J. Fixed Point Theory Appl. 19 (2017) 375–385
DOI 10.1007/s11784-016-0358-0
Published online November 8, 2016

Journal of Fixed Point Theory and Applications

Existence and nonexistence of solutions to pure supercritical p-Laplacian problems

Mónica Clapp and Sweta Tiwari

To Paul Rabinowitz with great esteem and admiration.

Abstract. We establish multiplicity and nonexistence of solutions to the quasilinear problem

$$-\Delta_p v = |v|^{q-2} v \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega,$$

in some bounded smooth domains Ω in $\mathbb{R}^N$, for $1 < p < N$ and some supercritical exponents $q > p^* := \frac{Np}{N-p}$. Multiplicity is established in domains arising from the Hopf maps. We show that, after a suitable change of metric, these maps become p-harmonic morphisms, i.e., they preserve the p-Laplace operator up to a factor. We use them to reduce the supercritical problem to an anisotropic quasilinear critical problem in a domain of lower dimension.

Mathematics Subject Classification. 35J92, 35B33, 53C43.

Keywords. Quasilinear supercritical problem, higher critical exponents, p-harmonic morphism, hopf maps.

1. Introduction and statement of results

This paper is concerned with the existence and nonexistence of solutions to the quasilinear supercritical problem

$$-\Delta_p v = |v|^{q-2} v \;\text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega, \qquad (\wp_q)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_p v := \text{div}(|\nabla v|^{p-2} \nabla v)$ is the p-Laplace operator, $1 < p < N$, and $q > p^* := \frac{Np}{N-p}$.

In the semilinear case $p = 2$ there has recently been some progress, and existence, nonexistence and multiplicity results have been established for particular classes of domains.

M. Clapp is supported by CONACYT grant 237661 and UNAM-DGAPA-PAPIIT grant IN104315 (Mexico). S. Tiwari is supported by a postdoctoral fellowship from UNAM-DGAPA (Mexico).

A remarkable result due to Bahri and Coron [1] establishes the existence of a positive solution to the critical problem $(\wp_{2^*})$ in every bounded domain Ω having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. Rabinowitz asked whether this remains true for supercritical exponents $q > 2^*$ as well; see [4, Question 2]. In [10,11] Passaseo gave a negative answer: for each $1 \leq k \leq N-3$ he exhibited a domain which has the homotopy type of a k-dimensional sphere, in which problem $(\wp_q)$ does not have a nontrivial solution for any $q \geq 2^*_{N,k} := \frac{2(N-k)}{N-k-2}$. The exponent $2^*_{N,k}$ has been called the $(k+1)$-st critical exponent in dimension N.

More general domains in which problem $(\wp_q)$ does not admit a solution for $p = 2$ and every $q \geq 2^*_{N,k}$ were introduced in [6]. These are domains of the form

$$\Omega := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}, \tag{1.1}$$

where Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ having the geometric property stated in the following definition.

Definition 1.1. A bounded smooth domain Θ in $(0,\infty)\times\mathbb{R}^{N-k-1}$ is said to be doubly starshaped if there exist two numbers $0 < t_0 < t_1$ such that $t \in (t_0, t_1)$ for every $(t, z) \in \Theta$ and Θ is strictly starshaped with respect to $\xi_0 := (t_0, 0)$ and $\xi_1 := (t_1, 0)$, i.e.,

$$\langle \sigma - \xi_i, \nu(\sigma)\rangle > 0 \qquad \forall \sigma \in \partial\Theta\backslash\{\xi_i\}, \quad i = 0, 1,$$

where $\nu(\sigma)$ is the outward-pointing unit normal to $\partial\Theta$ at σ.

The nonexistence results in [6] can be extended to the quasilinear case. We shall prove, e.g., the following one.

Theorem 1.2. *Let $1 \leq k < N - p$. If Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ which is doubly starshaped and Ω is given by (1.1), then problem $(\wp_q)$ does not have a nontrivial solution in Ω for any $q \geq p^*_{N,k} := \frac{(N-k)p}{N-k-p}$.*

The special case where $p = 2$ and Θ is a ball centered on $(0,\infty) \times \{0\}$ is the example given by Passaseo in [10,11]. Note that $p^*_{N,k} > p^*$ for every $1 \leq k < N - p$.

Until quite recently, only few existence results were known for supercritical problems. A fruitful approach which has been applied in recent years in the semilinear case, consists in reducing the supercritical problem to a more general elliptic critical or subcritical problem, either by considering rotational symmetries, or by means of maps which pull back local harmonic functions, or by a combination of both techniques. We refer the reader to the survey paper [7] for a detailed discussion.

In our recent paper [8] we considered domains Ω of the form (1.1) and, under some symmetry assumptions on Θ, we obtained multiplicity results for the supercritical quasilinear problem $(\wp_{p^*_{N,k}})$ by reducing it to an anisotropic quasilinear critical problem in the domain Θ.

