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New fixed point theorems for set-valued contractions in b-metric spaces

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Abstract. In this paper, we indicate a way to generalize a series of fixed point results in the framework of b-metric spaces and we exemplify it by extending Nadler's contraction principle for set-valued functions (see Nadler, Pac J Math 30:475–488, 1969) and a fixed point theorem for set-valued quasi-contraction functions due to Aydi et al. (see Fixed Point Theory Appl 2012:88, 2012).

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1. Introduction

In the last decades one can observe a remarkable amount of interest for the development of fixed point theory, since it has a huge number of applications.

Among the generalizations of the Banach–Caccioppoli–Picard principle—one of the central results of the above-mentioned theory, known also as the contraction principle—a central role is played by the following two:

- the one due to Nadler [22] who extended the contraction principle to set-valued functions and generated in this way many applications in control theory, convex optimization, etc. (see [17,31–34] and the references therein);
- the one due to Bakhtin [5] and Czerwik [13,14] who, motivated by the problem of the convergence of measurable functions with respect to measure, introduced b-metric spaces (a generalization of metric spaces) and proved the contraction principle in this framework. In the last period many mathematicians obtained fixed point results for single-valued or set-valued functions, in the setting of b-metric spaces (see, for example, [1,8–10,18,24,25,30,32,35] and the references therein).

In this paper, we indicate a way (see Lemma 2.2) to generalize a series of fixed point results in the framework of b-metric spaces and we exemplify it by extending Nadler's contraction principle for set-valued functions (see [22]) and a fixed point theorem for set-valued quasi-contraction functions due to Aydi et al. (see [4]).

2. Preliminary results

In this section, we sum up some basic facts that we are going to use later.

Definition 2.1. Given a non-empty set X and a real number $s \in [1, \infty)$, a function $d : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called b-metric space of constant s.

Remark 2.1. As when $s = 1$, a b-metric space is a metric space, we infer that the family of b-metric spaces is larger than the one of metric spaces. In other words, every metric space is a b-metric space. Note that Czerwik proved that the converse need not be true (see also [4,12,19,23,28]), so the family of b-metric spaces is effectively larger than the one of metric spaces.

Definition 2.2. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X, d) is called:

- convergent if there exists $l \in X$ such that $\lim_{n\to\infty} d(x_n, l) = 0$;
- Cauchy if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$, i.e. for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \in \mathbb{N}$, $m, n \geq n_\varepsilon$.

The b-metric space (X, d) is called complete if every Cauchy sequence of elements from (X, d) is convergent.

Beside the classical spaces $l^p(\mathbb{R})$ and $L^p[0, 1]$, where $p \in (0, 1)$, one can find examples of b-metric spaces in [4,6,10,13,14].

Remark 2.2. As in the case of metric spaces, a b-metric space can be endowed with the topology induced by its convergence and almost all the concepts and results which are valid for metric spaces can be extended to the framework of b-metric spaces.

An et al. [3] proved that every b-metric space is a semi-metrizable space (i.e. there exists a function $d : X \times X \to [0, \infty)$ such that: (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$; (iii) $x \in \overline{A}$ if and only if $d(x, A) = \inf\{d(x, y) \mid y \in A\} = 0$ for every $x \in X$ and every $A \subseteq X$). Consequently, many properties of b-metric spaces are obvious. In addition, they provided a sufficient condition for a b-metric space to be metrizable and gave an example showing that, in the framework of a b-metric space (X, d), there exists an open ball (i.e. a set of the form $\{y \in X \mid d(x, y) < r\}$, where $r > 0$) which is not open.

In a metric space (X, d), the function d is continuous (i.e. $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements from X and $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$).

The fact that this property is not valid for b-metric spaces of constant s (as $\frac{1}{s^2}d(x,y) \leq \underline{\lim}_{n\to\infty} d(x_n,y_n) \leq \overline{\lim}_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y)$ and $\frac{1}{s}d(x,y) \leq \underline{\lim}_{n\to\infty} d(x_n,y) \leq \overline{\lim}_{n\to\infty} d(x_n,y) \leq sd(x,y)$, see [21,23,26]) is a motivation of our Definition 3.2.

