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Nonoscillation and oscillation of second-order linear dynamic equations via the sequence of functions technique

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Abstract. We study nonoscillation/oscillation of the dynamic equation

$$\left(rx^{\Delta}\right)^{\Delta}(t)+p(t)\,x(t)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

where $t_0\in\mathbb{T}$, $\sup\mathbb{T}=\infty$, $r\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^{+})$, $p\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^{+})$. By using the Riccati substitution technique, we construct a sequence of functions which yields a necessary and sufficient condition for the nonoscillation of the equation. In addition, our results are new in the theory of dynamic equations and not given in the discrete case either. We also illustrate applicability and sharpness of the main result with a general Euler equation on arbitrary time scales. We conclude the paper by extending our results to the equation

$$\left(rx^{\Delta}\right)^{\Delta}(t)+p(t)\,x^{\sigma}(t)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

which is extensively discussed on time scales.

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1. Introduction

In this paper, we study nonoscillation and oscillation properties of solutions of the second-order linear dynamic equations

$$\left(rx^{\Delta}\right)^{\Delta}(t)+p(t)\,x(t)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}\tag{1.1}$$

and

$$\left(rx^{\Delta}\right)^{\Delta}(t)+p(t)\,x^{\sigma}(t)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\tag{1.2}$$

where $t_0\in\mathbb{T}$, $\sup\mathbb{T}=\infty$, $r\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^{+})$ and $p\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^{+})$.

Throughout the paper we assume that the following assumptions are satisfied:

(A1) $r \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$.

(a) $$\int_{t_0}^{\infty}\frac{1}{r(\eta)}\Delta\eta=\infty.$$

(A2) $p \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$.

(a) $$\int_{t_0}^{\infty}p(\eta)\Delta\eta=\infty.$$

(b) $$\int_{t_0}^{\infty}p(\eta)\Delta\eta<\infty.$$

Without further mentioning, we will assume that (A1) and (A2) hold.

By a solution of the linear second-order dynamic equation (1.1), we mean a function $x \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $rx^{\Delta} \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, which satisfies (1.1) identically on $[t_0,\infty)_{\mathbb{T}}$. We restrict our interest to those solutions of (1.1), which do not vanish in any neighborhood of infinity.

Now, we recall some brief history in order to formulate the motivation behind this work. Consider the second-order linear differential equation

$$x''(t)+p(t)x(t)=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{R}}, \tag{1.3}$$

where $p \in \mathrm{C}([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^+)$.

In 1918, Fite [7] proved the following well-known oscillation result.

Theorem A (See [7]). *Assume that $\int_{t_0}^{\infty}p(\eta)\,\mathrm{d}\eta=\infty$. Then (1.3) is oscillatory.*

This result is also proved by Wintner in 1949 [16] without assuming that p is positive.

It should be noted that Theorem A fails to apply Euler-type differential equations, i.e., equation (1.3) with $p(t)=\frac{\lambda}{t^2}$ for $t\in[t_0,\infty)_{\mathbb{R}}$, where $\lambda\in\mathbb{R}_0^+$.

The discovery of another famous test, which can be applied to Euler-type differential equations, is due to Kneser [12] in 1893. The result reads as follows.

Theorem B (See [12]).

(i) *If $p(t)\leq\frac{1}{4t^2}$ for all sufficiently large t, then (1.3) is nonoscillatory.*

(ii) *If there exists a constant $\lambda\in(\frac{1}{4},\infty)_{\mathbb{R}}$ such that $p(t)\geq\frac{\lambda}{t^2}$ for all sufficiently large t, then (1.3) is oscillatory.*

In 1948, Hille [10] proved the following result on oscillation and nonoscillation of (1.3) when the coefficient p is "integrally-small" (i.e., $\int_{t_0}^{\infty}p(\eta)\,\mathrm{d}\eta<\infty$), which includes Euler-type differential equations.

Theorem C (See [10]). *Assume that p is "integrally-small".*

(i) *If $\int_{t}^{\infty}p(\eta)\,\mathrm{d}\eta\leq\frac{1}{4t}$ for all sufficiently large t, then (1.3) is nonoscillatory.*

(ii) *If there exists a constant $\lambda\in(\frac{1}{4},\infty)_{\mathbb{R}}$ such that $\int_{t}^{\infty}p(\eta)\,\mathrm{d}\eta\geq\frac{\lambda}{t}$ for all sufficiently large t, then (1.3) is oscillatory.*

It is obvious that Theorem C improves Theorem B.

Wintner [17], in 1951, proved the following result improving the one due to Hille.

Theorem D (See [17]**).** *Assume that* p *is not identically zero eventually and is "integrally-small".*

(i) *If* $\left[\int_t^\infty p(\eta)\,\mathrm{d}\eta\right]^2 \leq \frac{1}{4}\,p(t)$ *for all sufficiently large* t*, then* (1.3) *is nonoscillatory.*
(ii) *If there exists a constant* $\lambda \in (\frac{1}{4},\infty)_{\mathbb{R}}$ *such that* $\left[\int_t^\infty p(\eta)\,\mathrm{d}\eta\right]^2 \geq \lambda p(t)$ *for all sufficiently large* t*, then* (1.3) *is oscillatory.*

In 1958, Opial [14] advanced the result of Wintner by proving the following result.