Here, we shall obtain some existence and multiplicity results in a different type of domains, arising from the Hopf maps. So let us recall what are these maps. For $N = 4, 8, 16$ we write $\mathbb{R}^N \equiv \mathbb{K} \times \mathbb{K}$, where $\mathbb{K}$ is either the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$. The Hopf map $\mathfrak{H} : \mathbb{R}^{2\dim\mathbb{K}} = \mathbb{K}\times\mathbb{K} \to \mathbb{K}\times\mathbb{R} = \mathbb{R}^{\dim\mathbb{K}+1}$ is defined by

$$\mathfrak{H}(z_1, z_2) := (2\overline{z_1}z_2,\ |z_1|^2 - |z_2|^2).$$

Its restriction to the unit sphere is the Hopf fibration $\mathfrak{f} : \mathbb{S}^{2\dim\mathbb{K}-1} \to \mathbb{S}^{\dim\mathbb{K}}$. So, in spherical coordinates,

$$\mathfrak{H}(s\vartheta) = s^2\mathfrak{f}(\vartheta) \ \text{ for } s \in [0,\infty), \vartheta \in \mathbb{S}^{2\dim\mathbb{K}-1}.$$

Hopf maps were used in [6] to obtain multiplicity results for problem $(\wp_q)$ with $p = 2$ in domains of the form $\Omega = \mathfrak{H}^{-1}(U)$ where U is a bounded smooth domain in $\mathbb{R}^{\dim\mathbb{K}+1}$. The main property of Hopf maps, which was used to obtain those results, is that they are harmonic morphisms, i.e., they preserve the Laplace operator, up to a factor. For $p \neq 2$ there is a similar notion: p-harmonic morphisms were introduced by Loubeau in [9] and further studied in [5]. They commute with the p-Laplace operator, up to a factor. The Hopf maps are not p-harmonic, but we shall prove that, after replacing the standard metric in the target space $\mathbb{R}^{\dim\mathbb{K}+1}$ by a suitable conformal metric, $\mathfrak{H}$ becomes a p-harmonic morphism. This will allow us to reduce problem $(\wp_q)$ in a domain of the form $\Omega = \mathfrak{H}^{-1}(U)$ to an anisotropic quasilinear problem in U. Then, we will apply the existence and multiplicity results obtained in [8] for the latter problem to derive existence and multiplicity results for the former one. The following theorem is one of them. Others will be given in Sect. 2.

The sphere of units $\mathbb{S}_\mathbb{K} := \{\zeta \in \mathbb{K} : |\zeta| = 1\}$ in $\mathbb{K}$ acts by multiplication on $\mathbb{K}\times\mathbb{K}$ and the Hopf map is $\mathbb{S}_\mathbb{K}$-invariant, i.e., $\mathfrak{H}(\zeta z_1, \zeta z_2) = \mathfrak{H}(z_1, z_2)$ for all $\zeta \in \mathbb{S}_\mathbb{K}$, $z_1, z_2 \in \mathbb{K}$. As usual, we denote by $O(n)$ the group of linear isometries of $\mathbb{R}^n$ and, for a subgroup Γ of $O(n)$, we denote by $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ the Γ-orbit of a point $x \in \mathbb{R}^n$ and by $\#\Gamma x$ its cardinality. Recall that a subset X of $\mathbb{R}^n$ is said to be Γ-invariant if $\gamma x \in X$ for every $\gamma \in \Gamma$ and $x \in X$, and a function $f\colon X \to \mathbb{R}$ is Γ-invariant if $f(\gamma x) = f(x)$ for every $\gamma \in \Gamma$ and $x \in X$.

Theorem 1.3. *Let $d = 2, 4, 8,\ \ p \in (1, d+1)$, and Ω be an $\mathbb{S}_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{R}^{2d}$. If $\mathfrak{H}(\Omega)$ is invariant under the action of a closed subgroup Γ of $O(d+1)$ and $\#\Gamma y = \infty$ for every $y \in \mathfrak{H}(\Omega)$, then problem $(\wp_{p^*_{2d,\,d-1}})$ has infinitely many $\mathbb{S}_\mathbb{K}$-invariant solutions in Ω.*

Note that $p^*_{2d,d-1} = \frac{(d+1)p}{d+1-p}$ is supercritical in dimension $2d$.

We wish to stress that it does not suffice that Ω is Γ-invariant and has infinite Γ-orbits, for some subgroup Γ of $O(N)$, to ensure that problem $(\wp_q)$ has a solution, as Theorem 1.2 shows. This stands in contrast to the critical case, where problem $(\wp_{p^*})$ has infinitely many solutions if the domain is Γ-invariant and all of its Γ-orbits are infinite; see [8, Theorem 2.1].

In the following section, we will prove this and other multiplicity results in domains arising from the Hopf maps, and we will give some examples. The proof of Theorem 1.2 is given in Sect. 3.