In the sequel, given a b-metric space (X,d):

- by $\mathcal{B}(X)$ we denote the set of non-empty bounded closed subsets of X
- for $A,B \in \mathcal{B}(X)$, we define the Hausdorff–Pompeiu distance between A and B by $h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$, where $d(x,C) = \inf_{c\in C} d(x,c)$ for every $x \in X$ and every $C \in \mathcal{B}(X)$
- given $T : X \to \mathcal{B}(X)$, for $c,d \in [0,1]$ and $x,y \in X$, we shall use the following notation:

$$N_{c,d}(x,y) = \max\{d(x,y), cd(x,T(x)), cd(y,T(y)), \frac{d}{2}(d(x,T(y))+d(y,T(x)))\}$$

- for a sequence $(x_n)_{n\in\mathbb{N}}$, of elements from X, sometimes, for the sake of brevity, we shall use the notation: $d_n = d(x_n,x_{n+1})$, where $n \in \mathbb{N}$.

Lemma 2.1. *For every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X,d) of constant s, the inequality*

$$d(x_0,x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i,x_{i+1}),$$

is valid for every $n \in \mathbb{N}$ and every $k \in \{1,2,3,...,2^n-1,2^n\}$.

Proof. We are going to use the method of mathematical induction. Denoting by $P(n)$ the statement: $d(x_0,x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i,x_{i+1})$ for every $x_0,x_1,...,x_{2^n} \in X$ and every $k \in \{1,2,3,...,2^n-1,2^n\}$, as the statements $P(0)$ and $P(1)$ are obvious, it remains to prove that $P(n) \Rightarrow P(n+1)$.

Indeed, the above-mentioned implication is true since, on the one hand, for every $k \in \{1,2,3,...,2^n-1,2^n\}$, using $P(n)$, we have

$$d(x_0,x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i,x_{i+1}).$$

On the other hand, for every $k \in \{2^n+1,2^n+2,...,2^{n+1}-1,2^{n+1}\}$, using again $P(n)$, we have

$$d(x_0,x_k) \leq s(d(x_0,x_{2^n}) + d(x_{2^n},x_k))$$
$$\leq s\left(s^n \sum_{i=0}^{2^n-1} d(x_i,x_{i+1}) + s^n \sum_{i=2^n}^{k-1} d(x_i,x_{i+1})\right) = s^{n+1} \sum_{i=0}^{k-1} d(x_i,x_{i+1}).$$

□

Lemma 2.2. *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X,d) of constant s, having the property that there exists $\gamma \in [0,1)$ such that*

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}),$$

for every $n \in \mathbb{N}$, is Cauchy.

Proof. First let us note that

$$d(x_{n+1}, x_n) \le \gamma^n d(x_1, x_0), \tag{1}$$

for every $n \in \mathbb{N}$.

For all $m, k \in \mathbb{N}$, with the notation $p = [\log_2 k]$, we have

$$\begin{aligned} d(x_{m+1}, x_{m+k}) &\le sd(x_{m+1}, x_{m+2}) + sd(x_{m+2}, x_{m+k}) \\ &\le sd(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+2^2}) + s^2 d(x_{m+2^2}, x_{m+k}) \\ &\le sd(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+2^2}) + s^3 d(x_{m+2^2}, x_{m+2^3}) \\ &\quad + s^3 d(x_{m+2^3}, x_{m+k}) \\ &\qquad \ldots \\ &\le \sum_{n=1}^{p} s^n d(x_{m+2^{n-1}}, x_{m+2^n}) + s^{p+1} d(x_{m+2^p}, x_{m+k}). \end{aligned}$$

Using Lemma 2.1 and (1), we obtain

$$\begin{aligned} &d(x_{m+1}, x_{m+k}) \\ &\le \sum_{n=1}^{p} s^{2n} \left(\sum_{i=m}^{m+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \right) + s^{2(p+1)} \left(\sum_{i=m}^{m+k-2^p-1} d(x_{2^p+i}, x_{2^p+i+1}) \right) \\ &\le \sum_{n=1}^{p+1} s^{2n} \left(\sum_{i=m}^{m+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \right) \le d(x_0, x_1) \sum_{n=1}^{p+1} s^{2n} \left(\sum_{i=0}^{2^{n-1}-1} \gamma^{m+2^{n-1}+i} \right) \\ &\le \frac{d(x_0, x_1)\gamma^m}{1-\gamma} \sum_{n=1}^{p+1} s^{2n} \gamma^{2^{n-1}} = \gamma^m \frac{d(x_0, x_1)}{1-\gamma} \sum_{n=1}^{p+1} \gamma^{2n \log_\gamma s + 2^{n-1}}. \end{aligned}$$

