J. Fixed Point Theory Appl. 19 (2017) 281–287
DOI 10.1007/s11784-016-0353-5
Published online November 12, 2016

Journal of Fixed Point Theory and Applications

A minimizing property of hyperbolic Keplerian orbits

Kuo-Chang Chen

Abstract. In this short note, we characterize hyperbolic Keplerian orbits as minimizing paths of the Keplerian action functional in the space of curves from a ray emanating from the attractive focus to a point in space. Variants of this result have been previously proved by different methods. Our proof based on hyperbolic anomaly is simple and informative.

Mathematics Subject Classification. 70F05, 70F15, 49J45, 49S05.

Keywords. Kepler problem, hyperbolic orbit, variational method.

1. Variational approaches for the Kepler problem

The equation of motions for the Kepler problem:

$$\mu\ddot{x} = -\alpha\frac{x}{|x|^3} \tag{1.1}$$

is the Euler–Lagrange equation for the Lagrangian action functional:

$$I_{\mu,\alpha,T}(x) = \int_0^T \frac{\mu}{2}|\dot{x}|^2 + \frac{\alpha}{|x|}\mathrm{d}t. \tag{1.2}$$

Here, μ and α are positive constants involving masses. We call solutions of (1.1) *Keplerian orbits* and $I_{\mu,\alpha,T}$ the *Keplerian action functional*. It is natural to regard Keplerian orbits as extrema of this functional in suitable function spaces. To include generalized solutions and allow more functional analytic tools, a natural choice of the function space would be $H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C})$ or $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{C})$.

Speaking of variational approaches for the Kepler problem, one should begin with the pioneering work of Gordon [17], in which elliptic Keplerian orbits, together with periodic collision–ejection orbits, were characterized as minimizers of the Keplerian action functional among non-contractible H^1-loops in the punctured complex plane. This observation played an important role in later discoveries of periodic solutions for several equivariant n-body problems, especially those which based on global estimates for the action functional. See [5,6,8,12–14,23] for examples and further references. Another

Birkhäuser

setting by the author characterizes some of these orbits as minimizing paths between two rays emanating from the attractive focus. This viewpoint is useful in showing existence of retrograde orbits of the three-body problem without the equal mass constraint [8,10].

Parabolic orbits for the n-body problem, as well as for some other singular Hamiltonian systems, arise naturally in the study of generalized solutions near collisions. They are interesting from variational perspective, because they can be characterized as free-time action-minimizers, and can, therefore, be linked with the Mather and weak KAM theories. Detailed discussions can be found in [2,3,18,19].

Given two different points with the same positive distance from the attractive focus, the parabolic collision–ejection path which connects them has larger action value than some direct and indirect Keplerian arcs which connects them with the same transfer time. This result was attributed to C. Marchal in [4] and proved in [16,21]. Similar proofs have also appeared in [20,22]. Here, we will give another proof, a proof which we think might be the simplest, more direct and informative. The proof is more informative, because it not only shows the existence of Keplerian arc will less action value, but also specifies eccentricities, semi-latus rectum, and action values of such hyperbolic Keplerian arcs. This result allows us to characterize hyperbolic Keplerian orbits as action minimizers among paths from a point to a ray emanating from the attractive focus. Details will be shown in Sect. 3.

2. Some preliminaries

For convenience, here, we briefly review some facts about hyperbolic Keplerian orbits. We refer readers to two introductory books [1, Sect. 4.4], [15, Chap 3] for details.

In terms of polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, solutions of (1.1) with pericenters on $\mathbb{R}_+ = \{0\} \times [0, \infty)$ are of the form:

$$r = \frac{\mathfrak{p}}{1 + \mathfrak{e}\cos\theta}, \tag{2.1}$$

where $\mathfrak{p} > 0$ is the semi-latus rectum and $\mathfrak{e} \geq 0$ is the eccentricity. This is Kepler's first law. Kepler's second law can be written:

$$r^2\dot{\theta} \;=\; \sqrt{\frac{\mathfrak{p}\alpha}{\mu}}. \tag{2.2}$$

The energy H of orbit is given by:

$$H \;=\; \frac{\alpha(\mathfrak{e}^2 - 1)}{2\mathfrak{p}}. \tag{2.3}$$

Conics described by (2.1) form a two-parameters' family. For each ξ in the upper half space $\mathbb{R}\times(0,\infty)$, there is a one-parameter family of such conics in the upper half space which emanate from their pericenters on $\mathbb{R}_+\backslash\{0\}$ and end at ξ. The corresponding Keplerian orbit is uniquely determined if we fix the transfer time.

