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On split generalised mixed equilibrium problems and fixed-point problems with no prior knowledge of operator norm

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Abstract. The purpose of this paper is to introduce an iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split generalised mixed equilibrium problem which is also a fixed point of a κ-strictly pseudocontractive mapping. Furthermore, a strong convergence theorem for approximating a common solution of a split generalised mixed equilibrium problem and a fixed-point problem for κ-strictly pseudocontractive mapping was stated and proved in the frame work of Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space and K a nonempty, closed and convex subset of H. A mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall \quad x, y \in K, \tag{1.1}$$

and *κ-strictly pseudocontractive* in the sense of Browder and Petryshyn [2] if for $0 \leq \kappa < 1$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \quad \forall \quad x, y \in K. \tag{1.2}$$

In a Hilbert space H, we can show that (1.2) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2. \tag{1.3}$$

A point $x \in K$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. So a fixed-point problem for T is to find $x \in F(T)$. It is a common knowledge that if T is κ-strictly pseudocontractive and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex. See [2,19,20,32] and references therein, for more details on strictly pseudocontractive mappings.

Let $g : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function and $B : C \to H$ be a nonlinear mapping. The *Generalised mixed equilibrium problem* is to find $u \in C$ such that

$$g(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C. \tag{1.4}$$

Denote the set of solutions of the problem (1.4) by $GMEP(g, \varphi, B)$. That is

$$GMEP(g, \varphi, B) = \{u \in C : g(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C\}. \tag{1.5}$$

If $B = 0$, then the generalised mixed equilibrium problem (1.4) reduces to the following *mixed equilibrium problem*, find $u \in C$ such that

$$g(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C. \tag{1.6}$$

If $\varphi = 0$, then the generalised mixed equilibrium problem (1.4) becomes the *generalised equilibrium problem*, find $u \in C$ such that

$$g(u, y) + \langle Bu, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

Again if $B = \varphi = 0$, then the generalised mixed equilibrium problem (1.4) becomes the *equilibrium problem*, find $u \in C$ such that

$$g(u, y) \geq 0, \quad \forall y \in C. \tag{1.8}$$

Equilibrium problems and their generalisations are well known to have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimisation problems, fixed-point problems and have been widely applied to physics, structural analysis, management sciences and economics, etc. (see, for example [1,5,22,23]). In solving equilibrium problem (1.8), the bifunction g is said to satisfy the following conditions:

- (A1) $g(x, x) = 0$ for all $x \in C$;
- (A2) g is monotone, i.e., $g(x, y) + g(y, x) \geq 0$ for all $x, y \in C$;
- (A3) for each $x, y \in C$, $\lim_{t\to 0} g(tz + (1 - t)x; y) \leq g(x; y)$;
- (A4) for each $x \in C$; $y \mapsto g(x, y)$ is convex and lower semicontinuous. It is known (see [31]), that if $g(x, y)$ satisfies $(A1)-(A4)$ then the function $F(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A1) - (A4)$ and $GMEP(g, B, \varphi,)$ is closed and convex. An interested reader may see [4,11,13–18,21,24–27,29,30] and the references there in for more information on equilibrium problem and its generalisations.

Let H_1 and H_2 be Hilbert spaces and C and Q nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $f_1 : C\times C \to \mathbb{R}$, $f_2 : Q\times Q \to \mathbb{R}$ be bifunctions, $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$, $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be functions and $B_1 : C \to H_1$, $B_2 : Q \to H_2$ be nonlinear mappings. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split generalised mixed equilibrium problem* is to find $x^* \in C$ such that

$$f_1(x^*, x) + \langle B_1x^*, x - x^*\rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{1.9}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q. \tag{1.10}$$

We shall denote the solution set of (1.9)–(1.10) by $\Omega = \{x^* \in GMEP(f_1, B_1, \varphi_1) : Ax^* \in GMEP(f_2, B_2, \varphi_2)\}$. If $B_1 = 0$ and $B_2 = 0$, then (1.9)–(1.10) reduces to the following split mixed equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{1.11}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q, \tag{1.12}$$

with solution set $\Omega_\varphi = \{x^* \in MEP(f_1, \varphi_1) : Ax^* \in MEP(f_2, \varphi_2)\}$. Again in (1.9)–(1.10) if $\varphi_1 = \varphi_2 = 0$, we obtain the following split generalised equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.13}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle \geq 0, \quad \forall y \in Q, \tag{1.14}$$

with solution set $\Omega_B = \{x^* \in GEP(f_1, B_1) : Ax^* \in GEP(f_2, B_2)\}$. Moreover, if $B_1 = B_2$ and $\varphi_1 = \varphi_2 = 0$, we have the split equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.15}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{1.16}$$

with solution set $\Omega_0 = \{x^* \in EP(f_1) : Ax^* \in EP(f_2)\}$.

Kazmi and Rizvi [12] studied the pair of equilibrium problems (1.15) and (1.16) called split equilibrium problem.

Recently, Bnouhachem [3] stated and proved the following strong convergence result.

Theorem 1.1. *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed and convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $A1 - A4$ and f_2 is upper semicontinuous in the first argument. Let $S, T : C \to C$ be a nonexpansive mapping such that $\Omega_0 \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a k-Lipschitzian mapping and η-strongly monotone and let $U : C \to C$ be τ-Lipschitzian mapping. For a given arbitrary $x_0 \in C$, let the iterative sequence $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{f_1}(x_n + \gamma A^*(T_{r_n}^{f_2} - I)Ax_n) \\ y_n = \beta_n S x_n + (1 - \beta_n) u_n \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu f)(T(y_n))] \ \forall n \geq 0 \end{cases} \tag{1.17}$$

where $\{r_n\} \subset (0, 2\zeta)$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Suppose the parameters satisfy $0 < \mu <$

$\left(\frac{2\eta}{k^2}\right)$, $0 \leq \rho\eta < \nu$, *where* $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\frac{\beta_n}{\alpha_n}) = 0$,

rm (c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$

(d) $\liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\zeta$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z \in \Omega_0 \cap F(T)$.

This result of Bnouhachem and other related results in literature depend on the prior knowledge of the operator norm.