2. Existence

Let $(\mathfrak{M}, h)$ and $(\mathfrak{N}, g)$ be Riemannian manifolds. A map ϕ: $(\mathfrak{M}, h) \to (\mathfrak{N}, g)$ is *horizontally weakly conformal* if, at each point $x \in \mathfrak{M}$ for which $\mathrm{d}\phi_x \neq 0$, the restriction $\mathrm{d}\phi_x$: $\mathcal{H}_x \to T_{\phi(x)}\mathfrak{N}$ to the horizontal space $\mathcal{H}_x := (\ker \mathrm{d}\phi_x)^\perp$ is surjective and conformal, i.e., there exists $\lambda(x) \in (0, \infty)$ such that

$$g(\mathrm{d}\phi_x(X), \mathrm{d}\phi_x(Y)) = \lambda^2(x) h(X, Y) \text{ for every } X, Y \in \mathcal{H}_x.$$

Defining $\lambda(x) := 0$ if $\mathrm{d}\phi_x = 0$, one obtains a continuous function λ: $\mathfrak{M} \to [0, \infty)$ which is called the *dilation*.

Let $p \in (1, \infty)$. A map $\phi : (\mathfrak{M}, h) \to (\mathfrak{N}, g)$ is *p-harmonic* if $\phi \mid K$ is a critical point of the p-energy

$$E_p(\psi, K) := \frac{1}{p} \int_K |\mathrm{d}\psi_x|^p \,\mathrm{d}x$$

for every compact subset K of $\mathfrak{M}$.

A map ϕ: $(\mathfrak{M}, h) \to (\mathfrak{N}, g)$ is a *p -harmonic morphism* if it pulls back local p-harmonic functions on $\mathfrak{N}$ to local p-harmonic functions on $\mathfrak{M}$. If $p = 2$ it is simply called a harmonic morphism; see [3]. The following useful characterizations were obtained by Burel and Loubeau in [5,9].

Theorem 2.1. *Let $\phi : (\mathfrak{M}, h) \to (\mathfrak{N}, g)$ be a $\mathcal{C}^2$-map and $p \in (1, \infty)$. Then, the following statements are equivalent:*

(a) *ϕ is a p-harmonic morphism;*
(b) *ϕ is horizontally weakly conformal and p-harmonic;*
(c) *There exists a (unique and continuous) function λ on $\mathfrak{M}$ such that*

$$\Delta_p^h(u \circ \phi) = \lambda^p \left[(\Delta_p^g u) \circ \phi\right]$$

for every $\mathcal{C}^2$-function u defined on a nonempty open subset of $\mathfrak{N}$, where Δ_p^h and Δ_p^g denote the p-Laplace operators on $(\mathfrak{M}, h)$ and $(\mathfrak{N}, g)$, respectively.

For $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, we set $d := \dim \mathbb{K}$ and we write $\mathbb{R}^{2d} \equiv \mathbb{K} \times \mathbb{K}$ and $\mathbb{R}^{d+1} \equiv \mathbb{K} \times \mathbb{R}$. The Hopf map $\mathfrak{H} : \mathbb{R}^{2d} \to \mathbb{R}^{d+1}$ defined by

$$\mathfrak{H}(z_1, z_2) := (2\overline{z_1} z_2,\, |z_1|^2 - |z_2|^2), \qquad z_1, z_2 \in \mathbb{K},$$

is a harmonic morphism with dilation $\lambda(x) = 2\,|x|$, but it is not a p-harmonic morphism for $p \neq 2$; cf. [5, Theorem 4.9.]. Next, we will show that, if we replace the standard metric in $\mathbb{R}^{d+1}$ by a suitable conformal metric, $\mathfrak{H}$ becomes a p-harmonic morphism. We start with the following lemmas.

Lemma 2.2. *Let ϕ: $(\mathfrak{M}^m, h) \to (\mathfrak{N}^n, g)$ be a $\mathcal{C}^1$-map without critical points (i.e., $\mathrm{d}\phi_x \neq 0$ for all $x \in \mathfrak{M}$) which is horizontally weakly conformal with dilation λ, $m \neq 2$ and $p \in (1, \infty)$. Then ϕ is p-harmonic if and only if ϕ is harmonic with respect to the conformally related metric on $\mathfrak{M}$ given by*

$$\widetilde{h} := (n\lambda^2(x))^{\frac{p-2}{m-2}} h.$$

Proof. Lemma 1.2 in [2] asserts that, under the above assumptions, ϕ is p-harmonic if and only if ϕ is harmonic with respect to the conformally related metric on $\mathfrak{M}$ given by

$$\widetilde{h} := |\mathrm{d}\phi_x|^{\frac{2(p-2)}{m-2}}\, h,$$

where $|\mathrm{d}\phi_x|$ is the Hilbert–Schmidt norm of $\mathrm{d}\phi_x$, which is defined by

$$|\mathrm{d}\phi_x|^2 := \sum_{i=1}^{m} g(\mathrm{d}\phi_x(e_i), \mathrm{d}\phi_x(e_i))$$

for any orthonormal basis $\{e_i\}$ of $T_x\mathfrak{M}$. As ϕ is horizontally weakly conformal with dilation λ, we get that $|\mathrm{d}\phi_x|^2 = n\lambda^2(x)$. □