A natural parametrization for hyperbolic orbits ($1 < \mathfrak{e}$) is by the *hyperbolic anomaly* $\psi_{\mathrm{H}} \in \mathbb{R}$:

$$r \;=\; \frac{\mathfrak{p}}{\mathfrak{e}^2 - 1}(\mathfrak{e}\cosh\psi_{\mathrm{H}} - 1). \tag{2.4}$$

The true anomaly θ and the hyperbolic anomaly ψ_{H} are related by:

$$\tan\frac{\theta}{2} = \sqrt{\frac{\mathfrak{e}+1}{\mathfrak{e}-1}}\tanh\frac{\psi_{\mathrm{H}}}{2},$$
$$\frac{\mathrm{d}\psi_{\mathrm{H}}}{\mathrm{d}\theta} = \frac{\mathfrak{e}\cosh\psi_{\mathrm{H}} - 1}{\sqrt{\mathfrak{e}^2 - 1}} \;=\; \frac{r}{\mathfrak{p}}\sqrt{\mathfrak{e}^2 - 1}. \tag{2.5}$$

The *hyperbolic Kepler equation* relates the hyperbolic anomaly with the transfer time τ, since the last pericenter passage:

$$\sqrt{\frac{\alpha(\mathfrak{e}^2-1)^3}{\mu\mathfrak{p}^3}}\,\tau = \mathfrak{e}\sinh\psi_{\mathrm{H}} - \psi_{\mathrm{H}}. \tag{2.6}$$

In particular, in the upper half space ($\psi_{\mathrm{H}} > 0$), we have $\mathfrak{e}\sinh\psi_{\mathrm{H}} - \psi_{\mathrm{H}} > 0$.

Using (2.2) and (2.5), ψ_{H} as a function of τ is proportional to the integration of the potential term on $[0, \tau]$:

$$\int_0^\tau \frac{\alpha}{r}\mathrm{d}t \;=\; \alpha\int_0^{\psi_{\mathrm{H}}(\tau)} \frac{1}{r}\frac{\mathrm{d}t}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\psi_{\mathrm{H}}}\,\mathrm{d}\psi_{\mathrm{H}} \;=\; \sqrt{\frac{\mu\alpha\mathfrak{p}}{\mathfrak{e}^2-1}}\,\psi_{\mathrm{H}}(\tau). \tag{2.7}$$

3. The main theorem

A simple rectilinear solution of (1.1) is the *parabolic ejection orbit* $x_{\mathrm{pe}}^{(\phi)}$ given by:

$$x_{\mathrm{pe}}^{(\phi)}(t) = c_{\mathrm{pe}}\, t^{\frac{2}{3}} e^{i\phi}, \quad c_{\mathrm{pe}} = \left(\frac{9\alpha}{2\mu}\right)^{\frac{1}{3}}. \tag{3.1}$$

It is easy to check that this is the only solution for (1.1) of the form $c\,t^{\lambda}e^{i\phi}$, it is indeed parabolic, and its action value is:

$$I_{\mu,\alpha,\tau}(x_{\mathrm{pe}}^{(\phi)}) \;=\; 2(6\mu\alpha^2)^{\frac{1}{3}}\tau^{\frac{1}{3}}. \tag{3.2}$$

Our main theorem compares these parabolic ejection orbits with some hyperbolic orbits:

