends any bounded set to an equicontinuous one, where q' *is the conjugate of* q, *i.e.,* $\frac{1}{q}+\frac{1}{q'}=1$.

Let $p > \frac{1}{\alpha}$, we define a linear operator

$$Q_\alpha : L^p(0,T;X) \to C([0,T];X),$$
$$Q_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)\mathrm{d}s. \tag{2.16}$$

Before proving some properties of the operator Q_α, we need the following result.

Proposition 2.4. [12] *Let* $D \subset L^1(0,T;X)$ *be such that*

(1) $\|f(t)\| \le \nu(t)$ *for a.e.* $t \in [0,T]$ *and for all* $f \in D$;
(2) $\chi(D(t)) \le \mu(t)$ *for a.e.* $t \in [0,T]$,

where $\nu, \mu \in L^1(0,T)$ *are nonnegative functions. Then, we have*

$$\chi\left(\int_0^t D(s)\mathrm{d}s\right) \le 4\int_0^t \chi(D(s))\mathrm{d}s, \quad t \in [0,T],$$

here

$$\int_0^t D(s)\mathrm{d}s = \left\{\int_0^t f(s)\mathrm{d}s : f \in D\right\}.$$

In what follows, with a sequence $\{z_n\}$ in a specific space, we use the notation $z_n \rightharpoonup z$ to indicate the weak convergence, and write $z_n \to z$ if $\{z_n\}$ converges strongly to z.

Definition 2.6. Let $p \ge 1$. A sequence $\{f_n\} \subset L^p(0,T;X)$ is said to be semicompact if there exist a function $\nu \in L^p(0,T)$ and a family of compact set $K(t), t \in [0,T]$, such that $\{f_n(t)\} \subset K(t)$ and $\|f_n(t)\| \le \nu(t)$ for a.e. $t \in [0,T]$.

Proposition 2.5. *Assume that the semigroup* $W(\cdot)$ *generated by* A *is norm-continuous, i.e., the map* $(0,\infty) \ni t \mapsto W(t) \in L(X)$ *is continuous. Then*

(1) *For each bounded set* $\Omega \subset L^p(0,T;X)$, $Q_\alpha(\Omega)$ *is an equicontinuous set in* $C([0,T];X)$. *Moreover, we have the following estimate:*

$$\chi_{\mathrm{PC}}(Q_\alpha(\Omega)) \le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi \cdot \chi(\Omega(s))\mathrm{d}s,$$

where $\|\cdot\|_\chi$ *is the* χ*-norm given by* (2.1).

(2) *If* $\{f_n\} \subset L^p(0,T;X), p > 1$, *is a semicompact sequence, then* $\{Q_\alpha(f_n)\}$ *is relatively compact in* $C([0,T];X)$. *Moreover, if* $f_n \rightharpoonup f$ *in* $L^p(0,T;X)$, *then* $Q_\alpha(f_n) \to Q_\alpha(f)$ *in* $C([0,T];X)$.

Proof. (1) Since $W(\cdot)$ is norm-continuous, so is $P_\alpha(\cdot)$ (see, e.g., [26]). Then, we deduce that $\Phi(t,s) = (t-s)^{\alpha-1}P_\alpha(t-s)$ satisfies $(\Phi1)-(\Phi2)$ in Proposition 2.3, which ensures the equicontinuity of $Q_\alpha(\Omega)$. Hence

$$\chi_{\rm PC}(Q_\alpha(\Omega)) = \sup_{t\in[0,T]} \chi(Q_\alpha(\Omega)(t)).$$

In view of Proposition 2.4, we have

$$\begin{aligned}\chi_{\rm PC}(Q_\alpha(\Omega)) &= \sup_{t\in[0,T]} \chi\left(\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)\Omega(s)\mathrm{d}s\right)\\ &\le 4\sup_{t\in[0,T]}\int_0^t \chi\left((t-s)^{\alpha-1}P_\alpha(t-s)\Omega(s)\right)\mathrm{d}s\\ &\le 4\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi\cdot\chi\left(\Omega(s)\right)\mathrm{d}s.\end{aligned}$$

(2) By the first part of this proposition, the sequence $\{Q_\alpha(f_n)\}$ is equicontinuous. In addition, one has

$$\begin{aligned}\chi\left(\{Q_\alpha(f_n)(t)\}\right) &= \chi\left(\left\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)\mathrm{d}s\right\}\right)\\ &\le 4\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi\cdot\chi\left(\{f_n(s)\}\right)\mathrm{d}s\\ &= 0,\end{aligned}$$

thanks to Proposition 2.4. So $\{Q_\alpha(f_n)(t)\}$, for each $t\in[0,T]$, is a relatively compact set. By the Arzelà–Ascoli theorem, $\{Q_\alpha(f_n)\}$ is relatively compact in $C([0,T];X)$. Then, the last assertion is justified as follows. At first, by the Hölder inequality, one sees that $Q_\alpha : L^p(0,T;X)\to C([0,T];X)$ is bounded and hence continuous. So it is continuous with respect to weak topology (see, e.g., [3, Theorem 3.10]). This implies $Q_\alpha(f_n)\rightharpoonup Q_\alpha(f)$ in $C([0,T];X)$. By the relative compactness of $\{Q_\alpha(f_n)\}$, the last convergence is in the norm of $C([0,T];X)$. The proof is complete. □

3. Existence of solutions on the half-line

In this section, we take $\varrho(t) = \mathrm{e}^{\delta t}$ for a fixed $\delta > 0$. Concerning problem (1.1)–(1.3), we give the following assumptions:

(**A**) The C_0-semigroup $\{W(t)\}_{t\ge0}$ generated by A is norm-continuous and globally bounded, i.e.,

$$\|W(t)x\| \le M_A\|x\|, \quad \forall t\ge0, \quad x\in X.$$

(**F**) The nonlinearity $F:\mathbb{R}^+\times X\times C([-h,0];X)\to K_v(X)$ satisfies:

(1) The multi-valued map $(v,w)\mapsto F(t,v,w)$ is u.s.c for each $t\in\mathbb{R}^+$;
(2) The multi-valued map $t\mapsto F(t,u(t),u_t)$ admits a strongly measurable selection for each $u\in \mathrm{PC}_\varrho$;

(3) There exists a function $m \in L^p_{\rm loc}(\mathbb{R}^+)$ such that
$\|F(t,v,w)\| = \sup\{\|\xi\| : \xi \in F(t,v,w)\} \le m(t)(\|v\| + \|w\|_h)$,
for all $(t,v,w) \in \mathbb{R}^+ \times X \times C([-h,0];X)$, here $\|w\|_h := \sup_{s\in[-h,0]} \|w(s)\|$ is the norm in $C([-h,0];X)$;

(4) If $W(\cdot)$ is noncompact, there is a function $k \in L^p_{\rm loc}(\mathbb{R}^+)$ such that

$$\chi(F(t,V,\Omega)) \le k(t)\left[\chi(V) + \sup_{t\in[-h,0]} \chi(\Omega(t))\right],$$

for a.e. $t \in \mathbb{R}^+$, and for all bounded sets $V \subset X, \Omega \subset C([-h,0];X)$.