Lemma 2.3. *Let $C \in (0, \infty)$, $\beta \in \mathbb{R}$. Then, the Hopf map*

$$\mathfrak{H} : (\mathbb{R}^{2d}\backslash\{0\},\, h) \to (\mathbb{R}^{d+1}\backslash\{0\},\, g)$$

is a harmonic morphism with respect to the conformally related metrics

$$h := (C\,|x|^2)^{\beta}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad \text{and } g := |y|^{2\beta}\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j.$$

Proof. Let

$$\gamma(s) := \frac{1}{2}\ln\left[\left(Cs^2\right)^{\beta}\right] \quad \text{and} \quad \mu(t) := \ln\left(t^{\beta}\right).$$

Then, $h = e^{2\gamma(|x|)}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and $g = e^{2\mu(|y|)}\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j$. Now, for each $s \in (0, \infty)$, $\mathfrak{H}$ maps the $(2d-1)$-sphere of radius s onto the d-sphere of radius s^2 via the scaled Hopf fibration, and has dilation $2s$. Therefore, the diagram

$$\begin{array}{ccc} \mathbb{R}^{2d} & \xrightarrow{\mathfrak{H}} & \mathbb{R}^{d+1} \\ f\downarrow & & \downarrow\overline{f} \\ [0,\infty) & \xrightarrow{\alpha} & [0,\infty) \end{array}$$

commutes, where $f(x) := |x|$, $\overline{f}(y) := |y|$ and $\alpha(s) := s^2$. These data satisfy the assumptions of Corollary 13.4.2 in [3], which says that $\mathfrak{H}$ is a harmonic morphism with respect to the conformally equivalent metrics h and g if and only if

$$(2d-2)\alpha'(s)\gamma'(s) = (d-1)4s^2\mu'(\alpha(s)),$$

i.e., if and only if γ and μ satisfy

$$\gamma'(s) = s\mu'(s^2),$$

as they clearly do. □

Proposition 2.4. *Let $p \in (1, \infty)$, $p \neq d+1$. Then, the Hopf map*

$$\mathfrak{H} : \mathbb{R}^{2d}\backslash\{0\} \to (\mathbb{R}^{d+1}\backslash\{0\},\, g)$$

is a p-harmonic morphism with respect to the standard metric on $\mathbb{R}^{2d}$ and the conformally related metric

$$g := |y|^{\frac{p-2}{d+1-p}}\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j$$

on $\mathbb{R}^{d+1}$. Its square dilation is given by $\lambda^2(x) = 4\,|x|^{\frac{2(d-1)}{d+1-p}}$.

Proof. Set $n := d+1$, $\beta := \frac{p-2}{2(n-p)}$ and $C := (4n)^{\frac{n-p}{d-1}}$. Then $g = |y|^{2\beta}\,\delta_{ij}\mathrm{d}y^i$ $\mathrm{d}y^j$, and the Hopf map

$$\mathfrak{H} : \mathbb{R}^{2d}\backslash\{0\} \to (\mathbb{R}^{d+1}\backslash\{0\},\, g)$$

is horizontally weakly conformal with dilation $\lambda^2(x) = 4(|x|^2)^{2\beta+1} = 4(|x|^2)^{\frac{d-1}{n-p}}$. By Lemma 2.2 and Theorem 2.1, it is a p-harmonic morphism iff

$$\mathfrak{H} : (\mathbb{R}^{2d}\backslash\{0\},\, h) \to (\mathbb{R}^{d+1}\backslash\{0\},\, g)$$

is a harmonic morphism with respect to the conformally related metric $h := (n\lambda^2(x))^{\frac{p-2}{2d-2}}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ on $\mathbb{R}^{2d}\backslash\{0\}$. This is true, by Lemma 2.3, because

$$(n\lambda^2(x))^{\frac{p-2}{2d-2}} = [4n(|x|^2)^{\frac{d-1}{n-p}}]^{\frac{p-2}{2(d-1)}} = (C\,|x|^2)^{\beta}$$

and therefore $h = (C\,|x|^2)^{\beta}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. □

Corollary 2.5. *Let $p \in (2, d+1)$, $\mathfrak{H} : \mathbb{R}^{2d} \to \mathbb{R}^{d+1}$ be the Hopf map, and U be a bounded smooth domain in $\mathbb{R}^{d+1}\backslash\{0\}$. Then, $u \in \mathcal{C}^2(U)$ satisfies*

$$\begin{cases} -div(|y|^{\frac{(p-2)(d-2p+3)}{2(d+1-p)}}\,|\nabla u|^{p-2}\,\nabla u) = \frac{1}{2^p|y|}\,|u|^{q-2}\,u & in\ U,\\ u = 0 & on\ \partial U,\end{cases} \qquad (\wp_q^{\#})$$

if and only if $v := u \circ \mathfrak{H}$ solves problem $(\wp_q)$ in $\Omega := \mathfrak{H}^{-1}(U)$.