Let us note that since $\lim_{n\to\infty}(2n \log_\gamma s + 2^{n-1} - n) = \infty$, for a fixed $M > 0$, there exists $n_0 \in \mathbb{N}$ such that $2n \log_\gamma s + 2^{n-1} - n \ge M$, i.e. $\gamma^{2n \log_\gamma s + 2^{n-1}} \le \gamma^M \gamma^n$ for each $n \in \mathbb{N}$, $n \ge n_0$, hence the series $\sum_{n=1}^{\infty} \gamma^{2n \log_\gamma s + 2^{n-1}}$ is convergent and denoting by S its sum, we come to the conclusion that

$$d(x_{m+1}, x_{m+k}) \le \gamma^m \frac{d(x_0, x_1)S}{1-\gamma},$$

for all $m, k \in \mathbb{N}$. Consequently, as $\lim_{n\to\infty} \gamma^n = 0$, we infer that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. □

Theorem 2.1. *Let (X, d) be a b-metric space of constant s and $T : X \to \mathcal{B}(X)$ having the property that there exist $c, d \in [0, 1]$ and $\alpha \in [0, 1)$ such that:*

(i) $\alpha ds < 1$;
(ii) $h(T(x), T(y)) \le \alpha N_{c,d}(x, y)$ *for all* $x, y \in X$.

Then, for every $x_0 \in X$, there exist $\gamma \in [0, 1)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that:

(a) $x_{n+1} \in T(x_n)$ *for every* $n \in \mathbb{N}$;
(b) $d(x_{n+1}, x_n) \le \gamma d(x_n, x_{n-1})$ *for every* $n \in \mathbb{N}$;
(c) $(x_n)_{n\in\mathbb{N}}$ *is Cauchy.*

Proof. Let us consider $\beta \in (\alpha, \min(1, \frac{1}{ds}))$, $\gamma = \max\{\beta, \frac{ds\beta}{2-ds\beta}\} < 1$, $x_0 \in X$ and $x_1 \in T(x_0)$.

If $x_1 = x_0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = x_0$ for every $n \in \mathbb{N}$ satisfies (a), (b) and (c).

Since, based on (ii), we have

$$d(x_1, T(x_1)) \leq h(T(x_0), T(x_1)) \leq \alpha N_{c,d}(x_0, x_1) < \beta N_{c,d}(x_0, x_1),$$

there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < \beta N_{c,d}(x_0, x_1)$.

If $x_2 = x_1$, then the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = x_1$ for every $n \in \mathbb{N}$, $n \geq 1$, satisfies (a), (b) and (c).

By repeating this procedure, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that $x_{n+1} \in T(x_n)$ and $0 < d_n < \beta N_{c,d}(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$, $n \geq 1$.

Because $d(x_{n-1}, T(x_{n-1})) \leq d_{n-1}$, $d(x_n, T(x_n)) \leq d_n$, $d(x_{n-1}, T(x_n)) \leq d(x_{n-1}, x_{n+1})$ and $d(x_n, T(x_{n-1})) = 0$, we have

$$\begin{aligned} 0 &< d_n < \beta N_{c,d}(x_{n-1}, x_n) \\ &\leq \beta \max\{d_{n-1}, cd_n, cd_{n-1}, \frac{d}{2} d(x_{n-1}, x_{n+1})\} \\ &\leq \beta \max\left\{d_{n-1}, cd_n, cd_{n-1}, \frac{ds}{2}(d_{n-1}+d_n)\right\} \leq \beta \max\left\{d_{n-1}, \frac{ds}{2}(d_{n-1}+d_n)\right\}, \end{aligned}$$

for every $n \in \mathbb{N}$, where the justification of the last inequality is the following: if, by reduction ad absurdum, $\max\{d_{n-1}, cd_n, cd_{n-1}, \frac{ds}{2}(d_{n-1} + d_n)\} = cd_n$, then we get that $0 < d_n < \beta cd_n \leq \beta d_n$, so we obtain the contradiction $1 < \beta$.