(**I**) The function $I_k : X \to X, k \in \Lambda$, is continuous and satisfies:

(1) There exists a nonnegative sequence $\{l_k\}_{k\in\Lambda}$ such that $\sum_{k\in\Lambda} l_k < \infty$ and

$$\|I_k(x)\| \le l_k\, \|x\|, \quad \text{for all } \ x \in X,\ k \in \Lambda.$$

(2) There exists a nonnegative sequence $\{\mu_k\}_{k\in\Lambda}$ such that

$$\chi(I_k(B)) \le \mu_k \chi(B),$$

for all bounded subsets $B \subset X$;

(3) The sequence $\{t_k\}_{k\in\Lambda}$ satisfies $\inf_{k\in\Lambda}(t_{k+1} - t_k) > 0$.

For given $\varphi \in C([-h,0];X)$, we define the space

$$\mathrm{PC}^\varphi_\varrho = \{v \in \mathrm{PC}_\varrho : v(0) = \varphi(0)\}.$$

For $v \in \mathrm{PC}^\varphi_\varrho$, let $v[\varphi]$ be a function given by

$$v[\varphi](t) = \begin{cases} \varphi(t) & \text{if } \ t \in [-h,0], \\ v(t) & \text{if } \ t > 0. \end{cases}$$

Now for $v \in \mathrm{PC}^\varphi_\varrho$, we denote

$$\mathcal{P}^p_F(v) = \{f \in L^p_{\rm loc}(\mathbb{R}^+;X) : f(t) \in F(t,v(t),v[\varphi]_t) \quad \text{for a.e. } t \in \mathbb{R}^+\}.$$

Definition 3.1. A function $u : [-h,+\infty) \to X$ is said to be an integral solution of problem (1.1)–(1.3) if and only if $u(t) = \varphi(t)$ for $t \in [-h,0]$, and there exists $f \in \mathcal{P}^p_F(u)$ such that

$$u(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k)) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)\mathrm{d}s,$$

for any $t > 0$.

Let $\mathcal{F} : \mathrm{PC}^\varphi_\varrho \to \mathcal{P}(\mathrm{PC}^\varphi_\varrho)$ be the multi-valued map defined by

$$\mathcal{F}(v)(t) = S_\alpha(t)v(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(v(t_k)) + \left\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)\mathrm{d}s : f \in \mathcal{P}^p_F(v)\right\}, \quad t > 0.$$

Then, v is a fixed point of *the solution operator* $\mathcal{F}$ iff $u = v[\varphi]$ is an integral solution of (1.1)–(1.3). To check the closedness of $\mathcal{F}$, we prove the following lemma.

Lemma 3.1. *Let* (***F***) *hold. If* $\{v_n\} \subset \mathrm{PC}_\varrho^\varphi$ *with* $v_n \to v^*$ *and* $f_n \in \mathcal{P}_F^p(v_n)$ *then* $f_n \rightharpoonup f^*$ *in* $L^p_{\text{loc}}(\mathbb{R}^+; X)$ *with* $f^* \in \mathcal{P}_F^p(v^*)$.

Proof. Let $\{v_n\} \subset \mathrm{PC}_\varrho^\varphi$ be such that $v_n \to v^*, f_n \in \mathcal{P}_F^p(v_n)$. We see that $\{f_n(t)\} \subset C(t) := \overline{F(t, \{v_n(t), v_n[\varphi]_t\})}$, which is a compact set for a.e. $t \in \mathbb{R}^+$, thanks to (**F**)(1). Let $T > 0$ be given. By **(F)**(3), we see that $\{f_n|_{[0,T]}\}$ is bounded by an L^p-integrable function. Thus, $\{f_n\}$ is a semicompact sequence and by [6, Corollary 3.3], it is weakly compact in $L^p(0,T;X)$. So one can assume that $f_n \rightharpoonup f^{1*} \in L^p(0,T;X)$. By Mazur's lemma (see, e.g., [3]), there exists a sequence $\tilde{f}_n \in \mathrm{co}\{f_i : i \geq n\}$ such that $\tilde{f}_n \to f^{1*}$ in $L^p(0,T;X)$ and then $\tilde{f}_n(t) \to f^{1*}(t)$ for a.e. $t \in [0,T]$. Since F has compact values, the upper semicontinuity of $F(t,\cdot,\cdot)$ means that for $\epsilon > 0$

$$F(t, v_n(t), v_n[\varphi]_t) \subset F(t, v^*(t), v^*[\varphi]_t) + B_\epsilon$$

for all large n, here B_ϵ denotes the ball in X centered at origin with radius ϵ. So

$$f_n(t) \in F(t, v^*(t), v^*[\varphi]_t) + B_\epsilon, \quad \text{for a.e. } t \in [0,T].$$

By the convexity of $F(t, v^*(t), v^*[\varphi]_t) + B_\epsilon$, the last inclusion still holds for $\tilde{f}_n(t)$ instead of $f_n(t)$. Consequently, $f^{1*}(t) \in F(t, v^*(t), v^*[\varphi]_t) + B_\epsilon$ for a.e. $t \in [0,T]$. Since ϵ is arbitrary, we obtain the inclusion $f^{1*}(t) \in F(t, v^*(t), v^*[\varphi]_t)$ for a.e. $t \in [0,T]$.

Repeating the above arguments for $t \in [(j-1)T, jT], j = 1, 2, ...,$ we get that $f_n \rightharpoonup f^{j*}$ in $L^p((j-1)T, jT; X)$ with $f^{j*}(t) \in F(t, v^*(t), v^*[\varphi]_t)$ for a.e. $t \in [(j-1)T, jT]$. Defining the function $f^* \in L^p_{\text{loc}}(\mathbb{R}^+; X)$ as follows:

$$f^*(t) = f^{j*}(t) \quad \text{if } t \in [(j-1)T, jT],$$

we obtain the conclusion of the lemma. □

We are now able to show the closedness of the solution operator.

Lemma 3.2. *Assume that* (***A***), (***F***) *and* (***I***) *are satisfied. Then, the solution operator* $\mathcal{F}$ *is closed.*

Proof. Let $\{v_n\} \subset \mathrm{PC}_\varrho^\varphi$ be a sequence converging to v^* and $z_n \in \mathcal{F}(v_n)$ be such that $z_n \to z^*$. We prove that $z^* \in \mathcal{F}(v^*)$. By the definition of $\mathcal{F}$, one can take $f_n \in \mathcal{P}_F^p(v_n)$ such that

$$z_n(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(v_n(t_k)) + Q_\alpha(f_n)(t), \quad t > 0, \tag{3.1}$$

where Q_α is defined in (2.16). By Lemma 3.1, $f_n \rightharpoonup f^*$ in $L^p_{\text{loc}}(\mathbb{R}^+; X)$ with $f^* \in \mathcal{P}_F^p(v^*)$. We will show that

$$z^*(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(v^*(t_k)) + Q_\alpha(f^*)(t), \quad t > 0. \tag{3.2}$$

Let $t > 0$, take $T > 0$ such that $t \leq T$ and consider the sequence $\{f_n|_{[0,T]}\}$. As argued in the proof of Lemma 3.1, this sequence is semicompact. Then by Proposition 2.5, $Q_\alpha(f_n) \to Q_\alpha(f^*)$ in $C([0,T];X)$ and, in particular, $Q_\alpha(f_n)(t) \to Q_\alpha(f^*)(t)$ in X. Thanks to the continuity of g and I_k, one can pass to the limit in (3.1) to get (3.2). The proof is complete. □

In the sequel, we show the condensivity of the solution operator.