Hendrickx and Oleshevsky [10] Observed that when $p = \infty$ or $p = 1$ the p-matrix norm is the largest of the row/column sums, and thus may be easily computed exactly. When $p = 2$, this problem reduces to computing an eigenvalue of $A^T A$ and thus can be solved in polynomial time in n, $log\frac{1}{\epsilon}$ and the bit-size of the entries of A.

Hendrickx and Oleshevsky [10] further stated and proved the following theorem.

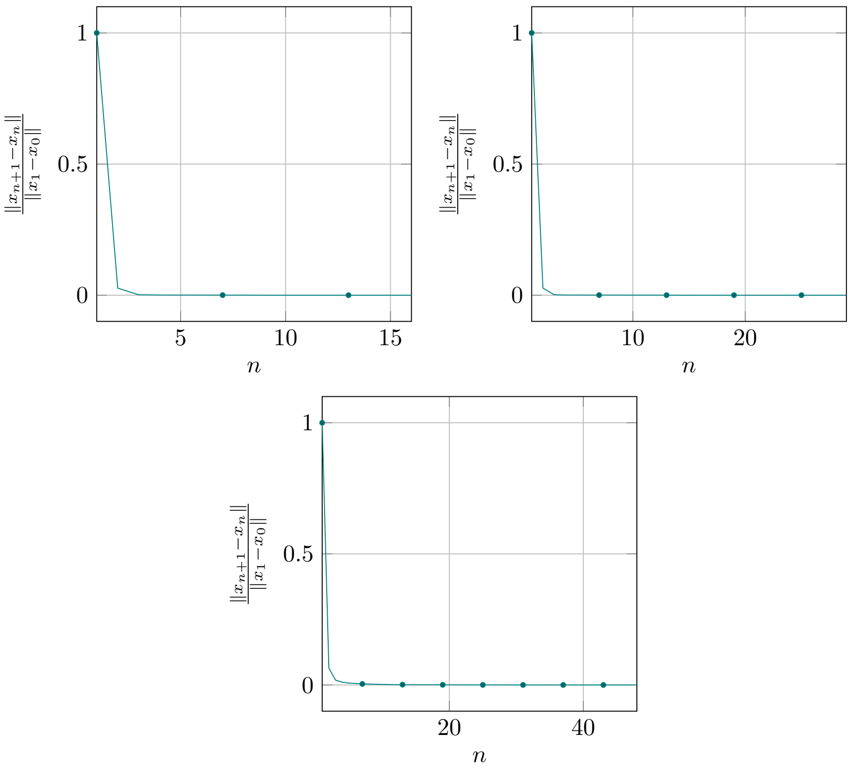

Figure 1. Errors: Case 1, $\epsilon = 10^{-4}$ (*top left*; 0:010 s); Case 2, $\epsilon = 10^{-4}$ (*top right*; 0:011 s); Case 3, $\epsilon = 10^{-4}$ (*bottom*; 0:013 s)

Theorem 1.2 (Hendrickx and Oleshevsky [10]). *Fix a rational $p \in [1, \infty)$ with $p \neq 1, 2$. Unless $P = NP$, there is no algorithm which, given input ϵ and a matrix M with entries in $\{-1, 0, 1\}$, computes $||M||_p$ to relative accuracy ϵ, in time which is polynomial in ϵ^{-1} and the dimensions of the matrix.*

The result Theorem 1.2 shows that sometimes it is very difficult if not impossible to calculate or even estimate the operator norm.

It is our intention here to introduce an iterative scheme which does not require any knowledge of the operator norm and obtain a strong convergence theorem for approximating solution of split generalised mixed equilibrium problem which also solves a fixed-point problem for κ-pseudocontractive mapping.

Precisely, we consider the following problem: find $x^* \in F(S)$ such that

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{1.18}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q, \tag{1.19}$$

where S is a strictly pseudocontractive mapping on C.

Many interesting practical problems (see [8]), can be formulated as fixed-point problems. The importance of equilibrium problem cannot be over emphasised as several mathematical problems (see [1]), such as optimisation problem, saddle points problem, Nash equilibria problem in noncooperative games, convex differentiable optimisation problem, variational operator inequalities problem, complementarity problems and variational inequalities with multivalued mappings can be formulated as equilibrium problems. It is easily observed that if we let $H_1 = H_2$, and S, A the identity operator, then this problem we are considering reduces to the generalised mixed equilibrium problem considered by Zhang [31], which in turn generalises equilibrium problems. Our problem also complements the work of He [9] and many other related results in the literature.

2. Preliminaries

We now state some important results that are vital to the proof of the main result.

Lemma 2.1 [6,7]. *Let H be a Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping, then for all $x, y \in H$,*

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2} \|(Tx - x) - (Ty - y)\|^2, \tag{2.1}$$

and consequently if $y \in F(T)$ then

$$\langle x - Tx, Ty - Tx \rangle \leq \frac{1}{2} \|Tx - x\|^2. \tag{2.2}$$

Lemma 2.2. *Let H be a real Hilbert space. Then the following result holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall\ x, y \in H. \tag{2.3}$$

Lemma 2.3. *Let H be a Hilbert space, then $\forall x, y \in H$ and $\alpha \in (0,1)$ we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.4}$$

Lemma 2.4 (Demiclosedness principle). *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be κ-strictly pseudocontractive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.5 [28]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n,\ n \geq 0, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([31]). *Let C be nonempty closed convex subset of a Hilbert space H. Let $B : C \to H$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and $f : C \times C \to \mathbb{R}$ be a bifunction that satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$; then there exists $u \in C$ such that*

$$f(u;y) + \langle Bu; y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0,\ \forall y \in C. \tag{2.6}$$

Define a mapping $T_r^f : C \to C$ as follows:

$$T_r^f(x) = \Big\{u \in C : f(u,y) + \langle Bu, y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \quad \forall y \in C\Big\}. \tag{2.7}$$

Then, the following conclusions hold:

1. *T_r^f is single-valued,*
2. *T_r^f is firmly nonexpansive, i.e., for any $x, y \in H$; $||T_r^f(x) - T_r^f(y)||^2 \leq \langle T_r^f(x) - T_r^f(y), x-y\rangle$,*
3. *$F(T_r^f) = GMEP(f;B;\varphi)$,*
4. *$GMEP(F;B;\varphi)$ is closed and convex.*