Consequently, $d_n < \beta d_{n-1}$ or $d_n < \beta \frac{ds}{2}(d_{n-1} + d_n)$, i.e. $d_n < \beta d_{n-1}$ or $d_n < \frac{ds\beta}{2-ds\beta} d_{n-1}$ for every $n \in \mathbb{N}$. Thus $d_n \leq \max\{\beta, \frac{ds\beta}{2-ds\beta}\} d_{n-1}$, i.e. $d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$ for every $n \in \mathbb{N}$.

Hence the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies (a) and (b). From Lemma 2.2 we deduce that it also satisfies (c). □

3. Main results

In this section, making use of Theorem 2.1, we present three fixed point theorems for set-valued functions.

Definition 3.1. A function $T : X \to \mathcal{B}(X)$, where (X, d) is a b-metric space, is called closed if for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements from X and $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $y_n \in T(x_n)$ for every $n \in \mathbb{N}$, we have $y \in T(x)$.

Theorem 3.1. *A function $T : X \to \mathcal{B}(X)$, where (X, d) is a complete b-metric space of constant s, has a fixed point, provided that it satisfies the following three conditions:*

(i) *T is closed;*
(ii) *there exist $c, d \in [0, 1]$ and $\alpha \in [0, 1)$ such that $h(T(x), T(y)) \leq \alpha N_{c,d}(x, y)$ for all $x, y \in X$;*
(iii) *$\alpha ds < 1$.*

Proof. Taking into account (ii) and (iii), by virtue of Theorem 2.1, there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that

$$x_{n+1} \in T(x_n), \tag{2}$$

for every $n \in \mathbb{N}$.

As the b-metric space (X, d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$ (so $\lim_{n\to\infty} x_{n+1} = u$). We combine (i) with (2) to see that $u \in T(u)$, i.e. u is a fixed point of T. □

Definition 3.2. Given a b-metric space (X, d), the b-metric d is called $*$- continuous if for every $A \in \mathcal{B}(X)$, every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} d(x_n, A) = d(x, A)$.

Our notion of $*$-continuity of d is stronger than the continuity of d in the first variable.

Theorem 3.2. *A function $T : X \to \mathcal{B}(X)$, where (X, d) is a complete b-metric space of constant s, has a fixed point, provided that it satisfies the following three conditions:*
- (i) *d is $*$-continuous;*
- (ii) *there exist $c, d \in [0, 1]$ and $\alpha \in [0, 1)$ such that $h(T(x), T(y)) \le \alpha N_{c,d}(x, y)$ for all $x, y \in X$;*
- (iii) $\alpha ds < 1$.

Proof. Based on (ii) and (iii), according to Theorem 2.1, there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that

$$x_{n+1} \in T(x_n), \tag{3}$$

for every $n \in \mathbb{N}$.

As the b-metric space (X, d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Then, using (ii) and (3), with the notation $d(x_n, u) = \delta_n$, we have

$$\begin{aligned} d(x_{n+1}, T(u)) &\le h(T(x_n), T(u)) \le \alpha N_{c,d}(x_n, u) \\ &= \alpha \max\left\{\delta_n, cd(x_n, T(x_n)), cd(u, T(u)), \frac{d}{2}(d(x_n, T(u)) + d(u, T(x_n)))\right\} \\ &\le \alpha \max\left\{\delta_n, cd_n, cd(u, T(u)), \frac{d}{2}(s(\delta_n + d(u, T(u))) + \delta_{n+1})\right\}, \end{aligned} \tag{4}$$

for every $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d_n = 0$ and $\lim_{n\to\infty} d(x_{n+1}, T(u)) = d(u, T(u))$ (as d is $*$-continuous and $\lim_{n\to\infty} x_{n+1} = u$), upon passing to limit, as $n \to \infty$, in (4), we get

$$d(u, T(u)) \le \max\left\{\alpha c, \frac{\alpha ds}{2}\right\} d(u, T(u)). \tag{5}$$

As $\max\{\alpha c, \frac{\alpha ds}{2}\} < 1$ (see (iii)), from (5), we conclude that $d(u, T(u)) = 0$, i.e. $u \in T(u)$. Hence T has a fixed point. □

Theorem 3.3. *A function $T : X \to \mathcal{B}(X)$, where (X, d) is a complete b-metric space of constant s, has a fixed point, provided that it satisfies the following two conditions:*

(i) *there exist* $c, d \in [0,1]$ *and* $\alpha \in [0,1)$ *such that* $h(T(x), T(y)) \leq \alpha N_{c,d}(x, y)$ *for all* $x, y \in X$;
(ii) $\max\{\alpha cs, \alpha ds\} < 1$.