Main theorem. *Fix $\phi \in (0, \pi)$ and $\tau > 0$. Let $x_{\mathrm{pe}}^{(\phi)}$ be the parabolic ejection orbit in (3.1). Let $x^{(\phi)} = re^{i\theta}$ be the Keplerian orbit such that $x^{(\phi)}(0) \in \mathbb{R}_+$ is its pericenter, and $x^{(\phi)}(\tau) = x_{\mathrm{pe}}^{(\phi)}(\tau)$. Then, $x^{(\phi)}$ is hyperbolic and*

$$I_{\mu,\alpha,\tau}(x^{(\phi)}) < I_{\mu,\alpha,\tau}(x_{\mathrm{pe}}^{(\phi)}).$$

Proof. The polar equation for $x^{(\phi)}$ is of the form (2.1), since the pericenter is on $\mathbb{R}_+$. The area of the sector swept by the vector from 0 to $x^{(\phi)}$ for $t \in [0, \tau]$ is

$$\frac{\mathfrak{p}^2}{2}\int_0^\phi \frac{1}{(1+\mathfrak{e}\cos\theta)^2}\mathrm{d}\theta \;=\; \frac{1}{2}\int_0^\phi r^2\,\mathrm{d}\theta \;=\; \frac{1}{2}\int_0^\tau r^2\dot{\theta}\,\mathrm{d}t \;=\; \frac{1}{2}\sqrt{\frac{\mathfrak{p}\alpha}{\mu}}\,\tau.$$

The last identity uses (2.2). Observe that

$$\frac{\mathfrak{p}}{1+\mathfrak{e}\cos\phi} \;=\; |x^{(\phi)}(\tau)| \;=\; |x^{(\phi)}_{\mathrm{pe}}(\tau)| \;=\; \left(\frac{9\alpha}{2\mu}\right)^{\frac{1}{3}}\tau^{\frac{2}{3}},$$

and so

$$\tau^2 = \left(\frac{2\mu}{9\alpha}\right)\left(\frac{\mathfrak{p}}{1+\mathfrak{e}\cos\phi}\right)^3. \tag{3.3}$$

Therefore,

$$\int_0^{\phi}\frac{1}{(1+\mathfrak{e}\cos\theta)^2}\mathrm{d}\theta = \frac{\sqrt{2}}{3}\frac{1}{(1+\mathfrak{e}\cos\phi)^{\frac{3}{2}}}. \tag{3.4}$$

By differentiating this identity with respect to ϕ, we find

$$\mathfrak{e}\sin\phi \;=\; \sqrt{2}\sqrt{1+\mathfrak{e}\cos\phi},$$

from which one can square both sides to deduce

$$(1+\mathfrak{e}\cos\phi)^2 = \mathfrak{e}^2-1. \tag{3.5}$$

This implies $\mathfrak{e}>1$; i.e. the orbit $x^{(\phi)}$ is hyperbolic.

In addition, from (3.3) and (3.5), we have:

$$\tau^2 = \left(\frac{2\mu}{9\alpha}\right)\left(\frac{\mathfrak{p}}{\sqrt{\mathfrak{e}^2-1}}\right)^3. \tag{3.6}$$

In terms of hyperbolic anomaly $\psi_{\mathrm{H}}=\psi_{\mathrm{H}}(\tau)$, by (2.4) and (3.5), we find:

$$\mathfrak{e}\cosh\psi_{\mathrm{H}}-1 \;=\; \frac{\mathfrak{e}^2-1}{1+\mathfrak{e}\cos\phi} \;=\; \sqrt{\mathfrak{e}^2-1}. \tag{3.7}$$

The action value for $x^{(\phi)}$ can be easily evaluated using the hyperbolic eccentric anomaly:

$$I_{\mu,\alpha,\tau}(x^{(\phi)}) = \frac{1}{2}\sqrt{\frac{\mathfrak{p}\mu\alpha}{\mathfrak{e}^2-1}}(\mathfrak{e}\sinh\psi_{\mathrm{H}}+3\psi_{\mathrm{H}}). \tag{3.8}$$

This follows easily by writing the Lagrangian as $2\alpha/r+H$ and using (2.3), (2.6), (2.7).