Lemma 3.3. *Let the hypotheses of Lemma 3.2 hold. If*

$$\ell := M_A \sum_{k\in\Lambda} \mu_k + 8 \sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)\mathrm{d}s < \infty, \tag{3.3}$$

then

$$\chi_\infty(\mathcal{F}(D)) \le \ell \cdot \chi_\infty(D),$$

for all bounded sets $D \subset \mathrm{PC}_\varrho^\varphi$.

Proof. Let $D \subset \mathrm{PC}_\varrho^\varphi$ be a bounded set. For $v \in D$, one can write $\mathcal{F}(v) = \mathcal{F}_1(v) + \mathcal{F}_2(v)$, where

$$\mathcal{F}_1(v)(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(v(t_k)),$$
$$\mathcal{F}_2(v)(t) = Q_\alpha \circ \mathcal{P}_F^p(v)(t).$$

Using the same arguments as those in [11, Lemma 3.1], we get

$$\chi_\infty(\mathcal{F}_1(D)) \le \left(M_A \sum_{k\in\Lambda} \mu_k \right) \chi_\infty(D), \tag{3.4}$$

where χ_∞ is the MNC defined in (2.2). Regarding $\mathcal{F}_2(D)$, we observe that $\Omega = \mathcal{P}_F^p(D)|_{[0,T]}$ is bounded in $L^p(0,T;X)$, then $\pi_T(\mathcal{F}_2(D)) = Q_\alpha(\Omega)$ obeys the following estimate:

$$\chi_{\mathrm{PC}}(\pi_T(\mathcal{F}_2(D))) \le 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi \cdot \chi(\Omega(s))\mathrm{d}s, \tag{3.5}$$

due to Proposition 2.5. Now deploying (**F**)(4), one has

$$\begin{aligned} \chi(\Omega(s)) &\le \chi(F(s, D(s), D[\varphi]_s)) \\ &\le k(s)[\chi(D(s)) + \sup_{r\in[-h,0]} \chi(D(s+r))] \\ &\le 2k(s)\chi_{\mathrm{PC}}(\pi_T(D)). \end{aligned}$$

Putting the last estimate in (3.5) yields

$$\chi_{\mathrm{PC}}(\pi_T(\mathcal{F}_2(D))) \le \left(8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)\mathrm{d}s \right) \chi_{\mathrm{PC}}(\pi_T(D)).$$

Therefore

$$\chi_\infty(\mathcal{F}_2(D)) \le \left(8 \sup_{t\ge 0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)\mathrm{d}s \right) \chi_\infty(D). \tag{3.6}$$

Combining (3.4) and (3.6), we arrive at

$$\chi_\infty(\mathcal{F}(D)) \le \ell \cdot \chi_\infty(D), \tag{3.7}$$

where ℓ is defined by (3.3). The proof is complete. □

Lemma 3.4. *Let the hypotheses of Lemma 3.2 hold. If*

$$\vartheta := \sup_{t>0} \int_0^{\sigma t} \frac{\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s < \infty, \tag{3.8}$$

$$\kappa := \sup_{t>0} \int_{\sigma t}^{t} \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s < \infty, \tag{3.9}$$

for some $\sigma \in (0,1)$, *then*

$$d_\infty(\mathcal{F}(D)) \le 2\kappa \cdot d_\infty(D), \tag{3.10}$$

for all bounded sets $D \subset \mathrm{PC}_\varrho^\varphi$.

Proof. Let $D \subset \mathrm{PC}_\varrho^\varphi$ be a bounded set. Using the decomposition $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ as in the proof of Lemma 3.3, we first demonstrate that

$$d_\infty(\mathcal{F}_1(D)) = 0.$$

To this end, for given $\epsilon > 0$, we have to prove the existence of $T > 0$ such that

$$\frac{\|\mathcal{F}_1(v)(t)\|}{\varrho(t)} < \epsilon, \quad \forall t \ge T, \quad v \in D.$$

Take $R > 0$ such that $\sup\{\|v\|_\varrho : v \in D\} \le R$ and $T_1 > 0$ such that

$$\frac{1}{\varrho(t)}\|S_\alpha(t)\|\|\varphi\|_h < \frac{\epsilon}{3}, \quad \forall t \ge T_1.$$

For $\mathcal{F}_1$, we choose $N_0 \in \Lambda$ such that

$$RM_A \sum_{k>N_0} l_k < \frac{\epsilon}{3}, \tag{3.11}$$

and, in addition, take $T_2 > 0$ such that

$$\frac{\|S_\alpha(t)\|}{\varrho(t)} R \sum_{k\le N_0} l_k < \frac{\epsilon}{3}, \forall t \ge T_2. \tag{3.12}$$

Then for any $v \in D$, using (**I**)(2) we get

$$\begin{aligned}
\frac{\|\mathcal{F}_1(v)(t)\|}{\varrho(t)} &\le \frac{1}{\varrho(t)}\|S_\alpha(t)\|\|\varphi\|_h + \frac{1}{\varrho(t)}\sum_{k\in\Lambda}\|S_\alpha(t-t_k)\|\|I_k(v(t_k))\| \\
&\le \frac{\epsilon}{3} + \frac{1}{\varrho(t)}\sum_{k\in\Lambda}\|S_\alpha(t-t_k)\|l_k\|v(t_k)\| \\
&\le \frac{\epsilon}{3} + \frac{R}{\varrho(t)}\sum_{k\le N_0}\|S_\alpha(t-t_k)\|l_k + \frac{RM_A}{\varrho(t)}\sum_{k>N_0} l_k \\
&\le \frac{\epsilon}{3} + R\sum_{k\le N_0}\frac{\|S_\alpha(t-t_k)\|}{\varrho(t-t_k)}l_k + RM_A\sum_{k>N_0} l_k \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \forall t \ge \max\{T_1, T_2 + t_{N_0}\},
\end{aligned}$$

thanks to (3.11) and (3.12) and the fact that $\varrho(\cdot)$ is nondecreasing and $\varrho(t) \ge 1, \forall t \ge 0$.