Proof. Making use of (i) and (ii), according to Theorem 2.1, there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that $x_{n+1} \in T(x_n)$, for every $n \in \mathbb{N}$. As the b-metric space (X, d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

First let us note that, as we have seen in (4), we have

$$\begin{aligned} &d(x_{n+1}, T(u)) \\ &\quad\leq \alpha \max\left\{\delta_n, cd(x_n, T(x_n)), cd(u, T(u)), \frac{d}{2}(d(x_n, T(u)) + d(u, T(x_n)))\right\} \\ &\quad\leq \alpha \max\left\{\delta_n, cd(x_n, T(x_n)), cd(u, T(u)), \frac{d}{2}(d(x_n, T(u)) + \delta_{n+1})\right\} \\ &\quad\leq \alpha \max\left\{\delta_n, cd_n, cd(u, T(u)), \frac{d}{2}(s(\delta_n + d(u, T(u))) + \delta_{n+1})\right\}, \end{aligned} \tag{6}$$

for every $n \in \mathbb{N}$.

We divide the discussion into two cases:

A. $d(u, T(u)) \leq \overline{\lim}_{n\to\infty} d(x_n, T(u))$;

and

B. $d(u, T(u)) > \overline{\lim}_{n\to\infty} d(x_n, T(u))$.

In case A, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ having the property that $\lim_{k\to\infty} d(x_{n_k+1}, T(u)) \geq d(u, T(u))$, so for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $d(u, T(u)) - \varepsilon \leq d(x_{n_k+1}, T(u))$, for every $k \in \mathbb{N}$, $k \geq k_\varepsilon$. Hence, taking into account (6), we get

$$\begin{aligned} &d(u, T(u)) - \varepsilon \\ &\quad\leq \alpha \max\left\{\delta_{n_k}, cd_{n_k}, cd(u, T(u)), \frac{d}{2}(s(\delta_{n_k} + d(u, T(u))) + \delta_{n_k+1})\right\}, \end{aligned}$$

for every $k \in \mathbb{N}$, $k \geq k_\varepsilon$. By passing to limit as $k \to \infty$ in the above inequality, we get that

$$\begin{aligned} d(u, T(u)) - \varepsilon &\leq \alpha \max\left\{cd(u, T(u)), \frac{sd}{2} d(u, T(u))\right\} \\ &= d(u, T(u)) \max\left\{\alpha c, \frac{\alpha sd}{2}\right\}, \end{aligned}$$

for every $\varepsilon > 0$, so

$$d(u, T(u)) \leq d(u, T(u)) \max\left\{\alpha c, \frac{\alpha sd}{2}\right\}.$$

Since $\max\{\alpha c, \frac{\alpha sd}{2}\} < 1$, from the above inequality, we conclude that $d(u, T(u)) = 0$, i.e. $u \in T(u)$, so T has a fixed point.

In case B, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, T(u)) \leq d(u, T(u)), \tag{7}$$

for every $n \in \mathbb{N}$, $n \geq n_0$. Since $d(u, T(u)) \leq s(\delta_{n+1} + d(x_{n+1}, T(u)))$, i.e. $\frac{d(u,T(u))}{s} - \delta_{n+1} \leq d(x_{n+1}, T(u))$, using (6) and (7), we get that

$$\begin{aligned}\frac{d(u,T(u))}{s} - \delta_{n+1} &\leq d(x_{n+1}, T(u)) \\ &\leq \alpha \max\{\delta_n, cd_n, cd(u, T(u)), \frac{d}{2}(d(x_n, T(u)) + \delta_{n+1}))\} \\ &\leq \alpha \max\{\delta_n, cd_n, cd(u, T(u)), \frac{d}{2}(d(u, T(u)) + \delta_{n+1}))\},\end{aligned}$$

for every $n \in \mathbb{N}$, $n \geq n_0$. By passing to limit as $n \to \infty$ in the above inequality, we obtain that

$$d(u, T(u)) \leq \alpha s \max\left\{cd(u, T(u)), \frac{d}{2} d(u, T(u))\right\} = \alpha s \max\left\{c, \frac{d}{2}\right\} d(u, T(u)).$$