3. Main results

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C\times C \to \mathbb{R}$ and $f_2 : Q\times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\varphi_1 : C \to \mathbb{R}\cup +\infty$ and $\varphi_2 : Q \to \mathbb{R}\cup +\infty$ be proper lower semicontinuous and convex function. Let $S : C \to C$ be a κ-strictly pseudocontraction, such that $\Omega \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in$*

$\left(\epsilon, \frac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ *for* $T^{f_2}_{r_n}Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$, *otherwise* (γ *being any nonnegative real number). Then the sequence* $\{w_n\}, \{x_n\}$ *and* $\{y_n\}$ *generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u \\ y_n = T^{f_1}_{r_n}(w_n + \gamma_n A^*(T^{f_2}_{r_n} - I)Aw_n) \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \ \forall n \geq 0 \end{cases} \tag{3.1}$$

converges strongly to a point $p \in \Omega \cap F(S)$, *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Proof. Let $p \in \Omega \cap F(S)$, then from (3.1) we have,

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n S y_n - p||^2 \\ &= ||(1-\beta_n)(y_n - p) + \beta_n(Sy_n - p)||^2 \\ &= (1-\beta_n)^2||y_n - p||^2 + \beta_n^2||Sy_n - p||^2 \\ &\quad +2\beta_n(1-\beta_n)\langle y_n - p, Sy_n - p\rangle \\ &\leq (1-\beta_n)^2||y_n - p||^2 + \beta_n^2[||y_n - p||^2 + \kappa||y_n - Sy_n||^2] \\ &\quad +2\beta_n(1-\beta_n)[||y_n - p||^2 \\ &\quad -\frac{1-\kappa}{2}||y_n - Sy_n||^2] \\ &= (1 - 2\beta_n + \beta_n^2)||y_n - p||^2 + \beta_n^2[||y_n - p||^2 + \kappa||y_n - Sy_n||^2] \\ &\quad +2\beta_n||y_n - p||^2 - 2\beta_n^2||y_n - p||^2 \\ &\quad -\beta_n(1-\beta_n)(1-\kappa)||y_n - Sy_n||^2 \\ &= ||y_n - p||^2 + [\beta_n^2\kappa - \beta_n(1-\beta_n)(1-\kappa)]||y_n - Sy_n||^2 \\ &= ||y_n - p||^2 + \beta_n[\kappa + \beta_n - 1]||y_n - Sy_n||^2 \\ &\leq ||y_n - p||^2. \end{aligned} \tag{3.2}$$

Again from (3.1),

$$\begin{aligned} ||y_n - p||^2 &= ||T^{f_1}_{r_n}(w_n + \gamma_n A^*(T^{f_2}_{r_n} - I)Aw_n) - p||^2 \\ &\leq ||w_n + \gamma_n A^*(T^{f_2}_{r_n} - I)Aw_n - p||^2 \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(T^{f_2}_{r_n} - I)Aw_n||^2 \\ &\quad +2\gamma_n\langle w_n - p, A^*(T^{f_2}_{r_n} - I)Aw_n\rangle, \end{aligned} \tag{3.3}$$

but from Lemma 2.2

$$\begin{aligned} 2\gamma_n\langle w_n - p, A^*(T^{f_2}_{r_n} - I)Aw_n\rangle &= 2\gamma_n\langle A(w_n - p) + (T^{f_2}_{r_n} - I)Aw_n \\ &\quad -(T^{f_2}_{r_n} - I)Aw_n, (T^{f_2}_{r_n} - I)Aw_n\rangle \\ &= 2\gamma_n[\langle T^{f_2}_{r_n}Aw_n - Ap, (T^{f_2}_{r_n} - I)Aw_n\rangle \\ &\quad -||(T^{f_2}_{r_n} - I)Aw_n||^2 \end{aligned}$$

$$\leq 2\gamma_n\Big[\frac{1}{2}||(T_{r_n}^{f_2}-I)Aw_n||^2-||(T_{r_n}^{f_2}-I)Aw_n||^2\Big]$$
$$= -\gamma_n||(T_{r_n}^{f_2}-I)Aw_n||^2. \tag{3.4}$$

Therefore, from (3.3), (3.4) and the condition $\gamma_n \in (\epsilon, \frac{||(T_{r_n}^{f_2}-I)Aw_n||^2}{||A^*(T_{r_n}^{f_2}-I)Aw_n||^2} - \epsilon)$, we obtain

$$\begin{aligned}||y_n-p||^2 &\leq ||w_n-p||^2+\gamma_n^2||A^*(T_{r_n}^{f_2}-I)Aw_n||^2-\gamma_n||(T_{r_n}^{f_2}-I)Aw_n||^2\\ &= ||w_n-p||^2+\gamma_n[\gamma_n||A^*(T_{r_n}^{f_2}-I)Aw_n||^2-||(T_{r_n}^{f_2}-I)Aw_n||^2\\ &\leq ||w_n-p||^2. \end{aligned} \tag{3.5}$$

Thus from (3.2) and (3.5) ,

$$\begin{aligned}||x_{n+1}-p|| &\leq ||w_n-p||\\ &= ||(1-\alpha_n)x_n+\alpha_n u-p||\\ &= ||(1-\alpha_n)(x_n-p)+\alpha_n(u-p)||\\ &\leq (1-\alpha_n)||x_n-p||+\alpha_n||u-p||\\ &\leq \max\{||x_n-p||,||u-p||\}\\ &\vdots\\ &\leq \max\{||x_0-p||,||u-p||\}. \end{aligned} \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded and so also are $\{y_n\},\{w_n\}$ and $\{Sy_n\}$ bounded.