We are in a position to evaluate $d_\infty(\mathcal{F}_2(D))$. Let $z \in \mathcal{F}_2(v), v \in D$. Taking $f \in \mathcal{P}_F^p(v)$ such that

$$z(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\mathrm{d}s, \quad t > 0,$$

we have

$$\begin{aligned} \frac{\|z(t)\|}{\varrho(t)} &\le \int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} \frac{\|f(s)\|}{\varrho(s)}\mathrm{d}s \\ &\le \left(\int_0^{\sigma t} + \int_{\sigma t}^t\right) \Theta(t,s)\mathrm{d}s, \end{aligned} \tag{3.13}$$

where

$$\Theta(t,s) = \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)}{\varrho(t-s)} \frac{\|v(s)\| + \|v[\varphi]_s\|_h}{\varrho(s)},$$

thanks to (**F**)(3). Let $t > 0$ such that $\sigma t - h > 0$. Then, we get, for $s \in [0, \sigma t]$,

$$\begin{aligned} \frac{\|v(s)\| + \|v[\varphi]_s\|_h}{\varrho(s)} &= \frac{1}{\varrho(s)}(\|v(s)\| + \sup_{\tau\in[-h,0]} \|v[\varphi](s+\tau)\|) \\ &\le \frac{1}{\varrho(s)}(\|v(s)\| + \sup_{\tau\in[-h,0]} \|\varphi(\tau)\| + \sup_{\tau\in[0,s]} \|v(\tau)\|) \\ &\le \|\varphi\|_h + \frac{1}{\varrho(s)}(\|v(s)\| + \sup_{\tau\in[0,s]} \|v(\tau)\|) \\ &\le \|\varphi\|_h + \frac{\|v(s)\|}{\varrho(s)} + \sup_{\tau\in[0,s]} \frac{\|v(\tau)\|}{\varrho(\tau)} \\ &\le \|\varphi\|_h + 2R. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{\sigma t} \Theta(t,s)\mathrm{d}s &\le 2R \int_0^{\sigma t} \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s \\ &\le \frac{\|\varphi\|_h + 2R}{[(1-\sigma)t]^{1-\alpha}} \int_0^{\sigma t} \frac{\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s \\ &\le \frac{(\|\varphi\|_h + 2R)\vartheta}{[(1-\sigma)t]^{1-\alpha}}, \end{aligned} \tag{3.14}$$

where ϑ is given in (3.8). On the other hand, for $s \ge \sigma t$ we see that

$$\begin{aligned} \frac{\|v(s)\| + \|v[\varphi]_s\|_h}{\varrho(s)} &= \frac{1}{\varrho(s)}(\|v(s)\| + \sup_{\tau\in[-h,0]} \|v[\varphi](s+\tau)\|) \\ &\le \frac{\|v(s)\|}{\varrho(s)} + \sup_{\tau\in[-h,0]} \frac{\|v[\varphi](s+\tau)\|}{\varrho(s+\tau)} \\ &\le \sup_{r\ge\sigma t} \frac{\|v(r)\|}{\varrho(r)} + \sup_{r\ge\sigma t-h} \frac{\|v(r)\|}{\varrho(r)} \le 2 \sup_{r\ge\sigma t-h} \frac{\|v(r)\|}{\varrho(r)}. \end{aligned}$$

Then

$$\int_{\sigma t}^{t} \Theta(t,s)\mathrm{d}s \le \left(\int_{\sigma t}^{t} \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s\right) 2 \sup_{r\ge \sigma t-h} \frac{\|v(r)\|}{\varrho(r)}$$
$$\le 2\kappa \sup_{r\ge \sigma t-h} \frac{\|v(r)\|}{\varrho(r)}, \tag{3.15}$$

where κ is defined by (3.9). Now using (3.14) and (3.15) in (3.13), we have

$$\frac{\|z(t)\|}{\varrho(t)} \le \frac{(\|\varphi\|_h + 2R)\vartheta}{[(1-\sigma)t]^{1-\alpha}} + 2\kappa \sup_{r\ge \sigma t-h} \frac{\|v(r)\|}{\varrho(r)},$$

for all $t > \frac{h}{\sigma}$, $v \in D, z \in \mathcal{F}_2(v)$. The last inequality implies

$$d_\infty(\mathcal{F}_2(D)) \le 2\kappa \cdot d_\infty(D).$$

The proof is complete. □

Combining Lemmas 3.3 and 3.4, we arrive at the following assertion.

Lemma 3.5. *Let the assumptions* ($\boldsymbol{A}$), ($\boldsymbol{F}$) *and* ($\boldsymbol{I}$) *hold. Then, the solution operator* $\mathcal{F}$ *is* χ^**-condensing provided that*

$$\ell = M_A \sum_{k\in\Lambda} \mu_k + 8 \sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)\mathrm{d}s < 1, \tag{3.16}$$

$$\vartheta = \sup_{t>0} \int_0^{\sigma t} \frac{\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s < \infty, \tag{3.17}$$

$$\kappa = \sup_{t>0} \int_{\sigma t}^{t} \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s < \frac{1}{2}, \tag{3.18}$$

for some $\sigma \in (0,1)$.

Proof. Using Lemmas 3.3 and 3.4, one gets, for all bounded sets $D \subset \mathrm{PC}_\varrho^\varphi$,

$$\chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D)) \le \max\{\ell, 2\kappa\} \cdot (\chi_\infty(D) + d_\infty(D)),$$

that is,

$$\chi^*(\mathcal{F}(D)) \le \max\{\ell, 2\kappa\} \cdot \chi^*(D).$$

If $\chi^*(\mathcal{F}(D)) \ge \chi^*(D)$ then $\chi^*(D) \le \max\{\ell, 2\kappa\} \cdot \chi^*(D)$, which ensures that $\chi^*(D) = 0$. Therefore, D is relatively compact due to Lemma 2.1. We get the conclusion as desired. □

Theorem 3.6. *Let the hypotheses of Lemma 3.5 hold. Assume that*

$$M_A \sum_{k\in\Lambda} l_k + 2 \sup_{t>0} \int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\, m(s)}{\varrho(t-s)}\mathrm{d}s < 1. \tag{3.19}$$

Then, problem (1.1)–(1.3) *has at least one integral solution in* PC_ϱ.

Proof. By Lemma 3.2, $\mathcal{F}$ is closed. In addition, $\mathcal{F}$ is χ^*-condensing due to Lemma 3.5. Moreover, $\mathcal{F}$ has compact values. Indeed, for $v \in \mathrm{PC}_\varrho^\varphi$, we have

$$\chi^*(\mathcal{F}(v)) \le \max\{\ell, 2\kappa\} \cdot \chi^*(\{v\}) = 0.$$

It follows that $\chi^*(\mathcal{F}(v)) = 0$ and then $\mathcal{F}(v)$ is a relatively compact set due to Lemma 2.1. Thanks to the closedness of $\mathcal{F}$, $\mathcal{F}(v)$ is compact.

To apply Theorem 2.2, it suffices to show that there exists $R > 0$ such that

$$\mathcal{F}(\mathbf{B}_R) \subset \mathbf{B}_R,$$

where $\mathbf{B}_R$ is the closed ball in $\mathrm{PC}^{\varphi}_{\varrho}$, centered at origin with radius R.