As $\alpha s \max\{c, \frac{d}{2}\} < 1$ (see (ii)), we infer that $d(u, T(u)) = 0$, so $u \in T(u)$, i.e. T has a fixed point. □

4. Remarks and comments

I. Let us recall the following result (see Lemma 3.1 from [29]):

Lemma 4.1. *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X, d) of constant s is Cauchy provided that:*

(i) *there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbb{N}$;

(ii) $s\gamma < 1$.

Obviously our Lemma 2.2 is a generalization of the above Lemma which is the corner stone of the results from [16,19,20,23,29].

II. The following definition is inspired by the definition of a multi-valued weakly Picard operator in the setting of a metric space from [7].

Definition 4.1. A function $T : X \to \mathcal{B}(X)$, where (X, d) is a b-metric space, is called a multi-valued weakly Picard operator if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$ and $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$ for every $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Let us mention that Theorems 3.1, 3.2 and 3.3 provide sufficient conditions for a function T to be multi-valued weakly Picard operator.

III. For $c = d = 0$ and $s = 1$ in Theorem 3.3 we obtain Theorem 5 from [22], i.e. Nadler's contraction principle for set-valued functions.

IV. Let us recall the following result (see Theorem 2.2 from [4]) which is a generalization of Theorem 1.2 from [2] which improves Theorem 3.3 from [15], Corollary 3.3. from [27], Corollary 4.3 from [29] and Theorem 1 from [11].

Theorem 4.1. *A function $T : X \to \mathcal{B}(X)$, where (X, d) is a complete b-metric space of constant s, has a fixed point, provided that it satisfies the following two conditions:*

(i) *there exists $a \in [0, 1)$ such that*

$$h(T(x), T(y)) \leq a \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\},$$

for all $x, y \in X$;

(ii) $a \leq \frac{1}{s+s^2}$.

Our Theorem 3.3 is a generalization of Theorem 4.1.

Indeed, we shall prove that if (i) and (ii) from Theorem 4.1 are satisfied, then (i) and (ii) from Theorem 3.3 (for $\alpha = 2a$ and $c = d = 1$) are satisfied. First let us note that, according to (ii) from Theorem 4.1, we have $0 \leq \alpha < \frac{2}{s+s^2} < \frac{1}{s}$, so $\alpha s < 1$. Consequently, $\alpha \in [0, \frac{1}{s}) \subseteq [0, 1)$ and (ii) from Theorem 3.3 is satisfied. Moreover, (i) from Theorem 4.1 implies that

$$\begin{aligned} &h(T(x), T(y)) \\ &\quad \leq \alpha \max\left\{\frac{d(x, y)}{2}, \frac{d(x, T(x))}{2}, \frac{d(y, T(y))}{2}, \frac{d(x, T(y))}{2}, \frac{d(y, T(x))}{2}\right\} \\ &\quad \leq \alpha \max\left\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{1}{2}(d(x, T(y)) + d(y, T(x)))\right\} \\ &\quad = \alpha N_{1,1}(x, y), \end{aligned}$$

for all $x, y \in X$, i.e. (i) from Theorem 3.3 is satisfied.

Now let us present a situation when Theorem 3.3 is applicable, but Theorem 4.1 is not.

We consider the b-metric space $(\mathbb{R}, d)$, where $d(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$, for which $s = 2$ and the function $f : \mathbb{R} \to \mathcal{B}(\mathbb{R})$ given by $f(x) = \{\frac{9}{10}x\}$ for every $x \in \mathbb{R}$. On the one hand, Theorem 3.3. is applicable taking $c = d = 0$ and $\alpha = \frac{9}{10}$. On the other hand, Theorem 4.1 is not applicable since (i) implies $\frac{9}{10} \leq \alpha$ and (ii) implies $\alpha \leq \frac{1}{6}$.

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