Theorem E (See [14]**).** *Assume that* p *is not identically zero eventually and is "integrally-small".*

(i) *If* $\int_t^\infty \left[\int_\zeta^\infty p(\eta)\,\mathrm{d}\eta\right]^2\mathrm{d}\zeta \leq \frac{1}{4}\int_t^\infty p(\eta)\,\mathrm{d}\eta$ *for all sufficiently large* t*, then* (1.3) *is nonoscillatory.*
(ii) *If there exists a constant* $\lambda \in (\frac{1}{4},\infty)_{\mathbb{R}}$ *such that* $\int_t^\infty \left[\int_\zeta^\infty p(\eta)\,\mathrm{d}\eta\right]^2\mathrm{d}\zeta \geq \lambda \int_t^\infty p(\eta)\,\mathrm{d}\eta$ *for all sufficiently large* t*, then* (1.3) *is oscillatory.*

Theorems B–E provide sufficient conditions for both oscillation and nonoscillation of (1.3), i.e., they do not provide necessary and sufficient conditions for nonoscillation/oscillation of solutions to (1.3). One would wish to know whether or not it is possible to give a necessary and sufficient condition for nonoscillation/oscillation of solutions to (1.3). An affirmative answer was given in [19] by Yan for the equation

$$(rx')'(t) + p(t)\,x(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.4}$$

where $r \in \mathrm{C}([t_0,\infty)_{\mathbb{R}},\mathbb{R}^+)$ and $p \in \mathrm{C}([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^+)$. Obviously, equation (1.4) includes (1.3) with $r(t) \equiv 1$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Below, we quote this nice result due to Yan (see also [13]).

Theorem F (See [19]**).** *Assume that* p *is "integrally-small". Define the sequence of functions* $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *by*

$$\alpha_k(t) := \begin{cases} \displaystyle\int_t^\infty p(\eta)\,\mathrm{d}\eta, & k = 0, \\ \displaystyle\int_t^\infty \frac{1}{r(\eta)}\big(\alpha_0(\eta)\big)^2\mathrm{d}\eta, & k = 1, \\ \displaystyle\int_t^\infty \frac{1}{r(\eta)}\big(\alpha_{k-1}(\eta) + \alpha_0(\eta)\big)^2\mathrm{d}\eta, & k \in \mathbb{N} \text{ with } k \geq 2, \end{cases}$$

for $t \in [t_1, \infty)_{\mathbb{R}}$. *Then* (1.4) *is nonoscillatory if and only if there exists* $t_1 \in [t_0, \infty)_{\mathbb{R}}$ *such that*

$$\lim_{k \to \infty} \alpha_k(t) =: \alpha(t) < \infty \quad \text{for } t \in [t_1, \infty)_{\mathbb{R}}.$$

It is shown in [19] that Theorems B–E are just consequences of Theorem F.

In [18], Tang, Yu and Peng studied oscillation and nonoscillation of

$$\Delta^2 x(n) + p(n)x(n) = 0 \quad \text{for } t \in [n_0, \infty)_{\mathbb{N}}, \tag{1.5}$$

where $\{p(n)\}$ is a nonnegative sequence of reals, and they proved the following result, which can be regarded as a discrete analogue of Theorem C.

Theorem G (See [10]**).**

(i) *If* $\sum_{j=n}^{\infty} p(j) \leq \frac{1}{4n}$ *for all sufficiently large* n, *then* (1.5) *is nonoscillatory.*

(ii) *If there exists a constant* $\lambda \in (\frac{1}{4}, \infty)_{\mathbb{R}}$ *such that* $\sum_{j=n}^{\infty} p(j) \geq \frac{\lambda}{n}$ *for all sufficiently large* n, *then* (1.5) *is oscillatory.*

However, we could not succeed to find references for discrete counterparts of Theorems D–F for (1.5). In this paper, we shall be looking for a dynamic generalization of Yan's result to dynamic equations of the form (1.1), which will also give us the discrete counterparts of those results for (1.5). Hence, to the best of our knowledge, our results will be new even for the discrete case.

The readers may refer to [1, 4, 5, 6, 9, 11, 15, 20] for papers focusing on nonoscillation, oscillation and disconjugacy properties of similar types of equations, which can be transformed into (1.1) under certain assumptions.

The paper is structured in the following way. In Section 2, we provide two auxiliary results for our main results; in Section 3, we present the time scales generalization of Theorem F for (1.1) together with a general example and some theorems, one of which states a comparison criteria. Section 4 includes some results for (1.2) and finally in Section 5, we show that under some additional conditions, we can extend Theorem F to (1.2) under the conditions assumed to hold while generalizing Theorem F for (1.1).

2. Auxiliary results

Theorem 2.1 (See [3, Theorem 3.1]**).** *The following conditions are equivalent.*

(i) *The second-order dynamic equation* (1.1) *has a nonoscillatory solution.*

(ii) *The second-order dynamic inequality*

$$\left(rx^{\Delta}\right)^{\Delta}(t) + p(t)x(t) \leq 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}$$

has an eventually positive solution.

(iii) *There exist a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and a function*

$$\Lambda \in \mathrm{C}^1_{\mathrm{rd}}\big([t_1, \infty)_{\mathbb{T}}, \mathbb{R}\big)$$

with $\frac{\Lambda}{r} \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying the first-order dynamic Riccati inequality*

$$\Lambda^{\Delta}(t) + \frac{1}{r(t)}\Lambda^{\sigma}(t)\Lambda(t) + p(t) \le 0 \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.1}$$

See also [20, Lemma 2.1].