By (3.6), we can rewrite (3.2) in terms of $\mathfrak{e}$, $\mathfrak{p}$:

$$I_{\mu,\alpha,\tau}\left(x^{(\phi)}_{\mathrm{pe}}\right) = 2\sqrt{2}\sqrt{\frac{\mathfrak{p}\mu\alpha}{\mathfrak{e}^2-1}}(\mathfrak{e}^2-1)^{\frac{1}{4}}.$$

Comparing with the above formula for $I_{\mu,\alpha,\tau}(x^{(\phi)})$, all we have to show is

$$\mathfrak{e}\sinh\psi_{\mathrm{H}}+3\psi_{\mathrm{H}} < 4\sqrt{2}(\mathfrak{e}^2-1)^{\frac{1}{4}}.$$

By (3.7), we have

$$(\mathfrak{e}\cosh\psi_{\mathrm{H}}-1)^2 = \mathfrak{e}^2-1.$$

Expanding the left-hand side yields

$$\mathfrak{e}\sinh\psi_{\mathrm{H}} \;=\; \sqrt{2}\sqrt{\mathfrak{e}\cosh\psi_{\mathrm{H}}-1} \;=\; \sqrt{2}\,(\mathfrak{e}^2-1)^{\frac{1}{4}}.$$

From the hyperbolic Kepler equation, we know $\mathfrak{e}\sinh\psi_{\mathrm{H}}-\psi_{\mathrm{H}}>0$. Therefore,

$$\mathfrak{e}\sinh\psi_{\mathrm{H}}+3\psi_{\mathrm{H}} \;<\; 4\mathfrak{e}\sinh\psi_{\mathrm{H}} \;=\; 4\sqrt{2}\,(\mathfrak{e}^2-1)^{\frac{1}{4}}$$

as desired. □

Identity (3.4) was also obtained in [16]. The main difference is that our argument bases on properties of the hyperbolic anomaly. As mentioned in the first section, this result in slightly variant forms can be found in [16,20–22]. Apart from these references, an application to the n-center problem can be found in [11]. The proof above is informative, because it identifies eccentricity, semi-latus rectum, hyperbolic anomaly, and action value of the Keplerian arc with smaller action value through equations (3.5)–(3.8).

The following corollary characterizes hyperbolic Keplerian orbits as minimizing paths of the Keplerian action functional in the space of curves from a ray emanating from the attractive focus to a point in space. An application to the four-body problem can be found in [9].

Corollary. *Given $\tau > 0$ and $\xi \in \mathbb{R} \times (0, \infty)$. Minimizers of the Keplerian action functional on the space*

$$\Gamma_\tau(\mathbb{R}_+, \xi) \;:=\; \{x \in H^1([0,\tau], \mathbb{C}) :\; x(0) \in \mathbb{R}_+,\; x(\tau) = \xi\}$$

are collision-free.

Proof. Existence of action minimizer follows from weak compactness of the space $\Gamma_\tau(\mathbb{R}_+, \xi)$, weak lower semi-continuity, and coercivity of the Keplerian action functional (see [7,17] for instances).

If an action minimizer has collision, then following a standard blow-up argument (see [14,21] for instances), a limiting minimizing path along the ray $\mathbb{R}_+\xi$ is a parabolic ejection orbit, but the main theorem tells us that such an orbit do not have least action value in spaces of the form $\Gamma_{\tau'}(\mathbb{R}_+, \xi)$. □

Acknowledgements

I feel privileged and honored to be invited to contribute in this special volume in honor of Professor Paul Rabinowitz. 16 years ago, on 2000/05/28 to be exact, I delivered a contributed talk on my analytic proof for the existence of figure-8 orbit [5] at University of Wisconsin, during the "Madison Conference on Nonlinear Analysis" in honor of Paul. That was the debut of my academic career. Ever since, I have kept working on variational methods for the n-body problem and have received continuous support and encouragements from Paul. I became regular participant for related conferences and workshops, such as the conference series in Nankai, and was invited by Paul to deliver lectures multiple times at Postech. I sincerely thank him for the his generous help and recognition, which were particularly important at the beginning stage of my career. I also thank Yiming Long for his invitation, and Guowei Yu for showing me related results. This work is supported in parts by the Ministry of Science and Technology (Grant NSC 102-2628-M-007-004-MY4) in Taiwan.