Since S is a κ-strictly pseudocontraction then,

$$\begin{aligned}&||Sx-p||^2 \leq ||x-p||^2+\kappa||x-Sx||^2\\ &\Rightarrow \langle Sx-p,Sx-p\rangle \leq \langle x-p,x-p\rangle+\kappa||x-Sx||^2\\ &\Rightarrow \langle Sx-p,Sx-x\rangle+\langle Sx-p,x-p\rangle \leq \langle x-p,x-p\rangle+\kappa||x-Sx||^2\\ &\Rightarrow \langle Sx-p,Sx-x\rangle \leq \langle x-Sx,x-p\rangle+\kappa||x-Sx||^2\\ &\Rightarrow \langle Sx-x,Sx-x\rangle+\langle x-p,Sx-x\rangle \leq \langle x-Sx,x-p\rangle+\kappa||x-Sx||^2\\ &\Rightarrow ||Sx-x||^2 \leq \langle x-p,x-Sx\rangle-\langle x-p,Sx-x\rangle++\kappa||x-Sx||^2\\ &\Rightarrow (1-\kappa)||Sx-x||^2 \leq 2\langle x-p,x-Sx\rangle. \end{aligned} \tag{3.7}$$

It follows from (3.1) and (3.7) that

$$\begin{aligned}||x_{n+1}-p||^2 &= ||(1-\beta_n)y_n+\beta_n Sy_n-p||^2\\ &= ||y_n-p+\beta_n(Sy_n-y_n)||^2\\ &= ||y_n-p||^2+\beta_n^2||Sy_n-p||^2-2\beta_n\langle y_n-p,Sy_n-y_n\rangle\\ &\leq ||y_n-p||^2+\beta_n(\beta_n-(1-\kappa))||Sy_n-p||^2\\ &\leq ||w_n-p||^2+\beta_n(\beta_n-(1-\kappa))||Sy_n-p||^2\\ &= ||(1-\alpha_n)x_n+\alpha_n u-p||^2+\beta_n(\beta_n-(1-\kappa))||Sy_n-p||^2\\ &= (1-\alpha_n)^2||x_n-p||^2+\alpha_n^2||u-p||^2\\ &\quad +2\alpha_n(1-\alpha_n)\langle x_n-p,u-p\rangle\\ &\quad +\beta_n(\beta_n-(1-\kappa))||Sy_n-p||^2. \end{aligned} \tag{3.8}$$

We now consider two cases to establish the strong convergence of $\{x_n\}$ to p.

Case 1. Assume that $\{||x_n - p||\}$ is monotonically decreasing sequence. Then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||. \tag{3.9}$$

Thus, from (3.8), we have

$$\begin{aligned}\beta_n((1-\kappa)-\beta_n)||Sy_n - y_n||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 - ||x_{n+1}-p||^2 \\ &\quad +\alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \\ &\quad \to 0, n\to\infty. \end{aligned} \tag{3.10}$$

Therefore,

$$||Sy_n - y_n|| \to 0, n \to \infty. \tag{3.11}$$

From (3.1),

$$||w_n - x_n|| = \alpha_n||u - x_n|| \to 0, n\to\infty. \tag{3.12}$$

Again from (3.1), we obtain

$$\begin{aligned}||x_{n+1}-p||^2 &= ||(1-\beta_n)y_n + \beta_n Sy_n - p||^2 \\ &= (1-\beta_n)^2||y_n - p||^2 + \beta_n^2||Sy_n - p||^2 + 2\beta_n(1-\beta_n)\langle y_n - p, Sy_n - p\rangle \\ &\leq ||y_n - p||^2 + \beta_n[-1+\kappa+\beta_n]||y_n - Sy_n||^2 \\ &\leq ||y_n - p||^2 \\ &= ||T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) - p||^2 \\ &\leq ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - \gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 \\ &\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \\ &\quad +\gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - \gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 \\ &\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \\ &\quad +\gamma_n[\gamma_n||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - ||(T_{r_n}^{f_2} - I)Aw_n||^2]. \end{aligned} \tag{3.13}$$

It then follows from (3.13) and the condition

$$\gamma_n \in \left(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right), \tag{3.14}$$

that

$$\begin{aligned}||x_{n+1}-p||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \\ &\quad -\epsilon||A^*(T_{r_n}^{f_2} - I)Aw_n||^2, \end{aligned} \tag{3.15}$$

which implies

$$\begin{aligned}\epsilon||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 - ||x_n - p||^2 \\ &\quad +\alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle. \end{aligned} \tag{3.16}$$

Hence,

$$\lim_{n\to\infty} ||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 = 0. \tag{3.17}$$

Furthermore, from (3.13) and (3.17)

$$\gamma_n||(T_{r_n}^{f_2}-I)Aw_n||^2 \leq (1-\alpha_n)^2||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||u-p||^2 \\ +\gamma_n^2||A^*(T_{r_n}^{f_2}-I)Aw_n||^2+2\alpha_n(1-\alpha_n)\langle x_n-p,u-p\rangle \\ \to 0, n\to\infty. \quad (3.18)$$

Therefore

$$\lim_{n\to\infty}||(T_{r_n}^{f_2}-I)Aw_n||=0. \quad (3.19)$$

On the other hand, if $T_{r_n}^{f_2}Aw_n=Aw_n$, then obviously,

$$\lim_{n\to\infty}||A^*(T_{r_n}^{f_2}-I)Aw_n||^2=0, \quad (3.20)$$

and

$$\lim_{n\to\infty}||(T_{r_n}^{f_2}-I)Aw_n||^2=0. \quad (3.21)$$

Also,

$$\begin{aligned}||y_n-p||^2 &= ||T_{r_n}^{f_1}(w_n+\gamma_nA^*(T_{r_n}^{f_2}-I)Aw_n)-p||^2 \\ &\leq \langle y_n-p, w_n+\gamma_nA^*(T_{r_n}^{f_2}-I)Aw_n-p\rangle \\ &= \frac{1}{2}[||y_n-p||^2+||w_n+\gamma_nA^*(T_{r_n}^{f_2}-I)Aw_n-p||^2-||y_n-p \\ &\quad -(w_n+\gamma_nA^*(T_{r_n}^{f_2}-I)Aw_n)-p)||^2] \\ &\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2+\gamma_n(\gamma_n||A^*(T_{r_n}^{f_2}-I)Aw_n||^2-|| \\ &\quad (T_{r_n}^{f_2}-I)Aw_n||^2)-||y_n-p \\ &\quad -(w_n+\gamma_nA^*(T_{r_n}^{f_2}-I)Aw_n-p)||^2] \\ &\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-(||y_n-w_n||^2+\gamma_n^2||A^*(T_{r_n}^{f_2}-I)Aw_n|| \\ &\quad -2\gamma_n\langle y_n-w_n, A^*(T_{r_n}^{f_2}-I)Aw_n\rangle)] \\ &\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-||y_n-w_n||^2+\gamma_n^2||A^*(T_{r_n}^{f_2}-I)Aw_n|| \\ &\quad +2\gamma_n||y_n-w_n||||A^*(T_{r_n}^{f_2}-I)Aw_n||]. \end{aligned} \quad (3.22)$$