Remark 2.1 (See [3, Theorem 3.1]**).** Assume that (A1) (a) holds, then the conclusion of Theorem 2.1 holds with $\Lambda \in \mathrm{C}^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ (this implies $\frac{\Lambda}{r} \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$).

For the next result, we define the monotone operator

$$\Gamma : \mathrm{C}_{\mathrm{rd}}\big([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+_0\big) \to \mathrm{C}_{\mathrm{rd}}\big([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+_0\big)$$

by

$$(\Gamma f)(t) := \int_t^{\infty} \frac{1}{r(\eta)} f^{\sigma}(\eta) f(\eta)\Delta\eta + \int_t^{\infty} p(\eta)\Delta\eta \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{2.2}$$

provided that the improper integrals on the right-hand side converge.

Lemma 2.1. *Assume that* (A1) (a) *and* (A2) (b) *hold. If the second-order dynamic equation* (1.1) *is nonoscillatory, then the solution* $\Lambda \in \mathrm{C}^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ *of* (2.1) *satisfies the following assertions:*

(i) Λ *tends to zero asymptotically, i.e.,* $\lim_{t\to\infty} \Lambda(t) = 0$*;*
(ii) $\int_{t_1}^{\infty} \frac{1}{r(\eta)}\Lambda^{\sigma}(\eta)\Lambda(\eta)\Delta\eta < \infty$*;*
(iii) Λ *is the fixed point of the operator* Γ *defined in* (2.2)*, i.e.,* $\Lambda(t) = (\Gamma\Lambda)(t)$ *for all* $t \in [t_1, \infty)_{\mathbb{T}}$*.*

Proof. It follows from (A2) (b) and (2.1) that

$$\frac{\Lambda^{\Delta}(t)}{\Lambda^{\sigma}(t)\Lambda(t)} + \frac{1}{r(t)} \le 0 \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

Integrating (2.3) over $[t_2, t)_{\mathbb{T}} \subset [t_2, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} 0 &\ge \int_{t_1}^{t} \frac{\Lambda^{\Delta}(\eta)}{\Lambda^{\sigma}(\eta)\Lambda(\eta)}\Delta\eta + \int_{t_1}^{t} \frac{1}{r(\eta)}\Delta\eta \\ &= \frac{1}{\Lambda(t_1)} - \frac{1}{\Lambda(t)} + \int_{t_1}^{t} \frac{1}{r(\eta)}\Delta\eta, \end{aligned}$$

which yields

$$\Lambda(t) \le \left(\int_{t_1}^{t} \frac{1}{r(\eta)}\Delta\eta\right)^{-1} \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}.$$

Thus, $\lim_{t\to\infty} \Lambda(t) = 0$ by (A1) (a). This completes the proof of part (i). Integrating (2.1) over $[t_1, \infty)_{\mathbb{T}}$ after dropping p, we obtain

$$\int_{t_1}^{\infty} \frac{1}{r(\eta)}\Lambda^{\sigma}(\eta)\Lambda(\eta)\Delta\eta \le \Lambda(t_1),$$

which proves (ii). Finally, integrating (2.1) over $[t,\infty)_{\mathbb{T}}\subset[t_1,\infty)_{\mathbb{T}}$ completes the proof of part (iii). The proof is completed. □

3. Main results

Theorem 3.1. *Assume that* (A1) (a) *and* (A2) (a) *hold. Then every solution of* (1.1) *is oscillatory.*

Proof. Assume to the contrary that (1.1) is nonoscillatory. By Theorem 2.1 and Remark 2.1, there exist a sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$ and a function $\Lambda\in \mathrm{C}^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ satisfying the first-order dynamic Riccati inequality (2.1). Integrating (2.1) over $[t_1,t)_{\mathbb{T}}\subset[t_1,\infty)_{\mathbb{T}}$ after dropping the positive term $\frac{1}{r}\Lambda^{\sigma}\Lambda$, we have

$$0\geq\Lambda(t)-\Lambda(t_1)+\int_{t_1}^{t}p(\eta)\Delta\eta\quad\text{for all } t\in[t_1,\infty)_{\mathbb{T}},$$

which implies $\lim_{t\to\infty}\Lambda(t)=-\infty$ by (A2) (a). This contradiction completes the proof. □

In what follows, we shall restrict our attention to the "integrally-small" case (A2) (b). To this end, we introduce formally the sequence of functions $\{\beta_k\}_{k\in\mathbb{N}_0}\subset \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ by

$$\beta_k(t):=\begin{cases}\displaystyle\int_t^{\infty}p(\eta)\Delta\eta, & k=0,\\ (\Gamma\beta_{k-1})(t), & k\in\mathbb{N},\end{cases}\tag{3.1}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where the operator Γ is defined by (2.2). Then, we have $\beta_0\geq0$ on $[t_0,\infty)_{\mathbb{T}}$, and by induction, we have $\beta_k\geq\beta_{k-1}$ on $[t_0,\infty)_{\mathbb{T}}$ for all $k\in\mathbb{N}$.