Proof. Set $n := d+1$ and $\beta := \frac{p-2}{2(n-p)}$. For the conformal metric $g := |y|^{2\beta}\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j$ on $\mathbb{R}^{d+1}\backslash\{0\}$ we have that $|g| := \det(g_{ij}) = |y|^{2\beta n}$ and

$$\begin{aligned}\Delta_p^g u &= \frac{1}{\sqrt{|g|}}\mathrm{div}\left(\sqrt{|g|}\,|y|^{-2\beta(p-1)}\,|\nabla u|^{p-2}\,\nabla u\right)\\ &= \frac{1}{|y|^{\beta n}}\mathrm{div}\left(|y|^{\beta(n-2p+2)}\,|\nabla u|^{p-2}\,\nabla u\right).\end{aligned}$$

Setting $v := u \circ \mathfrak{H}$ and $y := \mathfrak{H}(x)$, from Proposition 2.4 and Theorem 2.1 we obtain

$$\begin{aligned}\Delta_p(v)(x) &= \lambda^p(x)(\Delta_p^g u)(y)\\ &= 2^p\,|y|^{\frac{p(n-2)}{2(n-p)}-\frac{(p-2)n}{2(n-p)}}\,\mathrm{div}\left(|y|^{\frac{(p-2)(n-2p+2)}{2(n-p)}}\,|\nabla u(y)|^{p-2}\,\nabla u(y)\right)\\ &= 2^p\,|y|\,\mathrm{div}\left(|y|^{\frac{(p-2)(n-2p+2)}{2(n-p)}}\,|\nabla u|^{p-2}\,\nabla u\right)(y).\end{aligned}$$

This immediately yields the claim. □

The anisotropic critical problem $(\wp_q^{\#})$ with $q = \frac{(d+1)p}{d+1-p}$ was studied in [8] and multiplicity results were obtained under some symmetry assumptions. We combine them with Corollary 2.5 to obtain multiplicity results for problem $(\wp_q)$ in domains of the form $\Omega = \mathfrak{H}^{-1}(U)$.

Proof of Theorem 1.3. Theorem 2.1 in [8] says that, if U is a bounded smooth domain in $\mathbb{R}^{d+1}$ which is invariant under the action of a closed subgroup Γ of $O(d+1)$ and every Γ-orbit of U is infinite, then the anisotropic critical problem $(\wp_q^{\#})$ with $q = \frac{(d+1)p}{d+1-p}$ has infinitely Γ-invariant solutions in U. Theorem 1.3 now follows from Corollary 2.5.

Next, we fix $p \in (1, d+1)$, a closed subgroup $\widehat{\Gamma}$ of $O(d+1)$ and a $\widehat{\Gamma}$-invariant bounded smooth domain $\mathcal{D}$ in $\mathbb{R}^{d+1}$ such that $\#\widehat{\Gamma}y = \infty$ for every $y \in \mathcal{D}$. Then, the following statement holds true.

Theorem 2.6. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on $\widehat{\Gamma}$, $\mathcal{D}$ and p, with the following property: If Ω is an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{R}^{2d}$ such that $\mathcal{D} \subset \overline{\mathfrak{H}(\Omega)} \subset \mathbb{R}^{d+1}\backslash\{0\}$ and $\mathfrak{H}(\Omega)$ is Γ-invariant for some closed subgroup Γ of $\widehat{\Gamma}$ for which*

$$\min_{y\in\overline{\mathfrak{H}(\Omega)}} |y|^{\varrho} \, \#\Gamma y > 2^{p-d-1}\ell_m$$

with $\varrho = \frac{(d+1)(p-2)(d-2p+3)}{2p(d+1-p)} + \frac{(d+1)-p}{p}$, then the supercritical problem $(\wp_{p^*_{2d,\,d-1}})$ has at least m pairs of $\mathbb{S}_{\mathbb{K}}$-invariant solutions $\pm v_1, \ldots, \pm v_m$ in Ω of the form $v_j = u_j \circ \mathfrak{H}$; u_1 is positive, $u_2, \ldots, u_m$ change sign, and u_j is Γ-invariant and satisfies

$$\int_U \frac{1}{2^p\,|y|} \, |u_j|^{\frac{(d+1)p}{d+1-p}} \leq \ell_j S^{(d+1)/p} \text{ for every } j = 1, \ldots, m,$$

where S is the best Sobolev constant for the embedding $D^{1,p}(\mathbb{R}^{d+1}) \hookrightarrow L^{p^*}(\mathbb{R}^{d+1})$.

Proof. By Theorem 2.2 in [8], there exists a sequence (ℓ_m) as above with the property that, if U is a bounded smooth domain in $\mathbb{R}^{d+1}$ with $\mathcal{D} \subset \overline{U} \subset \mathbb{R}^{d+1}\backslash\{0\}$ which is invariant under the action of a closed subgroup Γ of Γ_∞ and

$$\min_{y\in\overline{U}} 2^{d+1-p} \, |y|^{\varrho} \, \#\Gamma y > \ell_m,$$

holds true, then the anisotropic critical problem $(\wp_q^{\#})$ with $q = \frac{(d+1)p}{d+1-p}$ has at least m pairs of Γ-invariant solutions $\pm u_1, \ldots, \pm u_m$ in U with the properties stated above. This, together with Corollary 2.5, yields the result. □

Theorems 1.3 and 2.6 extend some results proved in [6] for the semilinear case $p = 2$.