That is

$$||y_n-p||^2 \leq ||w_n-p||^2-||y_n-w_n||^2+2\gamma_n||y_n-w_n||||A^*(T_{r_n}^{f_2}-I)Aw_n||. \quad (3.23)$$

It then follows from (3.13) and (3.23) that

$$||x_{n+1}-p||^2 \leq ||w_n-p||^2-||y_n-w_n||^2+2\gamma_n||y_n-w_n||||A^*(T_{r_n}^{f_2}-I)Aw_n||, \quad (3.24)$$

which implies that

$$\begin{aligned}||y_n-w_n||^2 &\leq ||w_n-p||^2-||x_{n+1}-p||^2+2\gamma_n||y_n-w_n||||A^* \\ &\quad (T_{r_n}^{f_2}-I)Aw_n|| \\ &= ||(1-\alpha_n)x_n+\alpha_nu-p||^2-||x_{n+1}-p||^2+2\gamma_n||y_n-w_n||||A^*\end{aligned}$$

$$
\begin{aligned}
&(T_{r_n}^{f_2} - I)Aw_n|| \\
&\le ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||u - p||^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\
&\quad +2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n|| \to 0, n \to \infty. \qquad (3.25)
\end{aligned}
$$

From (3.12) and (3.25), we obtain that

$$||x_n - y_n|| \le ||x_n - w_n|| + ||w_n - y_n|| \to 0, n \to \infty. \qquad (3.26)$$

Let $u_n = w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n$.
Then

$$||u_n - w_n|| = \gamma_n||A^*(T_{r_n}^{f_2} - I)Aw_n|| \to 0, n \to \infty. \qquad (3.27)$$

Combining (3.25) and (3.27), we get

$$||y_n - u_n|| \le ||y_n - w_n|| + ||w_n - u_n|| \to 0, n \to \infty. \qquad (3.28)$$

It follows from (3.11) and Lemma 2.4 that $\{y_n\}$ converges weakly to $p \in F(T)$ and consequently $\{x_n\}$ and $\{w_n\}$ converges weakly to p.
Next, we show that $p \in GMEP(f_1, B_1, \varphi_1)$. Since $y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n)$, we have

$$
\begin{aligned}
&f_1(y_n, y) + \langle B_1 y_n, y - y_n\rangle + \varphi_1(y) - \varphi_1(y_n) \\
&\quad + \frac{1}{r_n}\langle y - y_n, y_n - w_n\rangle - \frac{1}{r_n}\langle y - y_n, \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n\rangle \ge 0, \quad \forall y \in C.
\end{aligned}
\qquad (3.29)
$$

Thus, from the monotonicity of $F_1(x, y) := f_1(x, y) + \langle B_1 x, y - x\rangle + \varphi_1(y) - \varphi_1(x)$, we have

$$
\begin{aligned}
&\frac{1}{r_n}\langle y - y_n, y_n - w_n\rangle - \frac{1}{r_n}\langle y - y_n, \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n\rangle \\
&\quad \ge f_1(y, y_n) + \langle B_1 y, y_n - y\rangle + \varphi_1(y_n) - \varphi_1(y), \qquad (3.30)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{1}{r_{n_k}}\langle y - y_{n_k}, y_{n_k} - w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - y_{n_k}, \gamma_n A^*(T_{r_{n_k}}^{f_2} - I)Aw_{n_k}\rangle \ge f_1(y, y_{n_k}) \\
&\quad + \langle B_1 y, y_{n_k} - y\rangle + \varphi_1(y_{n_k}) - \varphi_1(y). \qquad (3.31)
\end{aligned}
$$

Since $y_n \rightharpoonup p$, then it follows from (3.12), (3.19), (3.24), (3.26) and $A4$ that,

$$f_1(y, p) + \langle B_1 y, p - y\rangle + \varphi_1(p) - \varphi_1(y) \le 0, \forall y \in C. \qquad (3.32)$$

Now, for fixed $y \in C$, let $y_t := ty + (1-t)p$ for all $t \in (0, 1)$. This implies that $y_t \in C$. Thus from $A1$ and $A4$

$$
\begin{aligned}
0 &= f_1(y_t, y_t) + \langle B_1 y_t, y_t - y_t\rangle + \varphi_1(y_t) - \varphi_1(y_t) \\
&\le t[f_1(y_t, y) + \langle B_1 y_t, y - y_t\rangle + \varphi_1(y) - \varphi_1(y_t)] \\
&\quad +(1-t)[f_1(y_t, p) + \langle B_1 y_t, p - y_t\rangle + \varphi_1(p) - \varphi_1(y_t)] \\
&\le t[f_1(y_t, y) + \langle B_1 y_t, y - y_t\rangle + \varphi_1(y) - \varphi_1(y_t)]. \qquad (3.33)
\end{aligned}
$$

Therefore

$$f_1(y_t, y) + \langle B_1 y_t, y - y_t\rangle + \varphi_1(y) - \varphi_1(y_t) \ge 0. \qquad (3.34)$$

Furthermore, from $A3$, we have

$$f_1(p,y) + \langle B_1 p, y - p\rangle + \varphi_1(y) - \varphi_1(p) \geq 0, \tag{3.35}$$

which implies that $p \in GMEP(f_1, B_1, \varphi_1)$. Now we show that $Ap \in GMEP(f_2, B_2, \varphi_2)$. Since $\{w_n\}$ is bounded and $w_n \rightharpoonup p$, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to p$ and since A is a bounded linear operator, $Aw_{n_k} \to Ap$.