Let us define formally the limiting function β by

$$\beta(t):=\lim_{k\to\infty}\beta_k(t)\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}.\tag{3.2}$$

We are now ready to state our main result.

Theorem 3.2. *Assume that* (A1) (a) *and* (A2) (b) *hold. Then* (1.1) *is nonoscillatory if and only if there exists* $t_1\in[t_0,\infty)_{\mathbb{T}}$ *such that* β *defined by* (3.2) *is well defined on* $[t_1,\infty)_{\mathbb{T}}$.

Proof. ($\Longrightarrow$) Assume that (1.1) is nonoscillatory. By Theorem 3.1, there exist a sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$ and a function $\Lambda\in \mathrm{C}^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ satisfying the first-order dynamic Riccati inequality (2.1). From Lemma 2.1 (iii), it follows that

$$\Lambda(t)\geq\beta_0(t)\quad\text{for all } t\in[t_1,\infty)_{\mathbb{T}}.$$

Using Lemma 2.1 (iii) and the fact that the operator Γ defined in (2.2) is monotone, we have

$$\Lambda(t)=(\Gamma\Lambda)(t)\geq(\Gamma\beta_0)(t)=\beta_1(t)\quad\text{for all } t\in[t_1,\infty)_{\mathbb{T}}.$$

By induction and (3.1), we obtain

$$\Lambda(t) = (\Gamma\Lambda)(t) \geq \beta_k(t) \geq \beta_{k-1}(t) \geq \beta_0(t)$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$ and $k \in \mathbb{N}$, which implies that each function in the sequence $\{\beta_k\}_{k\in\mathbb{N}_0}$ is well defined on $[t_1, \infty)_{\mathbb{T}}$. Consequently, β is well defined on $[t_1, \infty)_{\mathbb{T}}$.

($\Longleftarrow$) Assume that each of the functions in the sequence $\{\beta_k\}_{k\in\mathbb{N}_0}$ defined by (3.1) exists on $[t_1, \infty)_{\mathbb{T}}$ and the sequence converges. It follows from the monotonicity of the sequence and (3.2) that

$$\beta(t) \geq \beta_k(t) \geq \beta_{k-1}(t) \geq \beta_0(t) \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}} \text{ and } k \in \mathbb{N},$$

where β is defined by (3.2). An application of Levi's monotone convergence theorem to (3.1) shows that $\beta(t) = (\Gamma\beta)(t)$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Differentiating this, we see that β satisfies

$$\beta^{\Delta}(t) + \frac{1}{r(t)}\beta^{\sigma}(t)\beta(t) + p(t) = 0 \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}},$$

showing that Theorem 2.1 holds with the positive function β, and therefore (1.1) is nonoscillatory. □

Remark 3.1. It should be mentioned that Theorem 3.2 generalizes Theorem F to arbitrary time scales since in the case $\mathbb{T} = \mathbb{R}$, we see that $\beta_0 = \alpha_0$ on $[t_0, \infty)_{\mathbb{R}}$, and it can be shown by induction that $\beta_k = \alpha_k + \alpha_0$ on $[t_0, \infty)_{\mathbb{R}}$ for $k \in \mathbb{N}$.

We have the following general example for testing the sharpness of Theorem 3.2.

Example 3.1. Let $t_0 \in \mathbb{R}^+$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_0^+$. Consider the following dynamic equation:

$$\begin{aligned}&\left(\frac{1}{\sum_{i=0}^{n-1} \cdot^{n-1-i}(\sigma(\cdot))^i}\, x^{\Delta}\right)^{\Delta}(t) \\ &\qquad + \frac{\lambda}{\sum_{i=0}^{n-1} t^{i+1}(\sigma(t))^{n-i}}\, x(t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.\end{aligned} \tag{3.3}$$

From [2, Theorem 1.24], we have

$$\beta_0(t) = \int_t^{\infty} \frac{\lambda}{\sum_{i=0}^{n-1} \eta^{i+1}(\sigma(\eta))^{n-i}}\,\Delta\eta = \frac{\lambda}{t^n} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}$$

and

$$\beta_1(t) = \int_t^{\infty} \left(\sum_{i=0}^{n-1} \eta^{n-1-i}\big(\sigma(\eta)\big)^i\right) \frac{\lambda}{\eta^n}\,\frac{\lambda}{(\sigma(\eta))^n}\,\Delta\eta = \frac{\lambda^2}{t^n} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Now, we define the sequence $\{\xi_k(\lambda)\}_{k\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ by

$$\xi_k(\lambda) := \begin{cases} \lambda, & k = 0, \\ \big(\xi_{k-1}(\lambda)\big)^2 + \lambda, & k \in \mathbb{N}. \end{cases} \tag{3.4}$$

By induction, we can compute that

$$\beta_k(t) = \frac{\xi_k(\lambda)}{t^n} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \text{ and } k \in \mathbb{N}.$$

Obviously,

$$\beta(t) := \lim_{k\to\infty} \beta_k(t) < \infty \quad \text{if and only if} \quad \xi(\lambda) := \lim_{k\to\infty} \xi_k(\lambda) < \infty,$$

which is equivalent to $\lambda \in [0, \frac{1}{4}]_{\mathbb{R}}$. Let us prove this. Using (3.4), we have

$$\xi_{k+1}(\lambda) - \xi_k(\lambda) = \big(\xi_k(\lambda) + \xi_{k-1}(\lambda)\big)\big(\xi_k(\lambda) - \xi_{k-1}(\lambda)\big) \quad \text{for all } k \in \mathbb{N}.$$