Theorem 2.6 gives many examples of domains in which the supercritical problem $(\wp_{p^*_{2d,\,d-1}})$ has a prescribed number m of solutions. Namely, write $\mathbb{R}^{d+1} \equiv \mathbb{C}^{d/2} \times \mathbb{R}$ and let $\Gamma_\infty := \{\mathrm{e}^{2\pi \mathrm{i}\theta} : \theta \in [0, 1)\}$ act on $\mathbb{R}^{d+1}$ by $\mathrm{e}^{\mathrm{i}\theta}(z, t) := (\mathrm{e}^{\mathrm{i}\theta}z, t)$ for $z \in \mathbb{C}^{d/2}$, $t \in \mathbb{R}$. Fix a bounded smooth domain $\mathcal{O}$ in $\mathbb{R}^2$ with $\overline{\mathcal{O}} \subset (0, \infty) \times \mathbb{R}$ and define

$$\mathcal{D} := \{(z, t) \in \mathbb{C}^{d/2} \times \mathbb{R} : (|z|, t) \in \mathcal{O}\}.$$

Then $\mathcal{D}$ is Γ_∞-invariant invariant and every Γ_∞-orbit in $\mathcal{D}$ is a circle. Now, fix a cylinder $\mathcal{Z} := \{(z, t) : R_1 < |z| < R_2\}$, $R_1, R_2 \in (0, \infty)$, and, for $r \in \mathbb{N}$, let $\Gamma_r := \{\mathrm{e}^{2\pi \mathrm{i}j/r} : j = 0, \ldots, r-1\}$. Then $\#\Gamma_r y = r$ for every $y \in \mathcal{Z}$. According to Theorem 2.6, for every bounded smooth domain U in $\mathbb{R}^{d+1}$ such that $\mathcal{D} \subset U \subset \mathcal{Z}$, which is Γ_r-invariant for some r satisfying

$$r > 2^{p-d-1} \max\{R_1^{-\varrho}, R_2^{-\varrho}\}\ell_m,$$

the supercritical problem $(\wp_{p^*_{2d,\,d-1}})$ has at least m solutions in $\Omega := \mathfrak{H}^{-1}(U)$.

Theorem 2.6 requires that all Γ-orbits in $\mathfrak{H}(\Omega)$ have large enough cardinality. Our next result allows $\mathfrak{H}(\Omega)$ to have small Γ-orbits, provided it is close enough but does not touch $\{0\}\times\mathbb{R}$. Although a more general result can be derived from Theorem 2.3 in [8], for the sake of clarity we state only the following special case.

Fix $p \in (1, d]$, $r \geq 2$ and a Γ_r-invariant bounded smooth domain $\widehat{U}$ in $\mathbb{R}^{d+1}\setminus\{0\}$, with Γ_r as before, such that $\widehat{U} \cap (\{0\}\times\mathbb{R}) \neq \emptyset$. For $\varepsilon > 0$ set

$$\widehat{U}_\varepsilon := \{y \in \widehat{U} : \operatorname{dist}(y, \{0\}\times\mathbb{R}) \geq \varepsilon\}.$$

Then, the following statement holds true.

Theorem 2.7. *Let m be the largest integer such that*

$$r \inf_{y\in\widehat{U}} |y|^\varrho > m \inf_{y\in\widehat{U}\cap(\{0\}\times\mathbb{R})} |y|^\varrho$$

with $\varrho = \frac{(d+1)(p-2)(d-2p+3)}{2p(d+1-p)} + \frac{(d+1)-p}{p}$. There exists $\varepsilon > 0$ with the following property: If Ω is an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{R}^{2d}$ such that $\mathfrak{H}(\Omega)$ is Γ_r-invariant,

$$\overline{\mathfrak{H}(\Omega)} \cap (\{0\}\times\mathbb{R}) = \emptyset \text{ and } \widehat{U}_\varepsilon \subset \mathfrak{H}(\Omega) \subset \widehat{U},$$

then the supercritical problem $(\wp_{p^*_{2d,\,d-1}})$ has at least m pairs of $\mathbb{S}_{\mathbb{K}}$-invariant solutions $\pm v_1, \ldots, \pm v_m$ in Ω of the form $v_j = u_j \circ \mathfrak{H}$; u_1 is positive, $u_2, \ldots, u_m$ change sign, and u_j is Γ_r-invariant.

Proof. This follows from Theorem 2.3 in [8] and Corollary 2.5. □

3. Nonexistence

In this section, we prove some nonexistence results for problem $(\wp_q)$ in domains of the following form: Fix $k_1, \ldots, k_m \in \mathbb{N}$ with $k := k_1 + \cdots + k_m$ and $1 \leq m \leq N - k - p$, and define Ω as

$$\Omega := \{(y^1, \ldots, y^m, z) \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times\mathbb{R}^{N-k-m} : \big(|y^1|, \ldots, |y^m|, z\big) \in \Theta\}, \tag{3.1}$$

where Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ whose closure is contained in $(0,\infty)^m \times \mathbb{R}^{N-k-m}$.