We first check that $\mathcal{F}(\mathrm{PC}^{\varphi}_{\varrho}) \subset \mathrm{PC}^{\varphi}_{\varrho}$. Let $v \in \mathrm{PC}^{\varphi}_{\varrho}$, then $d_\infty(\{v\}) = 0$. Using (3.10), we have $d_\infty(\mathcal{F}(v)) = 0$. Then it follows that $\mathcal{F}(v) \subset \mathrm{PC}^{\varphi}_{\varrho}$.

Now, we prove that $\mathcal{F}(\mathbf{B}_R) \subset \mathbf{B}_R$ for some $R > 0$. Assume to the contrary that for each $n \in \mathbb{N}$, there exists $v_n \in \mathbf{B}_n$ and $z_n \in \mathcal{F}(v_n)$ such that $\|z_n\|_\varrho > n$. Taking $f_n \in \mathcal{P}^p_F(v_n)$ such that

$$\begin{aligned} z_n(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(v_n(t_k)) \\ + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)\mathrm{d}s, \end{aligned}$$

we observe that, for all $t \geq 0$,

$$\begin{aligned} \frac{\|z_n(t)\|}{\varrho(t)} \leq \frac{\|S_\alpha(t)\|}{\varrho(t)}\|\varphi\|_h + \sum_{0<t_k<t} \frac{\|S_\alpha(t-t_k)\|}{\varrho(t-t_k)}\frac{\|v_n(t_k)\|}{\varrho(t_k)} l_k \\ + \int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)}\frac{\|v_n(s)\|+\|v_n[\varphi]_s\|_h}{\varrho(s)} m(s)\mathrm{d}s, \end{aligned}$$

thanks to (**I**)(2) and (**F**)(3). Noting that $\frac{\|S_\alpha(t)\|}{\varrho(t)} \leq \|S_\alpha(t)\| \leq M_A$, $\forall t \geq 0$, and

$$\begin{aligned} \frac{\|v_n(s)\|}{\varrho(s)} &\leq n, \qquad \forall s \geq 0, \\ \frac{\|v_n[\varphi]_s\|_h}{\varrho(s)} &\leq \frac{1}{\varrho(s)} \sup_{r\in[s-h,s]} \|v_n[\varphi](r)\| \\ &\leq \frac{1}{\varrho(s)}\left(\sup_{r\in[-h,0]}\|\varphi(r)\| + \sup_{r\in[0,s]}\|v_n(r)\|\right) \\ &\leq \|\varphi\|_h + \sup_{r\in[0,s]}\frac{1}{\varrho(r)}\|v_n(r)\| \\ &\leq \|\varphi\|_h + \|v_n\|_\varrho \leq \|\varphi\|_h + n, \end{aligned}$$

we get

$$\begin{aligned} \frac{\|z_n(t)\|}{\varrho(t)} \leq M_A\|\varphi\|_h + nM_A\sum_{k\in\Lambda} l_k \\ + (2n + \|\varphi\|_h)\int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)} m(s)\mathrm{d}s. \end{aligned}$$

This implies

$$1 < \frac{\|z_n\|_\varrho}{n} = \frac{1}{n}\sup_{t>0}\frac{\|z_n(t)\|}{\varrho(t)}$$
$$\le \frac{\|\varphi\|_h}{n}\left(M_A + \sup_{t>0}\int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)}m(s)\mathrm{d}s\right)$$
$$+ M_A\sum_{k\in\Lambda} l_k + 2\sup_{t>0}\int_0^t \frac{(t-s)^{\alpha-1}\|P_\alpha(t-s)\|}{\varrho(t-s)}m(s)\mathrm{d}s.$$

Passing to the limit in the last inequality, we get a contradiction with (3.19). The proof is complete. □

4. Weak stability result

In this section, we replace the assumptions (**A**) and (**F**) by stronger ones:

(**A***) The semigroup $W(\cdot)$ generated by A is norm-continuous and exponentially stable, i.e., there exists $\beta > 0$ such that

$$\|W(t)x\| \le M_A \mathrm{e}^{-\beta t}\|x\|, \quad \forall t \ge 0,\ x \in X.$$

(**F***) The multi-valued nonlinearity function F satisfies (**F**) with $m \in L^1(\mathbb{R}^+) \cap L^p_{\mathrm{loc}}(\mathbb{R}^+)$.

It is proved in [11] that, by (**A***) the fractional resolvent operators $S_\alpha(\cdot), P_\alpha(\cdot)$ are asymptotically stable, i.e.

$$\|S_\alpha(t)\|, \|P_\alpha(t)\| \to 0 \quad \text{as} \quad t \to +\infty.$$

By choosing $\varrho(t) \equiv 1$, we now consider the solution operator $\mathcal{F}$ on the space

$$\mathrm{PC}_0 = \left\{u \in \mathrm{PC}(\mathbb{R}^+; X) : \lim_{t\to+\infty} u(t) = 0\right\},$$

endowed with the norm $\|u\|_\infty = \sup_{t\ge 0}\|u(t)\|$.

Following the same arguments as in Sect. 3, we get the existence of global attracting solutions as follows.

Theorem 4.1. *Let (**A***), (**F***) and (**I**) hold. Then, the problem (1.1)–(1.3) admits an integral solution such that $\|u(t)\| = o(1)$ as $t \to +\infty$, provided that*

$$\ell = M_A\sum_{k\in\Lambda}\mu_k + 8\sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)\mathrm{d}s < 1, \tag{4.1}$$

$$\varpi = M_A\sum_{k\in\Lambda} l_k + 2\sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)\mathrm{d}s < 1. \tag{4.2}$$

Proof. We verify the assumptions of Lemma 3.5 and Theorem 3.6. Since $m \in L^1(\mathbb{R}^+)$, one sees that the condition (3.17) is fulfilled. In addition, condition (3.18) follows from (4.2), while condition (4.2) is exactly (3.19). □

Now, we state the main result of this section.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold. Then, the zero solution of (1.1) is weakly asymptotically stable.*

Proof. Let $\Sigma(\varphi)$ be the set of integral solutions of (1.1)–(1.3) with respect to the initial datum φ. Obviously, we have $0 \in \Sigma(0)$ thanks to the fact that $F(t,0,0) = 0$ and $I_k(0) = 0, k \in \Lambda$. By Theorem 4.1 we observe that, for each $\varphi \in C([-h,0];X)$ there exists $u \in \Sigma(\varphi)$ such that $\|u(t)\| \to 0$ as $t \to +\infty$. Then, we get $\|u_t\|_h \to 0$ as $t \to +\infty$, that is, the zero solution is weakly attractive. It remains to show that this solution is stable.