Iterating the above procedure, we see that

$$\xi_{k+1}(\lambda) - \xi_k(\lambda) = \left(\prod_{i=1}^{k} \big(\xi_i(\lambda) + \xi_{i-1}(\lambda)\big)\right)\lambda^2 \geq 0 \quad \text{for all } k \in \mathbb{N},$$

which shows that the sequence $\{\xi_k(\lambda)\}_{k\in\mathbb{N}_0}$ is nondecreasing. Hence, for any $\lambda \in \mathbb{R}_0^+$, the sequence $\{\xi_k(\lambda)\}_{k\in\mathbb{N}_0}$ is monotonic, i.e., $\xi(\lambda)$ exists (finite or infinite). We shall show that $\xi(\lambda) < \infty$ if and only if $\lambda \in [0, \frac{1}{4}]_{\mathbb{R}}$. Then, by taking limit in (3.4) as $k \to \infty$, we see that

$$\xi(\lambda) = \big(\xi(\lambda)\big)^2 + \lambda \quad \text{or} \quad \big(\xi(\lambda)\big)^2 - \xi(\lambda) + \lambda = 0.$$

The discriminant of the quadratic form is $(1 - 4\lambda)$, which is nonnegative if and only if $\lambda \in [0, \frac{1}{4}]_{\mathbb{R}}$. This implies that $\xi(\lambda) < \infty$ if and only if $\lambda \in [0, \frac{1}{4}]_{\mathbb{R}}$. So that (3.3) is nonoscillatory if and only if $\lambda \in [0, \frac{1}{4}]_{\mathbb{R}}$.

As an immediate consequence of Theorems 3.1 and 3.2, we have the following result.

Theorem 3.3. *Equation* (1.1) *is nonoscillatory if and only if one of the following conditions holds:*

(i) *there exists* $n \in \mathbb{N}_0$ *such that for* $k \in [0, n)_{\mathbb{Z}}$ *the functions* β_k *are well defined, but* β_n *does not exist;*

(ii) *the sequence of functions* $\{\beta_k\}_{k\in\mathbb{N}_0} \subset \mathrm{C_{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ *introduced in* (3.1) *is well defined, but there exists an increasing unbounded sequence of points* $\{\xi_k\}_{k\in\mathbb{N}_0} \subset [t_0, \infty)_{\mathbb{T}}$ *such that* $\beta(\xi_k) = \infty$ *for each* $k \in \mathbb{N}_0$, *where* β *is defined by* (3.2).

4. Complementary results

We start this section by mentioning that under conditions (A1) and (A2), oscillation of (1.1) implies oscillation of (1.2) (see [3, Theorem 4.10 and Remark 4.11]).

Lemma 4.1. *If* x *is a solution of* (1.2), *then*

$$\begin{aligned}\big(rx^{\Delta}\big)^{\sigma}(t) - r(t)\left(1 - \big(\mu(t)\big)^2 \frac{p(t)}{r(t)}\right)x^{\Delta}(t)&\\ + \mu(t)p(t)x(t) = 0 \quad \textit{for all } t \in [t_0, \infty)_{\mathbb{T}}.&\end{aligned} \tag{4.1}$$

Proof. Let x be a solution of (1.2), then multiplying (1.2) by μ and using the so-called *simple useful formula* (see (6.1)), we get

$$\big(rx^{\Delta}\big)^{\sigma}(t) - \Big(r(t) - \big(\mu(t)\big)^{2}p(t)\Big)x^{\Delta}(t) + \mu(t)p(t)x(t) = 0 \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}.$$

This completes the proof. □

Since we are concerned with the eventual behavior of the solution, we may assume without loss of generality that either one of the following conditions holds:

(A3) (a) $-\mu\frac{p}{r} \notin \mathcal{R}^{+}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$;

(b) $-\mu\frac{p}{r} \in \mathcal{R}^{+}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

With the conditions above, we mean that the regressivity property does not hold on $[s,\infty)_{\mathbb{T}}$ for any $s \in [t_0,\infty)_{\mathbb{T}}$.

Theorem 4.1. *Assume that* (A1) (a) *and* (A3) (a) *hold. Then every solution of* (1.2) *oscillates.*

Proof. Assume to the contrary that (1.2) is nonoscillatory. Let $x(t) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$. By (A1) (a), we have

$$x^{\Delta}(t) > 0 \quad \text{for all } t \in [t_1,\infty)_{\mathbb{T}}$$

(see [3, Remark 3.2]). But (4.1) implies that either $x^{\Delta\sigma}(t_2) \leq 0$ or $x^{\Delta}(t_2) \leq 0$ or $x(t_2) \leq 0$ for some $t_2 \in [t_1,\infty)_{\mathbb{T}}$, which is a contradiction. This completes the proof. □

Lemma 4.2. *Assume that* (A1) *and* (A3) (b) *hold. Equation* (1.2) *is equivalent to*

$$\big(\varphi rx^{\Delta}\big)^{\Delta}(t) + \varphi^{\sigma}(t)p(t)x(t) = 0 \quad \textit{for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{4.2}$$

where $\varphi \in \mathrm{C}^{1}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *is defined by*

$$\varphi(t) := \mathrm{e}_{\ominus(-\mu\frac{p}{r})}(t,t_0) \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.3}$$