Fix $\tau_1, \ldots, \tau_m \in (0, \infty)$ and, for each $i = 1, \ldots, m$, let φ_i be the solution to the problem

$$\begin{cases} \varphi_i'(t)t + (k_i + 1)\varphi_i(t) = 1, & t \in (0, \infty), \\ \varphi_i(\tau_i) = 0. \end{cases}$$

Then φ_i is given by $\varphi_i(t) = \frac{1}{k_i+1}[1 - (\frac{\tau_i}{t})^{k_i+1}]$, so it is strictly increasing in $(0, \infty)$. Now, for $y^i \neq 0$ we consider the vector field

$$\chi(y^1, \ldots, y^m, z) := (\varphi_1(|y^1|)y^1, \ldots, \varphi_m(|y^m|)y^m, z). \tag{3.2}$$

Lemma 3.1. *χ has the following properties:*

(i) $div\chi = N - k$,

(ii) *For every* $y^i \in \mathbb{R}^{k_i+1}\backslash\{0\}$, $z \in \mathbb{R}^{N-k-m}$, $\xi \in \mathbb{R}^N$, $p \in (1, \infty)$,

$$\left\langle \mathrm{d}\chi(y^1, \ldots, y^m, z)[\xi], \, |\xi|^{p-2}\xi \right\rangle \leq \max\left\{1 - k_1\varphi_1(|y^1|), \ldots, 1 - k_m\varphi_1(|y^m|), 1\right\} \, |\xi|^p.$$

Proof. These statements follow immediately from Lemma 4.2 in [6]. □

Theorem 1.2 is proved by the same argument used in [6] for the semilinear case. We give the details for the reader's convenience.

Proof of Theorem 1.2. Let t_0 be as in Definition 1.1 and let χ be the vector field defined in (3.2) with $m = 1$ and $\tau_1 = t_0$, i.e., $k_1 = k$, $\varphi(t) = \frac{1}{k+1}[1 - (\frac{t_0}{t})^{k+1}]$ and

$$\chi(y, z) := (\varphi(|y|)y, z) \qquad \text{for } (y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1}, y \neq 0.$$

Since $1 - k\varphi(t) < 1$ for $t \in (t_0, \infty)$, Lemma 3.1 yields

$$\langle \mathrm{d}\chi(x)[\xi], |\xi|^{p-2}\xi\rangle \leq |\xi|^p \qquad \forall x \in \Omega, \xi \in \mathbb{R}^N.$$

Therefore, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ is a solution to problem $(\wp_q)$, the variational identity (4) in Pucci and Serrin's paper [12] applied to the function $\mathcal{F}(u, X) := \frac{1}{p}|X|^p - \frac{1}{q}|u|^q$, $(u, X) \in \mathbb{R} \times \mathbb{R}^N$, together with Lemma 3.1, implies that

$$\begin{aligned} \frac{p-1}{p}\int_{\partial\Omega} |\nabla u|^p \, (\chi \cdot \nu_\Omega) \, \mathrm{d}\sigma &= \int_\Omega (\mathrm{div}\chi) \left(\frac{1}{q}|u|^q - \frac{1}{p}|\nabla u|^p\right) \mathrm{d}x + \int_\Omega \langle \mathrm{d}\chi[\nabla u], \, |\nabla u|^{p-2}\nabla u\rangle \, \mathrm{d}x \\ &\leq \int_\Omega (N - k) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{N-k}\right) |\nabla u|^p \, \mathrm{d}x, \end{aligned} \tag{3.3}$$

where ν_Ω is the outward-pointing unit normal to $\partial\Omega$.

Next, we claim that

$$\langle(\varphi(t)t, z), \, \nu_\Theta(t, z)\rangle > 0 \qquad \forall (t, z) \in \partial\Theta\backslash\{\xi_0, \xi_1\} \tag{3.4}$$

where $\nu_\Theta(t, z)$ is the outward-pointing unit normal to $\partial\Theta$ at (t, z), which we write as $\nu_\Theta(t, z) = (\nu_1(t, z), \nu_2(t, z)) \in \mathbb{R} \times \mathbb{R}^{N-k-1}$, and $\xi_0 := (t_0, 0)$ and $\xi_1 := (t_1, 0)$ are as in Definition 1.1.

To prove this claim, let $(t, z) \in \partial\Theta\backslash\{\xi_0, \xi_1\}$. Since Θ is doubly star-shaped we have that

$$(t - t_i)\nu_1(t, z) + \langle z, \nu_2(t, z)\rangle > 0 \text{ for } i = 0, 1.$$

Therefore, setting $\psi(t) := \varphi(t)t - t$, we obtain

$$\begin{aligned} \langle(\varphi(t)t, \, z), \nu_\Theta(t, z)\rangle &= \varphi(t)t\nu_1(t, z) + \langle z, \nu_2(t, z)\rangle \\ &> (\varphi(t)t - t + t_i)\nu_1(t, z) = (\psi(t) + t_i)\nu_1(t, z) \text{ for } i = 0, 1. \end{aligned}$$