Let $\varphi \in C([-h,0];X)$ and $u \in \Sigma(\varphi)$. Then, there exists $f \in \mathcal{P}^p_F(u)$ such that

$$u(t) = S_\alpha(t)\varphi(0) + \sum_{0<t_k<t} S_\alpha(t-t_k)I(u(t_k)) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)\mathrm{d}s, \quad t > 0.$$

Now, using the assumptions ($\mathbf{F}^*$) and ($\mathbf{I}$), we get

$$\|u(t)\| \leq M_A\|\varphi\|_h + M_A\|u\|_\infty \sum_{k\in\Lambda} l_k + (2\|u\|_\infty + \|\varphi\|_h)\sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\mathrm{d}s, \quad t > 0.$$

The last estimate implies

$$\|u\|_\infty \leq \left[M_A\sum_{k\in\Lambda} l_k + 2\sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\mathrm{d}s\right]\|u\|_\infty + (M_A+1)\|\varphi\|_h.$$

Hence

$$\|u_t\|_h \leq \|\varphi\|_h + \|u\|_\infty \leq \left(1 + \frac{1+M_A}{1-\varpi}\right)\|\varphi\|_h, \quad \forall t > 0,$$

where ϖ is given by (4.2). The last inequality ensures the stability of the zero solution. The proof is complete. □

5. Application

In this section, the obtained abstract results will be demonstrated in the following lattice differential system

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}u_i(t) = (\mathrm{Au}(t))_i + f_i(t), \quad t > 0, \ \ t \neq t_k, \ \ k \in \mathbb{N}, \tag{5.1}$$

$$f_i(t) \in [f_{1i}(t,u_i(t),u_i(t-\rho(t))), f_{2i}(t,u_i(t),u_i(t-\rho(t)))], \tag{5.2}$$

$$\Delta u_i(t_k) = I_{ik}(u_i(t_k)), \tag{5.3}$$

$$u_i(s) = \varphi_i(s), \quad s \in [-h,0], \tag{5.4}$$

where $u = (u_i) : [-h,+\infty) \to \ell^2$ is the unknown function, $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is the Caputo derivative of order $\alpha \in (0,1)$, $A : \ell^2 \to \ell^2$ is the linear operator given by

$$(Av)_i = v_{i+1} - (2+\lambda)v_i + v_{i-1}, \quad v \in \ell^2,$$

$\rho : \mathbb{R}^+ \to [0,h]$ is a continuous function, and λ is a positive number. Here, ℓ^2 is the space of sequences $(v_i)_{i\in\mathbb{Z}}$ satisfying $\sum_{i\in\mathbb{Z}} v_i^2 < \infty$, which becomes a Hilbert space with the inner product $(u,v)_{\ell^2} = \sum_{i\in\mathbb{Z}} u_i v_i$, and $[f_1, f_2] = \{\tau f_1 + (1-\tau) f_2 : \tau \in [0,1]\}$.

Lattice differential systems like (5.1)–(5.4) come from various problems such as image processing, pattern recognition, electrical engineering, ... In particular, (5.1)–(5.4) can be seen as a model of semi-discretization for the fractional partial differential inclusion

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) - \lambda u(x,t) + f(x,t), \quad x \in \mathbb{R}, \quad t > 0,\\
&\quad f(x,t) \in [f_1(x,t,u(x,t),u(x,t-\rho(t))), f_2(x,t,u(x,t),u(x,t-\rho(t)))],\\
&\Delta u(x,t_k) = I_k(x, u(x,t_k)),\\
&\quad u(x,s) = \varphi(x,s), \quad s \in [-h,0],
\end{aligned}$$

where the discretizing is made in spatial variable x.

Let $B : \ell^2 \to \ell^2$ be the linear operator defined by $(Bv)_i = v_{i+1} - v_i$, then its adjoint operator B^* is given by $(B^*v)_i = v_{i-1} - v_i$. In addition, if $\tilde{A} : \ell^2 \to \ell^2$ is the operator defined by $(\tilde{A}v)_i = v_{i+1} - 2v_i + v_{i-1}$ then

$$-\tilde{A} = BB^* = B^*B.$$

It should be noted that $A = \tilde{A} - \lambda I$ is a bounded operator on ℓ^2. Then, the semigroup $\{\mathrm{e}^{tA} : t \geq 0\}$ is uniformly continuous (see, e.g., [7]) and hence norm-continuous. However, this semigroup is non-compact, since it can be extended to a group $\{\mathrm{e}^{tA} : t \in \mathbb{R}\}$ and the identity operator $I = \mathrm{e}^{tA}\mathrm{e}^{-tA}$ is non-compact.

To obtain the exponential stability of $\{\mathrm{e}^{tA} : t \geq 0\}$, we consider the system

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \tilde{A}v(t) - \lambda v(t), \qquad v(t) \in \ell^2.$$

Multiplying by v, we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 &= (\tilde{A}v(t), v(t)) - \lambda\|v(t)\|^2\\
&= -(B^*Bv(t), v(t)) - \lambda\|v(t)\|^2 = -\|Bv(t)\|^2 - \lambda\|v(t)\|^2\\
&\leq -\lambda\|v(t)\|^2.
\end{aligned}$$

Then by the Gronwall lemma, we get

$$\|v(t)\| \leq \mathrm{e}^{-\lambda t}\|v(0)\|,$$

and, therefore, one has the estimate $\|\mathrm{e}^{tA}\| \leq \mathrm{e}^{-\lambda t}, t \geq 0$, i.e., the semigroup $\{\mathrm{e}^{tA} : t \geq 0\}$ is exponentially stable. The assumption ($\mathbf{A}^*$) is verified.

Before going to the further settings for (5.1)–(5.4), we recall a result on the Hausdorff MNC in ℓ^2 (see, e.g., [2, Theorem 4.2]). Let $R_n : \ell^2 \to \ell^2$ be the linear operator defined by

$$R_n(v) = \sum_{|i|>n} v_i e_i,$$

where $e_i = (\delta_{ij})_{j\in\mathbb{Z}}$. Then, the map $\chi : 2^{\ell^2} \to \mathbb{R}^+$ defined by

$$\chi(B) = \limsup_{n\to+\infty}[\sup_{v\in B} \|R_n(v)\|] = \limsup_{n\to+\infty}\left[\sup_{v\in B}\left(\sum_{|i|>n}|v_i|^2\right)^{\frac{1}{2}}\right]$$

is the Hausdorff MNC in ℓ^2.

Now, we give the following assumptions:

(N1) The functions $f_{1i}, f_{2i} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}, i \in \mathbb{Z}$, are continuous and satisfy

$\max\{|f_{1i}(t, y, z)|^2, |f_{2i}(t, y, z)|^2\} \le m^2(t)(|y|^2 + |z|^2), \forall(t, \eta, z) \in \mathbb{R}^+ \times \mathbb{R}^2,$

where $m \in C(\mathbb{R}^+; \mathbb{R}^+)$ satisfies

$$m(t) \le \frac{C_m}{1 + t^{\alpha+1}} \quad \text{for some } C_m > 0.$$

(N2) $I_{ik} : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}, k \in \mathbb{N}$, are continuous functions such that

$$|I_{ik}(y)| \le l_k|y|,$$

where $\{l_k : k \in \mathbb{N}\}$ is a sequence of non-negative numbers such that $\sum_{k\in\mathbb{N}} l_k < \infty$.