In other words, both (1.2) *and* (4.2) *have the same solutions.*

Proof. Using the *simple useful formula* (see (6.1)), we get

$$\big(rx^{\Delta}\big)^{\Delta}(t) + \mu(t)p(t)x^{\Delta}(t) + p(t)x(t) = 0 \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.4}$$

Let us assume that $\varphi \in \mathrm{C}^{1}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ solves the initial value problem

$$\begin{cases} \varphi^{\Delta}(t) = \mu(t)\dfrac{p(t)}{r(t)}\varphi^{\sigma}(t) & \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ \varphi(t_0) = 1. \end{cases} \tag{4.5}$$

It follows from [2, Theorem 2.74] that $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is defined by (4.3). Multiplying both sides of (4.4) by φ^σ, we get

$$\begin{aligned}0 &= \varphi^\sigma(t)\big(rx^\Delta\big)^\Delta(t) + \varphi^\sigma(t)\mu(t)p(t)x^\Delta(t) + \varphi^\sigma(t)p(t)x(t)\\ &= \varphi^\sigma(t)\big(rx^\Delta\big)^\Delta(t) + \varphi^\Delta(t)r(t)x^\Delta(t) + \varphi^\sigma(t)p(t)x(t)\\ &= \big(\varphi rx^\Delta\big)^\Delta(t) + \varphi^\sigma(t)p(t)x(t)\end{aligned}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. This completes the proof. □

Remark 4.1. If (A3) (b) holds, then we have $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ by [2, Theorem 2.74]. It also follows from (4.5) that

$$\varphi^\Delta(t) \geq 0 \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}},$$

which shows that $\varphi(t) \geq \varphi(t_0) = 1$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Now, we have new additional assumptions.

(A1) (b) $\displaystyle\int_{t_0}^{\infty} \frac{\mathrm{e}_{-\mu\frac{p}{r}}(\eta,t_0)}{r(\eta)}\Delta\eta = \infty.$

(A2) (c) $\displaystyle\int_{t_0}^{\infty} \frac{p(\eta)}{\mathrm{e}^{\sigma_1}_{-\mu\frac{p}{r}}(\eta,t_0)}\Delta\eta = \infty;$

(d) $\displaystyle\int_{t_0}^{\infty} \frac{p(\eta)}{\mathrm{e}^{\sigma_1}_{-\mu\frac{p}{r}}(\eta,t_0)}\Delta\eta < \infty.$

Remark 4.2. When (A1), (A2) and (A3) (b) hold, due to Remark 4.1, (A1) (b), (A2) (a) and (A2) (d) imply (A1) (a), (A2) (c) and (A2) (b), respectively.

Theorem 4.2. *Assume that* (A1) (b), (A2) (c) *and* (A3) (b) *hold. Then every solution of* (1.2) *oscillates.*

Proof. The proof follows from Lemma 4.2 and Theorem 3.1. □

Let us define the operator

$$\Psi : \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0) \to \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$$

by

$$\begin{aligned}(\Psi f)(t) :=& \int_t^{\infty} \frac{\mathrm{e}_{-\mu\frac{p}{r}}(\eta,t_0)}{r(\eta)} f^\sigma(\eta)f(\eta)\Delta\eta\\ &+ \int_t^{\infty} \frac{p(\eta)}{\mathrm{e}^{\sigma}_{-\mu\frac{p}{r}}(\eta,t_0)}\Delta\eta \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.\end{aligned}$$

Let us define formally the sequence of functions $\{\gamma_k\}_{k\in\mathbb{N}_0} \subset \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ by

$$\gamma_k(t) := \begin{cases} \displaystyle\int_t^{\infty} \frac{p(\eta)}{\mathrm{e}^{\sigma_1}_{-\mu\frac{p}{r}}(\eta,t_0)}\Delta\eta, & k = 0,\\ (\Psi\gamma_{k-1})(t), & k \in \mathbb{N}, \end{cases} \tag{4.6}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, and the formal limiting function γ by

$$\gamma(t) := \lim_{k\to\infty} \gamma_k(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.7}$$

Theorem 4.3. *Assume that* (A1) (b)*,* (A2) (d) *and* (A3) (b) *hold. Then,* (1.2) *is nonoscillatory if and only if there exists* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that* γ *is well defined on* $[t_1, \infty)_{\mathbb{T}}$*.*

Proof. The proof follows from Lemma 4.2 and Theorem 3.2. □

Theorem 4.4. *Equation* (1.2) *is nonoscillatory if and only if one of the following conditions holds:*

(i) *there exists* $n \in \mathbb{N}_0$ *such that for* $k \in [0, n)_{\mathbb{Z}}$ *the functions* γ_k (*introduced in* (4.6)) *are well-defined, but* γ_n *does not exist;*

(ii) *the sequence of functions* $\{\gamma_k\}_{k\in\mathbb{N}_0} \subset \mathrm{C}_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ *is well defined, but there exists an increasing unbounded sequence of points* $\{\xi_k\}_{k\in\mathbb{N}_0} \subset [t_0, \infty)_{\mathbb{T}}$ *such that* $\gamma(\xi_k) = \infty$ *for each* $k \in \mathbb{N}_0$*, where* γ *is defined by* (3.2).