References

[1] Battin, R.H.: An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition. AIAA Education Series, USA (1999)

[2] Barutello, V., Terracini, S., Verzini, G.: Entire minimal parabolic trajectories: the planar anisotropic Kepler problem. Arch. Ration. Mech. Anal. **207**, 583–609 (2013)

[3] Barutello, V., Terracini, S., Verzini, G.: Entire parabolic trajectories as minimal phase transitions. Cal. Var. Partial Differ. Equations **49**, 329–491 (2014)

[4] Chenciner, A.: Symmetries and "simple" solutions of the classical n-body problem. XIVth International Congress on Mathematical Physics, pp. 4–20 (2005)

[5] Chen, K.-C.: On Chenciner-Montgomery's orbit in the three-body problem. Discrete Contin. Dynam. Syst. **7**, 85–90 (2001)

[6] Chen, K.-C.: Binary decompositions for the planar N-body problem and symmetric periodic solutions. Arch. Ration. Mech. Anal. **170**, 247–276 (2003)

[7] Chen, K.-C.: Removing collision singularities from action minimizers for the N-body problem with free boundaries. Arch. Ration. Mech. Anal. **181**, 311–331 (2006)

[8] Chen, K.-C.: Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses. Ann. Math. **167**, 325–348 (2008)

[9] Chen, K.-C.: Keplerian action functional, convex optimization, and an application to the four-body problem (2013) (preprint)

[10] Chen, K.-C., Lin, Y.-C.: On action-minimizing retrograde and prograde orbits of the three-body problem. Commun. Math. Phys. **1291**, 403–441 (2009)

[11] Chen, K.-C., Yu, G.: Syzygy sequences of the N-center problem. Ergodic Theory Dynam. Systems (2016) (to appear)

[12] Chenciner, A., Montgomery, R.: A remarkable periodic solution of the three body problem in the case of equal masses. Ann. Math. **152**, 881–901 (2000)

[13] Chenciner, A., Venturelli, A.: Minima de l'intégrale d'action du Problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites 'hip-hop'. Celest. Mech. Dynam. Astron. **77**(2), 139–151 (2000)

[14] Ferrario, D., Terracini, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem. Invent. Math. **155**, 305–362 (2004)

[15] Fitzpatrick, R.: An Introduction to Celestial Mechanics. Cambridge University Press, Cambridge (2012)

[16] Fusco, G., Gronchi, G.F., Negrini, P.: Platonic polyhedra, topological constraints and periodic solutions of the classical N-body problem. Invent. Math. **185**, 283–332 (2011)

[17] Gordon, W.: A minimizing property of Keplerian orbits. Am. J. Math. **99**, 961–971 (1977)

[18] Maderna, E.: On weak KAM theory of N-body problems. Ergodic Theory Dynam. Syst. **32**, 1019–1041 (2012)

[19] Maderna, E.: Venturelli, A, Globally minimizing parabolic motions in the Newtonian N-body problem. Arch. Ration. Mech. Anal. **194**, 283–313 (2009)

[20] Soave, N., Terracini, S.: Symbolic dynamics for the N-centre problem at negative energies. Discrete Contin. Dyn. Syst. Ser. A **32**, 3245–3301 (2012)

[21] Terracini, S., Venturelli, A.: Symmetric trajectories for the $2N$-body problem with equal masses. Arch. Ration. Mech. Anal. **184**, 465–493 (2007)

[22] Yu, G.: Periodic solutions of the planar N-center problem with topological constraints. Discrete Contin. Dynam. Syst. A **36**, 5131–5162 (2016)

[23] Zhang, S., Zhou, Q.: Variational methods for the choreography solution to the three-body problem. Sci. China Ser. A **45**, 594–597 (2002)

Kuo-Chang Chen
Department of Mathematics
National Tsing Hua University
Hsinchu 30013
Taiwan
e-mail: kchen@math.nthu.edu.tw