Let $f_1, f_2 : \mathbb{R}^+ \times \ell^2 \times C([-h, 0]; \ell^2) \to \ell^2$ be functions defined by

$$f_1(t, v, w) = (f_{1i}(t, v_i, w_i(-\rho(0))))_{i\in\mathbb{Z}},$$
$$f_2(t, v, w) = (f_{2i}(t, v_i, w_i(-\rho(0))))_{i\in\mathbb{Z}}.$$

Then, one can check that f_1, f_2 are continuous. Furthermore, it follows from (N1) that

$$\begin{aligned}\|f_1(t, v, w)\|^2 &= \sum_{i\in\mathbb{Z}} |f_{1i}(t, v_i, w_i(-\rho(0)))|^2 \\ &\le m^2(t)\sum_{i\in\mathbb{Z}}(|v_i|^2 + |w_i(-\rho(0))|^2) \\ &= m^2(t)(\|v\|^2 + \|w(-\rho(0))\|^2) \\ &\le m^2(t)(\|v\|^2 + \sup_{s\in[-h,0]} \|w(s)\|^2).\end{aligned}$$

Similarly, we have

$$\|f_2(t, v, w)\|^2 \le m^2(t)(\|v\|^2 + \sup_{s\in[-h,0]} \|w(s)\|^2).$$

Now putting

$$F(t, v, w) = [f_1(t, v, w), f_2(t, v, w)], v \in \ell^2, w \in C([-h, 0]; \ell^2),$$

we see that

$$\|F(t, v, w)\| \le m(t)(\|v\| + \|w\|_h).$$

In addition, F is a multimap with convex and compact values. Indeed, it is easily seen that for each $(t, v, w) \in \mathbb{R}^+ \times \ell^2 \times C([-h, 0]; \ell^2)$, $F(t, v, w)$ is a closed convex set. Moreover, $F(t, v, w) \subset \text{span}\{f_1(t, v, w), f_2(t, v, w)\}$, i.e., $F(t, v, w)$ is a bounded set lying in a two-dimensional subspace of ℓ^2, thus $F(t, v, w)$ is a compact set. Since f_1, f_2 are continuous, the multimap

$(v,w) \mapsto F(t,v,w)$ is closed. This implies that $F(t,\cdot,\cdot)$ is u.s.c for each $t \in \mathbb{R}$. Noting that for each $u \in \mathrm{PC}_0$, $\tau \in [0,1]$, the function

$$f(t) = \tau f_1(t,u(t),u(t-\rho(t))) + (1-\tau) f_2(t,u(t),u(t-\rho(t))), \tau \in [0,1]$$

is a (strongly) measurable selection of F. Thus, (**F**)(1)–(**F**)(3) are fulfilled.

We now evaluate $\chi(F(t,V,W))$ for bounded sets $V \subset \ell^2, W \subset C([-h,0];\ell^2)$. One sees that

$$\begin{aligned}
\sup_{(v,w)\in V\times V} \|R_n[f_1(t,v,w)]\| &= \left(\sum_{|i|>n} |f_{1i}(t,v_i,w_i(-\rho(0)))|^2\right)^{\frac{1}{2}} \\
&\le m(t) \sup_{(v,w)\in V\times V} \left(\sum_{|i|>n} \left[|v_i|^2 + |w_i(-\rho(0))|^2\right]\right)^{\frac{1}{2}} \\
&\le m(t) \sup_{(v,w)\in V\times V} \left[\left(\sum_{|i|>n} |v_i|^2\right)^{\frac{1}{2}} + \left(\sum_{|i|>n} |w_i(-\rho(0))|^2\right)^{\frac{1}{2}}\right] \\
&\le m(t)\left[\sup_{v\in V}\left(\sum_{|i|>n} |v_i|^2\right)^{\frac{1}{2}} + \sup_{w\in W}\left(\sum_{|i|>n} |w_i(-\rho(0))|^2\right)^{\frac{1}{2}}\right] \\
&= m(t)\left[\sup_{v\in V}\|R_n(v)\| + \sup_{w\in W}\|R_n(w(-\rho(0)))\|\right].
\end{aligned}$$

Passing to the limit in the last inequality yields

$$\begin{aligned}
\chi(f_1(t,V,W)) &\le m(t)\big[\chi(V) + \chi(W(-\rho(0)))\big] \\
&\le m(t)\big[\chi(V) + \sup_{s\in[-h,0]} \chi(W(s))\big].
\end{aligned}$$

By the same arguments for f_2, we have

$$\chi(f_2(t,V,W)) \le m(t)\left[\chi(V) + \sup_{s\in[-h,0]} \chi(W(s))\right].$$

Observing that

$$F(t,V,W) \subset \mathrm{co}\{f_1(t,V,W) \cup f_2(t,V,W)\},$$

one gets

$$\begin{aligned}
\chi(F(t,V,W)) &\le \chi\left(f_1(t,V,W) \cup f_2(t,V,W)\right) \\
&\le \max\{\chi\left(f_1(t,V,W)\right), \chi\left(f_2(t,V,W)\right)\} \\
&\le m(t)\left[\chi(V) + \sup_{s\in[-h,0]} \chi(W(s))\right].
\end{aligned}$$

Then, (**F**)(4) is satisfied and, therefore, (**F***) is testified with $k = m$.

Now, consider the maps $I_k : \ell^2 \to \ell^2, k \in \mathbb{N}$, defined by

$$I_k(v) = (I_{ik}(v_i))_{i\in\mathbb{Z}}.$$

Then, by the continuity of I_{ik}, one gets the continuity of I_k. Moreover, by (N2) we have

$$\begin{aligned}\|I_k(v)\| &= \left(\sum_{i\in\mathbb{Z}} |I_{ik}(v_i)|^2\right)^{\frac{1}{2}} \le l_k \left(\sum_{i\in\mathbb{Z}} |v_i|^2\right)^{\frac{1}{2}} \\ &= l_k\|v\|.\end{aligned}$$

Hence, (**I**)(1) is satisfied. Now, let $V \subset \ell^2$ be a bounded set. Then

$$\begin{aligned}\sup_{v\in V} \|R_n(I_k(v))\| &= \sup_{v\in V} \left(\sum_{|i|>n} |I_{ik}(v_i)|^2\right)^{\frac{1}{2}} \\ &\le l_k \sup_{v\in V} \left(\sum_{|i|>n} |v_i|^2\right)^{\frac{1}{2}} = l_k \sup_{v\in V} \|R_n(v)\|.\end{aligned}$$

Taking the limit of the last inequality as $n \to +\infty$, we arrive at

$$\chi(I_k(V)) \le l_k \chi(V).$$

So the assumption (**I**) holds with $\mu_k = l_k, k \in \mathbb{N}$, provided that $\inf\{t_{k+1} - t_k : k \in \mathbb{N}\} > 0$.