5. Final comments

In this section, we provide some assumptions for the equivalence of conditions (A1) (a), (A1) (b) and (A2) (b), (A2) (d). Therefore, oscillation of (1.1) and (1.2) are equivalent.

It follows from [3, Theorem 5.5] that both (1.1) and (1.2) are oscillatory if (A1) (a) and (A2) (a) hold. Hence, below, we focus on conditions (A1) (a) and (A2) (b). It follows from Theorem 4.1 that we may also consider (A3) (b), which implies

$$\int_{t_0}^{\infty} \mu(\eta) \frac{p(\eta)}{r(\eta)} \Delta\eta < \infty \iff \lim_{t\to\infty} \mathrm{e}_{-\mu\frac{p}{r}}(t, t_0) > 0.$$

We proceed with proving this fact.

Lemma 5.1. *Let* $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}_0^+)$ *with* $-f \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$*. Then, the following conditions are equivalent:*

(i) $\int_s^\infty f(\eta)\Delta\eta < \infty$*;*

(ii) $\lim_{t\to\infty} \mathrm{e}_{-f}(t, s) > 0$.

Proof. First, we let

$$y(t) := \mathrm{e}_{-f}(t, s) > 0 \quad \text{for } t \in [s, \infty)_{\mathbb{T}}.$$

Then, we see that

$$y^{\Delta}(t) = -f(t)y(t) < 0 \quad \text{for all } t \in [s, \infty)_{\mathbb{T}}, \tag{5.1}$$

which implies that y is decreasing on $[s, \infty)_{\mathbb{T}}$, i.e., $\ell := \lim_{t\to\infty} y(t)$ exists.

(i)$\Longrightarrow$(ii). There exists $r \in [s, \infty)_{\mathbb{T}}$ such that

$$\int_r^\infty f(\eta)\Delta\eta < \frac{1}{2}.$$

Integrating (5.1), we get

$$y(r) - y(t) = \int_r^t f(\eta)y(\eta)\Delta\eta \le y(r)\int_r^t f(\eta)\Delta\eta \le \frac{1}{2}\,y(r)$$

for all $t \in [r,\infty)_{\mathbb{T}}$, which yields

$$\frac{1}{2}\,y(r) \le y(t) \quad \text{for all } t \in [r,\infty)_{\mathbb{T}},$$

and thus (ii) holds.

(ii)$\Longrightarrow$(i). Then, $y(t) > \ell$ for all $t \in [s,\infty)_{\mathbb{T}}$, where $\ell > 0$. It follows from (5.1) that

$$y(s) - y(t) = \int_s^t f(\eta)y(\eta)\Delta\eta > \ell\int_s^t f(\eta)\Delta\eta \quad \text{for all } t \in [s,\infty)_{\mathbb{T}},$$

which proves (i) by letting $t \to \infty$.

Hence, the proof is completed. □

Now, consider the following additional condition:

(A4) (a) $\limsup_{t\to\infty} \dfrac{\mu(t)}{r(t)} < \infty$,

(b) $\liminf_{t\to\infty} \dfrac{\mu(t)}{r(t)} > 0$.

Thus, if in addition (A4) (a) holds, we see that

$$\int_{t_0}^{\infty} p(\eta)\Delta\eta < \infty \implies \int_{t_0}^{\infty} \mu(\eta)\,\frac{p(\eta)}{r(\eta)}\,\Delta\eta < \infty$$

(reverse implication holds under (A4) (b)) and thus,

$$\int_{t_0}^{\infty} p(\eta)\Delta\eta < \infty \implies \lim_{t\to\infty} \mathrm{e}_{-\mu\frac{p}{r}}(t,t_0) > 0.$$

Remark 5.1. If (A3) (b) and (A4) (a) hold, then (A1) (a) and (A2) (b) imply (A1) (b) and (A2) (d).

Combining Remarks 4.2 and 5.1, we infer that if (A3) (b) and (A4) (a) hold, then (A1) (a) and (A2) (b) are equivalent to (A1) (b) and (A2) (d). Hence, we can give the following theorem.

Theorem 5.1. *Assume that* (A1) (a), (A2) (b), (A3) (b) *and* (A4) *hold. Then* (1.2) *is nonoscillatory if and only if there exists* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that* γ *defined by* (4.7) *is well defined on* $[t_1,\infty)_{\mathbb{T}}$.

Proof. The proof follows from Theorem 4.3 and Remark 5.1. □

Theorem 5.2. *Assume that* (A1) (a), (A2) (b), (A3) (b) *and* (A4) (a) *hold. Then* (1.2) *is nonoscillatory if and only if one of the following conditions holds:*

(i) *there exists* $n \in \mathbb{N}_0$ *such that for* $k \in [0,n)_{\mathbb{Z}}$ *the functions* γ_k (*introduced in* (4.6)) *are well defined, but* γ_n *does not exist;*

(ii) *the sequence of functions* $\{\gamma_k\}_{k\in\mathbb{N}_0} \subset \mathrm{C}_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ *is well defined, but there exists an increasing unbounded sequence of points* $\{\xi_k\}_{k\in\mathbb{N}_0} \subset [t_0,\infty)_{\mathbb{T}}$ *such that* $\gamma(\xi_k)=\infty$ *for each* $k\in\mathbb{N}_0$*, where* γ *is defined by* (3.2).