Note that $\psi'(t) = -k\varphi(t) < 0$ if $t > t_0$. So, as $t \in (t_0, t_1)$ for every $(t, z) \in \Theta$, we have that

$$\varphi(t_1)t_1 - t_1 = \psi(t_1) \leq \psi(t) \leq \psi(t_0) = -t_0 \qquad \forall (t, z) \in \partial\Theta.$$

If $\nu_1(t,z) \leq 0$, then

$$\langle(\varphi(t)t, z), \nu_\Theta(t,z)\rangle > (\psi(t) + t_0)\nu_1(t,z) \geq 0$$

and if $\nu_1(t,z) \geq 0$, then

$$\langle(\varphi(t)t, z), \nu_\Theta(t,z)\rangle > (\psi(t) + t_1)\nu_1(t,z) \geq \varphi(t_1)t_1\nu_1(t,z) \geq 0.$$

This proves inequality (3.4).

Since $\Omega = \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}$, from inequality (3.4) we conclude that

$$\langle\chi(\sigma), \nu_\Omega(\sigma)\rangle > 0 \qquad \forall\sigma \in \partial\Omega \backslash \{(y,0) \in \partial\Omega : |y| \neq t_0, |y| \neq t_1\}. \tag{3.5}$$

If $u \neq 0$, this inequality combined with (3.3) gives

$$0 < \int_\Omega (N-k)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{N-k}\right)|\nabla u|^p \,\mathrm{d}x,$$

which implies that $q < p^*_{N,k}$. □

It is an open question, even in the semilinear case, whether Theorem 1.2 holds true for a domain of the form (3.1) with $m > 1$. A partial result, which was proved in [6] for $p = 2$, can be extended to the quasilinear case as follows.

Theorem 3.2. *Given $\varepsilon > 0$ and $\tau = (\tau_1, \ldots, \tau_m) \in (0,\infty)^m$ there exists $\varrho \in (0, \min_{i=1,\ldots,m} \tau_i)$ such that, if $\Theta := B^{N-k}_\varrho(\tau)$ is the ball in $(0,\infty)^m \times \mathbb{R}^{N-k-m}$ centered at $(\tau, 0)$ of radius ϱ and Ω is defined as in (3.1), then problem $(\wp_q)$ does not have a nontrivial solution in Ω for every $q \geq p^*_{N,k} + \varepsilon$.*

Proof. The proof is a straightforward adaptation of the proof of Theorem 1.3 in [6]. □

Note that, if Θ is a ball, Ω has the homotopy type of the product of spheres $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$. It turns out that the radius $\varrho \to 0$ as $\varepsilon \to 0$, so Theorem 3.2 does not provide an example of a domain of the form (3.1) with $m > 1$ in which problem $(\wp_q)$ does not have a nontrivial solution for $q = p^*_{N,k}$.

References

[1] Bahri, A., Coron, J.M.: On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Commun. Pure Appl. Math. **41**(3), 253–294 (1988)

[2] Baird, P., Gudmundsson, Sigmundur: p-harmonic maps and minimal submanifolds. Math. Ann. **294**(4), 611–624 (1992)

[3] Baird, P., Wood, J.C.: Harmonic morphisms between Riemannian manifolds. London Mathematical Society Monographs. New Series, 29. The Clarendon Press, Oxford University Press, Oxford (2003)

[4] Brezis, H.: Elliptic equations with limiting Sobolev exponents—the impact of topology. Frontiers of the mathematical sciences: 1985 (New York, 1985). Commun. Pure Appl. Math. **39**(S), S17–S39 (1986)

[5] Burel, J.M., Loubeau, E.: $p-$harmonic morphisms: the $1<2<p$ case and some non-trivial examples. Differential geometry and integrable systems (Tokyo, 2000), 21–37, Contemp. Math., 308, Am. Math. Soc., Providence, RI (2002)

[6] Clapp, M., Faya, J., Pistoia, A.: Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents. Calc. Var. Partial Differ. Equ. **48**(3–4), 611–623 (2013)

[7] Clapp, M., Pistoia, Angela: Symmetries, Hopf fibrations and supercritical elliptic problem. Contemp. Math. **656**, 1–12 (2016)

[8] Clapp, M., Tiwari, S.: Multiple solutions to a pure supercritical problem for the p-Laplacian. Calc. Var. Partial Differ. Equ. **55**(1), 23 (2016). (Art. 7)

[9] Loubeau, E.: On p-harmonic morphisms. Differ. Geom. Appl. **12**(3), 219–229 (2000)

[10] Passaseo, Donato: Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains. J. Funct. Anal. **114**, 97–105 (1993)

[11] Passaseo, D.: New nonexistence results for elliptic equations with supercritical nonlinearity. Differ. Integral Equ. **8**, 577–586 (1995)

[12] Pucci, P., Serrin, James: A general variational identity. Indiana Univ. Math. J. **35**(3), 681–703 (1986)

Mónica Clapp and Sweta Tiwari
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
04510 Mexico D.F.
Mexico
e-mail: monica.clapp@im.unam.mx;
swetatiwari@im.unam.mx