Finally, we give an estimate for the integral

$$I(t) = \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \mathrm{d}s.$$

Observe that in our case $\|\mathrm{e}^{tA}\| \le 1$, then $\|P_\alpha(t)\| \le \frac{1}{\Gamma(\alpha)}, \forall t \ge 0$. Then

$$\begin{aligned}I(t) &\le \frac{C_m}{\Gamma(\alpha)} \left(\int_0^{\frac{t}{2}} \frac{(t-s)^{\alpha-1}}{1+s^{\alpha+1}} \mathrm{d}s + \int_{\frac{t}{2}}^t \frac{(t-s)^{\alpha-1}}{1+s^{\alpha+1}} \mathrm{d}s\right) \\ &\le \frac{C_m}{\Gamma(\alpha)} \left(\left(\frac{t}{2}\right)^{\alpha-1} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{1+s^{\alpha+1}} + \frac{1}{1+\left(\frac{t}{2}\right)^{\alpha+1}} \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} \mathrm{d}s\right) \\ &\le \frac{C_m}{\Gamma(\alpha)} \left(J(t) + \frac{1}{\alpha}\right),\end{aligned}$$

where

$$J(t) = \left(\frac{t}{2}\right)^{\alpha-1} \int_0^{\frac{t}{2}} \frac{\mathrm{d}s}{1+s^{\alpha+1}}.$$

Noting that

$$\lim_{t\to 0} J(t) = \lim_{t\to+\infty} J(t) = 0,$$

we get $\sup_{t>0} J(t) < \infty$, then $\sup_{t>0} I(t) < \infty$. It follows that the conditions (4.1) and (4.2) are testified with C_m, l_k small, and we get the weakly asymptotic stability of zero solution to (5.1) and (5.2).

6. Conclusion

A unified approach is proposed to prove the global solvability and weakly asymptotic stability for the semilinear fractional differential inclusion involving impulsive effects given by (1.1)–(1.3). That is, we analyze the fixed point set of the solution operator $\mathcal{F}$ associated with our problem on PC_ϱ, the space of piecewise continuous functions on the half-line with weighted function ϱ, on which the MNC χ^* is defined to ensure a compactness condition. In the case when the semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is exponentially stable, the existence of decay solutions to (1.1)–(1.3) implies the weak attractivity of the zero solution that leads to the weakly asymptotic stability as mentioned. In further analysis, one can estimate the decay rate of solutions provided that the impulsive condition (1.2) is relaxed or the Caputo derivative in (1.1) is replaced by a substantial Caputo derivative (see, e.g [5]).

Acknowledgements

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.02-2015.18.

References

[1] Aubin, J.P.: Viability Theory. Birkhäuser, Basel (1991)

[2] Ayerbe Toledano, J.M., Domnguez Benavides, T., Lpez Acedo, G.: Measures of noncompactness in metric fixed point theory. In: Operator Theory: Advances and Applications, vol. 99. Birkhäuser, Basel (1997)

[3] Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. Springer, New York (2011)

[4] Cernea, A.: On the existence of mild solutions for nonconvex fractional semilinear differential inclusions. Electron. J. Qual. Theory Differ. Equ. **64**, 1–15 (2012)

[5] Chen, M., Deng, W.: Discretized fractional substantial calculus. ESAIM Math. Model. Numer. Anal. **49**, 373–394 (2015)

[6] Diestel, J., Ruess, W.M., Schachermayer, W.: Weak compactness in $L^1(\mu, X)$. Proc. Am. Math. Soc. **118**, 447–453 (1993)

[7] Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (2000)

[8] Ji, S., Wen, S.: Nonlocal Cauchy problem for impulsive differential equations in Banach spaces. Int. J. Nonlinear Sci. **10**, 88–95 (2010)

[9] Kamenskii, M., Obukhovskii, V., Zecca, P.: Condensing multivalued maps and semilinear differential inclusions in Banach spaces. In: de Gruyter Series in Nonlinear Analysis and Applications, vol. 7. Walter de Gruyter, Berlin (2001)

[10] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)

[11] Ke, T.D., Lan, D.: Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces. Fract. Calc. Appl. Anal. **17**, 96–121 (2014)

[12] Ke, T.D., Lan, D.: Global attractor for a class of functional differential inclusions with Hille–Yosida operators. Nonlinear Anal. **103**, 72–86 (2014)

[13] Liu, Z., Lv, J., Sakthivel, R.: Approximate controllability of fractional functional evolution inclusions with delay in Hilbert spaces. IMA J. Math. Control Inform. **31**, 363–383 (2014)

[14] Ren, Y., Lanying, H.: Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay. J. Comput. Appl. Math. **235**, 2603–2614 (2011)

[15] Sakthivel, R., Ganesh, R., Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. **225**, 708–717 (2013)

[16] Sakthivel, R., Ganesh, R., Ren, Y., Anthoni, S.M.: Approximate controllability of nonlinear fractional dynamical systems. Commun. Nonlinear Sci. Numer. Simul. **18**, 3498–3508 (2013)

[17] Sakthivel, R., Ren, Y.: Approximate controllability of fractional differential equations with state-dependent delay. Results Math. **63**, 949–963 (2013)

[18] Samoilenko, A.M., Perestyuk, N.A.: Impulsive Differential Equations. World Scientific Publishing Co, River Edge (1995). (translated from the Russian)

[19] Seidman, T.I.: Invariance of the reachable set under nonlinear perturbations. SIAM J. Control Optim. **25**(5), 1173–1191 (1987)

[20] Stamova, I.: Stability Analysis of Impulsive Functional Differential Equations. Walter de Gruyter, Berlin (2009)

[21] Wang, R.-N., Ma, Q.-H.: Some new results for multi-valued fractional evolution equations. Appl. Math. Comput. **257**, 285–294 (2014)

[22] Wang, J.R., Ibrahim, A.G., Feckan, M.: Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces. Appl. Math. Comput. **257**, 103–118 (2015)

[23] Wang, R.-N., Zhu, P.-X., Ma, Q.-H.: Multi-valued nonlinear perturbations of time fractional evolution equations in Banach spaces. Nonlinear Dyn. **80**, 1745–1759 (2015)

[24] Wang, J.R., Zhou, Y.: Existence and controllability results for fractional semilinear differential inclusions. Nonlinear Anal. Real World Appl. **12**, 3642–3653 (2011)

[25] Wang, R.N., Xiang, Q.M., Zhu, P.X.: Existence and approximate controllability for systems governed by fractional delay evolution inclusions. Optimization **63**, 1191–1204 (2014)

[26] Wang, R.-N., Chena, D.-H., Xiao, T.-J.: Abstract fractional Cauchy problems with almost sectorial operators. J. Differ. Equ. **252**, 202–235 (2012)

[27] Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)

[28] Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comp. Math. Appl. **59**, 1063–1077 (2010)

[29] Zhou, Y., Vijayakumar, V., Murugesu, R.: Controllability for fractional evolution inclusions without compactness. Evol. Equ. Control Theory **4**, 507–524 (2015)

Tran Dinh Ke
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Cau Giay
Hanoi
Vietnam
e-mail: ketd@hnue.edu.vn

Do Lan
Faculty of Computer Engineering and Science
Thuyloi University
175 Tay Son, Dong Da
Hanoi
Vietnam
e-mail: dolan@tlu.edu.vn