Set $z_{n_k} = Aw_{n_k} - T^{f_2}_{r_{n_k}} Aw_{n_k}$. Then we have that $Aw_{n_k} - z_{n_k} = T^{f_2}_{r_{n_k}} Aw_{n_k}$, and from (3.19), we have

$$\lim_{n\to\infty} z_{n_k} = 0. \tag{3.36}$$

Therefore, from the definition of $T^{f_2}_{r_{n_k}}$, we observe that

$$\begin{aligned} & f_2(Aw_{n_k} - z_{n_k}, y) + \langle B_2 w_{n_k} - z_{n_k}, y - w_{n_k} + z_{n_k}\rangle + \varphi_2(y) - \varphi_2(w_{n_k} - z_{n_k}) \\ & \quad + \frac{1}{r_{n_k}} \langle y - (w_{n_k} - z_{n_k}), (w_{n_k} - z_{n_k}) - w_{n_k}\rangle \geq 0, \forall y \in C. \end{aligned} \tag{3.37}$$

Since f_2 is upper semicontinuous in first argument, then F_2 defined as

$$F_2(x,y) := f_2(x,y) + \langle B_2 x, y - x\rangle + \varphi_2(y) - \varphi_2(x) \tag{3.38}$$

is also upper semicontinuous in first argument. Thus, taking lim sup to the inequality (3.37) as $k \to \infty$ and using assumption A3, we have

$$f_2(Ap, y) + \langle B_2 Ap, y - Ap\rangle + \varphi_2(y) - \varphi_2(Ap) \geq 0 \ \forall y \in C, \tag{3.39}$$

which implies $Ap \in GMEP(f_2, B_2, \varphi_2)$. Hence $p \in \Omega \cap F(S)$.
We now show that $\{x_n\}$ converges strongly to p.

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n S y_n - p||^2 \\ &\leq ||y_n - p||^2 \\ &\leq ||w_n - p||^2 \\ &= ||(1-\alpha_n)x_n + \alpha_n u - p||^2 \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n(u-p)||^2 \\ &= (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\ &\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n[\alpha_n||u-p||^2 + 2(1-\alpha_n)\langle x_n - p, u - p\rangle]. \end{aligned} \tag{3.40}$$

Therefore, by Lemma 2.5, we obtain $x_n \to p$, $n \to \infty$.

Case 2. Assume that $\{||x_n - p||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}. \tag{3.41}$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$. It follows from (3.8) that

$$0 \leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2$$

$$\leq ||x_{\tau(n)+1} - p||^2 - (1-\alpha_{\tau(n)})^2||x_{\tau(n)} - p||^2$$
$$\leq \alpha_n^2||u-p||^2 + \alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle$$
$$+ \beta_n(\beta_n - (1-\kappa))||y_n - Sy_n||^2. \quad (3.42)$$

That is,

$$\beta_{\tau(n)}((1-\kappa) - \beta_{\tau(n)})||Sy_{\tau(n)} - y_{\tau(n)}||^2 \leq \alpha_{\tau(n)}^2||u-p||^2$$
$$+ 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, u-p\rangle$$
$$\to 0, \quad as \quad n \to \infty. \quad (3.43)$$

By the same argument as (3.11) to (3.28) in case 1, we conclude that $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $p \in F(S) \cap \Omega$. Now for all $n \geq n_0$,

$$0 \leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2$$
$$\leq (1-\alpha_{\tau(n)})||x_{\tau(n)} - p||^2 + \alpha_{\tau(n)}^2||u-p||^2 + 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)}\langle x_{\tau(n)}$$
$$-p, u-p\rangle - ||x_{\tau(n)} - p||^2$$
$$= \alpha_{\tau(n)}[\alpha_{\tau(n)}||u-p||^2 + 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, u-p\rangle - ||x_{\tau(n)} - p||^2]. \quad (3.44)$$

Therefore,

$$||x_{\tau(n)} - p||^2 \leq \alpha_{\tau(n)}||u-p||^2 + 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, u-p\rangle$$
$$\to 0, \; n \to \infty. \quad (3.45)$$

Thus,

$$\lim_{n\to\infty} ||x_{\tau(n)} - p||^2 = 0, \quad (3.46)$$

and

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}. \quad (3.47)$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \quad (3.48)$$

So $\lim_{n\to\infty} \Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $p \in F(S) \cap \Omega$, $\forall n \geq 0$. □

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be continuous and monotone mappings, $\varphi_1 : C \to \mathbb{R} \cup +\infty$ and $\varphi_2 : Q \to \mathbb{R} \cup +\infty$ be proper lower semicontinuous and convex function. Let $S : C \to C$ be a nonexpansive mapping, such that $\Omega \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \left(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right)$ for $T_{r_n}^{f_2}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$ otherwise (γ*

being any nonnegative real number). Then the sequence $\{w_n\}, \{x_n\}$ *and* $\{y_n\}$ *generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \ \forall n \geq 0 \end{cases} \tag{3.49}$$

converges strongly to a point $p \in \Omega \cap F(S)$ *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Corollary 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces, let* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A*. Let* $f_1 : C \times C \to \mathbb{R}$ *and* $f_2 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* $(A1)$–$(A4)$ *and* f_2 *is upper semicontinuous in first argument. Let* $B_1 : H_1 \to H_1$ *and* $B_2 : H_2 \to H_2$ *be continuous and monotone mappings. Let* $S : C \to C$ *be a* κ *strictly pseudocontraction, such that* $\Omega_B \cap F(S) \neq \emptyset$*. Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$*;* $\gamma_n \in \left(\epsilon, \dfrac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right)$ *for* $T_{r_n}^{f_2}Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$*, otherwise (*γ *being any nonnegative real number). Then the sequence* $\{w_n\}, \{x_n\}$ *and* $\{y_n\}$ *generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \ \forall n \geq 0 \end{cases} \tag{3.50}$$

converges strongly to a point $p \in \Omega_B \cap F(S)$ *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Corollary 3.4. *Let* H_1 *and* H_2 *be two real Hilbert spaces, let* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A*. Let* $f_1 : C \times C \to \mathbb{R}$ *and* $f_2 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* $(A1) - -(A4)$ *and* f_2 *is upper semicontinuous in first argument. Let* $\varphi_1 : C \to \mathbb{R} \cup +\infty$ *and* $\varphi_2 : Q \to \mathbb{R} \cup +\infty$ *be proper lower semicontinuous and convex function. Let* $S : C \to C$ *be a* κ *strictly pseudocontraction, such that* $\Omega_\varphi \cap F(S) \neq \emptyset$*. Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$*;* $\gamma_n \in \left(\epsilon, \dfrac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right)$ *for* $T_{r_n}^{f_2}Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$*, otherwise (*γ *being any nonnegative real number). Then the sequence* $\{w_n\}, \{x_n\}$ *and* $\{y_n\}$ *generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \quad \forall n \geq 0 \end{cases} \tag{3.51}$$