Let us give the following remark, which may cover the discrete case $\mathbb{T}=\mathbb{Z}$.

Remark 5.2. Note also that the following assumptions imply (A4) (a):

(A5) $\limsup_{t\to\infty} \mu(t) < \infty$;

(A6) $\liminf_{t\to\infty} r(t) > 0$.

6. Appendix: Time scales essentials

A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Throughout the paper, a time scale is denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t\in\mathbb{T}$, we define the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\,(t,\infty)_{\mathbb{T}}$, while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t):=\sup\,(-\infty,t)_{\mathbb{T}}$, and the *graininess function* $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is defined to be $\mu(t):=\sigma(t)-t$. A point $t\in\mathbb{T}$ is called *right-dense* if $\sigma(t)=t$ and/or equivalently $\mu(t)=0$ holds; otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered* points are defined with respect to the backward jump operator. For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^\kappa$, the Δ-derivative $f^\Delta(t)$ of f at the point t is defined to be the number, provided it exists, with the property that, for any $\varepsilon>0$, there is a neighborhood U of t such that

$$\big|[f^\sigma(t)-f(s)]-f^\Delta(t)[\sigma(t)-s]\big| \le \varepsilon\,|\sigma(t)-s| \quad \text{for all } s\in U,$$

where $\mathbb{T}^\kappa := \mathbb{T}\backslash\{\sup\mathbb{T}\}$ if $\sup\mathbb{T}=\max\mathbb{T}$ and satisfies $\rho(\max\mathbb{T})\neq\max\mathbb{T}$; otherwise, $\mathbb{T}^\kappa:=\mathbb{T}$, and $f^\sigma := f\circ\sigma$ on $\mathbb{T}$. Unless otherwise specified, we mean by “derivative” the “Δ-derivative” of a function. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$, and it has a finite limit at left-dense points, and the *set of rd-continuous functions* is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The set of functions $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ includes the functions whose derivative is in $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ too. For a function $f\in\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the so-called *simple useful formula* holds, i.e.,

$$f^\sigma(t)=f(t)+\mu(t)f^\Delta(t) \quad \text{for all } t\in\mathbb{T}^\kappa. \tag{6.1}$$

For $s,t\in\mathbb{T}$ and a function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t)-F(s) \quad \text{for } s,t\in\mathbb{T},$$

where $F\in\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is an antiderivative of f, i.e., $F^\Delta=f$ on $\mathbb{T}^\kappa$.

A function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is called *regressive* if $1+\mu f\neq 0$ on $\mathbb{T}^\kappa$, and *positively regressive* if $1+\mu f>0$ on $\mathbb{T}^\kappa$. The *set of regressive functions* and the *set of positively regressive functions* are denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$ and $\mathcal{R}^+(\mathbb{T},\mathbb{R})$, respectively, and $\mathcal{R}^-(\mathbb{T},\mathbb{R})$ is defined similarly.

Let $f \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then the *exponential function* $\mathrm{e}_f(\cdot, s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t)x(t) & \text{for } t \in \mathbb{T}^{\kappa}, \\ x(s) = 1 \end{cases} \tag{6.2}$$

for some fixed $s \in \mathbb{T}$. For $h \in \mathbb{R}^+$, set

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\},$$
$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}$$

and

$$\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}.$$

For $h \in \mathbb{R}_0^+$, we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) := \begin{cases} z, & h = 0, \\ \dfrac{1}{h} \mathrm{Log}(1 + hz), & h > 0, \end{cases}$$

for $z \in \mathbb{C}_h$, then the exponential function can also be written in the form

$$\mathrm{e}_f(t, s) := \exp\left\{ \int_s^t \xi_{\mu(\eta)}\big(f(\eta)\big)\Delta\eta \right\} \quad \text{for } s, t \in \mathbb{T}.$$

If $f \in \mathcal{R}^+([s, \infty)_{\mathbb{T}}, \mathbb{R})$, the exponential function $\mathrm{e}_f(\cdot, s)$ is strictly positive on $[s, \infty)_{\mathbb{T}}$, while $\mathrm{e}_f(\cdot, s)$ alternates in sign at right-scattered points of the interval $[s, \infty)_{\mathbb{T}}$ provided that $f \in \mathcal{R}^-([s, \infty)_{\mathbb{T}}, \mathbb{R})$. For $h \in \mathbb{R}_0^+$, let $z, w \in \mathbb{C}_h$, the *circle plus* $\oplus_h$ and the *circle minus* $\ominus_h$ are defined by

$$z \oplus_h w := z + w + hzw \quad \text{and} \quad z \ominus_h w := \frac{z - w}{1 + hw},$$

respectively. Further throughout the paper, we abbreviate the operations $\oplus_\mu$ and $\ominus_\mu$ simply by $\oplus$ and $\ominus$, respectively. It is also known that $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ is a subgroup of $\mathcal{R}(\mathbb{T}, \mathbb{R})$, i.e., $0 \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, $f, g \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ implies $f \oplus_\mu g \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ and $\ominus_\mu f \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, where $\ominus_\mu f := 0 \ominus_\mu f$ on $\mathbb{T}$.

The readers are referred to [2] for further interesting details on the time scale theory.

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