converges strongly to a point $p \in \Omega_\varphi \cap F(S)$ *where* $\{\alpha_n\}_{n=1}^\infty$ *and* $\{\beta_n\}_{n=1}^\infty$ *are real sequences in* $(0,1)$ *satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Corollary 3.5. *Let* H_1 *and* H_2 *be two real Hilbert spaces, let* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A*. Let* $f_1 : C \times C \to \mathbb{R}$ *and* $f_2 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* $(A1)-(A4)$ *and* f_2 *is upper semicontinuous in first argument. Let* $S : C \to C$ *be a* κ *strictly pseudocontraction, such that* $\Omega_0 \cap F(S) \neq \emptyset$*. Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$*;* $\gamma_n \in \left(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right)$ *for* $T_{r_n}^{f_2} Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$*, otherwise (*γ *being any nonnegative real number). Then the sequence* $\{w_n\}, \{x_n\}$ *and* $\{y_n\}$ *generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \ \forall n \geq 0 \end{cases} \tag{3.52}$$

converges strongly to a point $p \in \Omega_0 \cap F(S)$ *where* $\{\alpha_n\}_{n=1}^\infty$ *and* $\{\beta_n\}_{n=1}^\infty$ *are real sequences in* $(0,1)$ *satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

4. Numerical example and application

We present here in this section an example, a numerical result and an application to split convex minimisation problem .

4.1. Example

Let $H_1 = H_2 = L^2([0,1])$ with inner product given as $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$. Now take $f_1(x,y) := ||y||_{L^2} - ||x||_{L^2}$; $B_1 x := 2x$; $\varphi_1(x) = ||x||_{L^2}$ and $Sx = x$. Suppose $A : L^2([0,1]) \to L^2([0,1])$ is defined by

$$Ax(s) = \int_0^1 V(s,t)x(t)\mathrm{d}t, \forall x \in L^2([0,1]),$$

where $V : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Then A is a bounded linear operator and the adjoint A^* of A is defined by

$$A^*x(s) = \int_0^1 V(t,s)x(t)\mathrm{d}t, \quad \forall x \in L^2([0,1]).$$

Here we take $V(s,t) = e^{st}$. Finally take $f_2(x,y) := ||y||^2_{L^2} - ||x||^2_{L^2}$; $B_2x := 3x$; $\varphi_2(x) = ||x||^2_{L^2}$. We consider the problem; find $x^* \in H_1$ such that

$$Sx^* = x^*, \tag{4.1}$$

$$f_1(x^*,x) + \langle B_1x^*, x - x^*\rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \ \forall x \in H_1, \tag{4.2}$$

and $y^* = Ax^* \in H_2$ solves

$$f_2(y^*,y) + \langle B_2y^*, y - y^*\rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \ \forall y \in H_2. \tag{4.3}$$

The set of solutions of problem (4.1)–(4.3) is nonempty (since $x(t) = 0$, a.e. is in the set of solutions). Take $\alpha_n = \frac{1}{n+3}$, $\beta_n = \frac{1}{2}(1 - \frac{1}{n+2})$ and let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number) in iterative scheme (3.1) to obtain

$$\begin{cases} w_n = \left(1 - \frac{1}{n+3}\right)x_n + \frac{1}{n+3}u \\ y_n = T^{f_1}_{r_n}(w_n + \gamma_n A^*(T^{f_2}_{r_n} - I)Aw_n) \\ x_{n+1} = \left(1 - \frac{1}{2}\left(1 - \frac{1}{n+2}\right)\right)y_n + \frac{1}{2}\left(1 - \frac{1}{n+2}\right)y_n, \quad \forall n \geq 0. \end{cases} \tag{4.4}$$

4.2. Example with numerical computation

Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = \mathbb{R}$. Let $f_1(x,y) = -5x^2 + xy + 4y^2$, $\phi_1(x) = x^2$ and $B_1(x) = 4x$, then $T^{f_1}_r(x) = \dfrac{x}{15r+1}$. Also Let $f_2(x,y) = -3x^2 + xy + 2y^2$, $\phi_2(x) = 2x^2$ and $B_2(x) = 2x$, then $T^{f_2}_r(x) = \dfrac{x}{11r+1}$. Furthermore, let $Ax = 8x$, $A^*x = 8x$ and $S(x) = -2x$. We make difference choices of x_0, u and use $\frac{||x_{n+1} - x_n||}{||x_1 - x_0||} < 10^{-4}$ for stopping criterion. Take $\alpha_n = \frac{1}{n+2}$, $\beta_n = \frac{1}{6}(1 - \frac{1}{n+2})$, $r_n = \dfrac{n}{n+1}$ and let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and γ_n any positive real number otherwise, in iterative scheme (3.1) to obtain

$$\begin{cases} w_n = \left(1 - \frac{1}{n+2}\right)x_n + \frac{1}{n+2}u \\ y_n = \dfrac{w_n + \gamma_n 8\left(\frac{-11\left(\frac{n}{n+1}\right)w_n}{11\left(\frac{n}{n+1}\right)+1}\right)}{15\left(\frac{n}{n+1}\right) + 1} \\ x_{n+1} = \left(1 - \frac{1}{6}\left(1 - \frac{1}{n+2}\right)\right)y_n - \frac{1}{3}\left(1 - \frac{1}{n+2}\right)y_n, \quad \forall n \geq 0. \end{cases} \tag{4.5}$$

Case 1. $x_0 = 2$, $u = 1$ and $\gamma_n \in \left(\epsilon, \frac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0000021$ otherwise.

Case 2. $x_0 = 6$, $u = 3$ and $\gamma_n \in \left(\epsilon, \frac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0000222$ otherwise.

Case 3. $x_0 = 1$, $u = 8$ and $\gamma_n \in \left(\epsilon, \frac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0003$ otherwise.

The Mathlab version used is R2014a and the execution times are as follows:

(1) (case 1, $\varepsilon = 10^{-4}$) and execution time is 0.010 s.
(2) (case 2, $\varepsilon = 10^{-4}$) and execution time is 0.011 s.
(3) (case 3, $\varepsilon = 10^{-2}$) and execution time is 0.013 s (Fig. 1).

4.3. Applications to split convex minimisation problem

Here, we apply our result to study the following split convex minimisation problem: find

$$x^* \in F(S) \text{ such that } x^* = \arg\min_{x\in C}(h_1(x) + \phi_1(x) + \varphi_1(x)), \tag{4.6}$$

and such that

$$Ax^* = \arg\min_{y\in Q}(h_2(x) + \phi_2(x) + \varphi_2(x)), \tag{4.7}$$

where C and Q are nonempty closed and convex subset of H_1 and H_2. Also $h_1, \varphi_1 : C \to \mathbb{R}$ and $h_2, \varphi_2 : Q \to \mathbb{R}$ are four convex and lower semi-continuous functionals. Furthermore, $\phi_1 : C \to \mathbb{R}$ and $\phi_2 : Q \to \mathbb{R}$ are convex continuously differentiable functions and $A : H_1 \to H_2$ a bounded linear operator. Let $f_i(x, y) = h_i(y) - h_i(x)$ and $B_i = \nabla\phi_i$, $i = 1, 2$ and $\nabla\phi$ denote the gradient of ϕ.
Then the split convex minimisation problem (4.6)–(4.7) can be formulated as the following split generalised mixed equilibrium problem: find $x^* \in F(S)$ such that

$$h_1(x) - h_1(x^*) + \langle\nabla\phi_1 x^*, x - x^*\rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{4.8}$$

and $y^* = Ax^* \in Q$ solves

$$h_2(y) - h_2(y^*) + \langle\nabla\phi_2 y^*, y - y^*\rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q. \tag{4.9}$$

Thus Theorem 3.1 provides a strong convergence theorem for solving split convex minimisation problem (4.6)–(4.7).

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References

[1] Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)

[2] Browder, F.E., Petryshyn, W.V.: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**, 197–228 (1967)

[3] Bnouhachem, A.: Strong Convergence Algorithm for split Equilibrium problems and Hierarchical fixed points problems. Sci. World J. Article ID 390956, 12 (2014). doi:10.1155/2014/390956

[4] Cholamjiak, P., Suantai, S.: Convergence analysis for a system of equilibrium problems and a countable family of relatively quasi-nonexpansive mappings in Banach spaces. Abstr. Appl. Anal. Article ID 141376, 17 (2010). doi:10.1155/2010/141376

[5] Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)

[6] Crombez, G.: A hierarchical presentation of operators with fixed points on Hilbert spaces. Numer. Funct. Anal. Optim. **27**, 259–277 (2006)

[7] Crombez, G.: A geometrical look at iterative methods for operators with fixed points. Numer. Funct. Anal. Optim. **26**, 157–175 (2005)

[8] Ferreira, P.J.S.G.: Fixed point problems—an introduction. REVISTA DO DETUA, vol. 1, No 6, SETEMBRO (1996)

[9] He, Z.: The split equilibrium problem and its convergence algorithms. J. Inequal. Appl. **2012**, 162 (2012)

[10] Hendrickx, J.M., Olshevsky, A.: Matrix P-Norms are NP-Hard TO Approximate if $P \neq 1, 2, \infty$.. arXiv:0908.1397v3 [cs.CC] (2010)

[11] Kassay, G., Reich, S., Sabach, S.: Iterative methods for solving systems of variational inequalities in reflexive Banack spaces. SIAM J. Optim. **21**, 1319–1344 (2011)

[12] Kazmi, K.R., Rizvi, S.H.: Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. J. Egypt. Math. Soc. **21**, 44–51 (2013)

[13] Kumam, P.: Strong convergence theorems by an extragradient method for solving variational inequalities and equilibrium problems in a Hilbert space. Turk. J. Math. **33**(1), 85–98 (2009)

[14] Kumam, P.: A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive mapping. Nonlinear Anal. Hybrid Syst. **2**(4), 1245–1255 (2008)

[15] Kumam, P.: A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping. J. Appl. Math. Comput. **29**(1–2), 263–280 (2009)

[16] Kumam, P., Jaiboon, C.: A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings

with application to optimization problems. Nonlinear Anal. Hybrid Syst. **3**(4), 510–530 (2009)

[17] Kumam, P., Katchang, P.: A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mappings. Nonlinear Anal. Hybrid Syst. **3**(4), 475–486 (2009)

[18] Liu, Y.: A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. **71**, 4852–4861 (2009)

[19] López-Acedo, G., Xu, H.-K.: Iterative methods for strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. Theory Methods Appl. **67**(7), 2258–2271 (2007)

[20] Marino, G., Xu, H.-K.: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**(1), 336–346 (2007)

[21] Moudafi, A., Thera, M.: Proximal and dynamical approaches to equilibrium problems. In: Lecture note in Economics and Mathematical Systems, pp. 187–201, Springer, New York (1999)

[22] Qin, X., Kang, S.M., Cho, Y.J.: Convergence theorems on generalized equilibrium problems and fixed point problems with applications. Proc. Estonian Acad. Sci. **58**, 170–318 (2009)

[23] Qin, X., Cho, Y.J., Kang, S.M.: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20–30 (2009)

[24] Reich, S., Sabach, S.: Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces. Contemp. Math. **568**, 225–240 (2012)

[25] Shehu, Y., Ogbuisi, F.U.: Approximation of a common fixed point of left Bregman strongly nonexpansive mappings and solution of equilibrium problems. J. Appl. Anal. doi:10.1515/jaa-2015-0007 (2015)

[26] Wang, Z., Su, Y.: Strong convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Appl. Math. Inform. **28**(3–4), 783–796 (2010)

[27] Wangkeeree, R.: Strong convergence of the iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems of an infinite family of nonexpansive mappings. Nonlinear Anal. Hybrid Syst. **3**(4), 719–733 (2009)

[28] Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **2**, 240–256 (2002)

[29] Yao, Y., Cho, Y.J., Chen, R.: An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems. Nonlinear Anal. Theory Methods Appl. **71**(7–8), 3363–3373 (2009)

[30] Yao, Y., Noor, M.A., Zainab, S., Liou, Y.C.: Mixed equilibrium problems and optimization problems. J. Math. Anal. Appl. **354**(1), 319–329 (2009)

[31] Zhang, S.: Generalized mixed equilibrium problem in Banach spaces. Appl. Math. Mech. (English Edition) **30**, 1105–1112 (2009)

[32] Zhou, H.Y.: Convergence theorems of fixed points for strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. Theory Methods Appl. **69**(2), 456–462 (2008)

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