J. Fixed Point Theory Appl. 19 (2017) 1083–1102
DOI 10.1007/s11784-016-0329-5
Published online September 27, 2016

Journal of Fixed Point Theory and Applications

How rigid the finite ultrametric spaces can be?

O. Dovgoshey, E. Petrov and H.-M. Teichert

Abstract. A metric space X is rigid if the isometry group of X is trivial. The finite ultrametric spaces X with $|X| \geq 2$ are not rigid since for every such X there is a self-isometry having exactly $|X|-2$ fixed points. Using the representing trees we characterize the finite ultrametric spaces X for which every self-isometry has at least $|X| - 2$ fixed points. Some other extremal properties of such spaces and related graph theoretical characterizations are also obtained.

Mathematics Subject Classification. 54E35.

Keywords. Finite ultrametric space, rigidness of metric spaces, extremal problems in finite ultrametric spaces, fixed point, representing tree of ultrametric space.

1. Introduction

Recall some definitions from the theory of metric spaces. A *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x)$,
(ii) $(d(x, y) = 0) \Leftrightarrow (x = y)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric space (X, d) is *ultrametric* if the *strong triangle inequality*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

holds for all $x, y, z \in X$. In this case, d is called an ultrametric on X and (X, d) is an ultrametric space. The *spectrum* of a metric space (X, d) is the set

$$\mathrm{Sp}(X) = \{d(x, y)\colon x, y \in X\}.$$

In 2001 at the Workshop on General Algebra [17] the attention of experts on the theory of lattices was guided to the following problem of I. M. Gelfand: using graph theory describe up to isometry all finite ultrametric spaces. An appropriate representation of ultrametric spaces by rooted trees was proposed in [10, 11, 13, 18].

An application of the representation from [10, 11] is a structural characteristic of all finite ultrametric spaces X for which the Gomory–Hu inequality

$$|\operatorname{Sp}(X)| \leq |X|$$

becomes an equality (see [7, 20]). The purpose of this paper is to describe the structure of finite ultrametric spaces which have maximum rigidity.

Recall that a *graph* is a pair (V, E) consisting of a nonempty set V and a (probably empty) set E whose elements are unordered pairs of different points from V. For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called the *set of vertices* and the *set of edges*, respectively. A graph is *complete* if $\{x, y\} \in E(G)$ for all distinct $x, y \in V(G)$. A *path* in a graph G is a subgraph P of G for which

$$V(P) = \{x_0, x_1, \dots, x_k\}, \quad E(P) = \{\{x_0, x_1\}, \dots, \{x_{k-1}, x_k\}\},$$

where all x_i are distinct. A graph G is *connected* if any two distinct vertices of G can be joined by a path. A finite graph C is a cycle if $|V(C)| \geq 3$ and there exists an enumeration $(v_1, \dots, v_n)$ of its vertices such that

$$\big(\{v_i, v_j\} \in E(C)\big) \iff \big(|i - j| = 1 \text{ or } |i - j| = n - 1\big).$$

A connected graph without cycles is called a *tree*. A tree T may have a distinguished vertex called the *root*; in this case T is called a *rooted tree*. For a rooted tree T, we denote by L_T the set of leaves of T. We denote by (G, v, l) a graph G with a *distinguished vertex* $v \in V(G)$ and a *labeling function* $l \colon V(G) \to L$. In what follows we usually suppose that the set L coincides with $\mathbb{R}^+$.

Let $k \geq 2$. A graph G is called *complete k-partite* if its vertices can be divided into k disjoint nonempty sets $X_1, \dots, X_k$ so that there are no edges joining the vertices of the same set X_i and any two vertices from different X_i, X_j, $1 \leq i, j \leq k$, are adjacent. In this case we write $G = G[X_1, \dots, X_k]$. We shall say that G is a *complete multipartite graph* if there exists $k \geq 2$ such that G is complete k-partite; cf. [3].

2. The representing trees of finite ultrametric spaces

For every metric space (X, d) we write

$$\operatorname{diam} X = \sup\{d(x, y) : x, y \in X\}.$$

Definition 2.1 (See [4]). Let (X, d) be a finite ultrametric space, $|X| \geqslant 2$. Define the *diametrical* graph G_X as follows:

$$V(G_X) = X$$

and, for all u, $v \in X$,

$$\big(\{u, v\} \in E(G_X)\big) \iff \big(d(u, v) = \operatorname{diam} X\big).$$

Lemma 2.2 (See [4]). *Let (X, d) be a finite ultrametric space, $|X| \geq 2$. Then $G_X = G_X[X_1, \dots, X_k]$ for some $k \geq 2$.*

With a finite nonempty ultrametric space (X, d), we can associate a labeled rooted tree T_X by the following rule. The root of T_X is, by definition, the set X. If $X = \{x\}$ is a one-point set, then T_X is a tree consisting of one node $\{x\}$ which has the label 0. Let $|X| \geq 2$. According to Lemma 2.2 we have $G_X = G_X[X_1, \dots, X_k]$. In this case, we say that, the root of the tree T_X is labeled by $\operatorname{diam} X$ and T_X has k nodes $X_1, \dots, X_k$ of the first level with the labels

$$l(X_i) := \operatorname{diam} X_i, \quad i = 1, \dots, k. \tag{2.1}$$

The nodes of the first level indicated by zero are leaves, and those indicated by positive numbers are internal nodes of T_X. If the first level has no internal nodes, then the tree T_X is constructed. Otherwise, by repeating the above-described procedure with X_i corresponding to internal nodes of the first level instead of X, we obtain the nodes of the second level, etc. Since X is finite, all vertices on some level will be leaves, and the construction of T_X is completed.

The above-constructed labeled rooted tree T_X is called the *representing tree* of the ultrametric space (X, d).

Let (T, v^*) be a rooted tree. For every node u^* of (T, v^*) define a rooted subtree T_{u^*} of (T, v^*) as follows: u^* is the root of T_{u^*} and a vertex $w \in T$ belongs to $V(T_{u^*})$ if and only if u^* lies on the path joining v^* and w in T, moreover

$$\big(\{u, v\} \in E(T_{u^*})\big) \Longleftrightarrow \big(\{u, v\} \in E(T)\big)$$

for all u, $v \in V(T_{u^*})$.

The following lemma was formulated in [20] for finite ultrametric spaces X satisfying the equality $|X| = |\operatorname{Sp}(X)|$ but its proof is also true for arbitrary finite ultrametric spaces.

Lemma 2.3. *Let (X, d) be a finite ultrametric space, $|X| \geq 2$, and let a and b be two different leaves of the tree T_X. If $(x_1, x_2, \dots, x_n)$, $x_1 = a$, $x_n = b$, is the path joining a and b in T_X, then*

$$d(a, b) = \max_{1 \leq i \leq n} l(x_i). \tag{2.2}$$

Let (X, d) be a metric space. Recall that a subset B of X is a *ball* in (X, d) if there is $r \geq 0$ and $t \in X$ such that

$$B = \{x \in X \colon d(x, t) \leq r\}.$$

In this case we write $B = B_r(t)$. By $\mathbf{B}_X$ we denote the set of all balls in (X, d).

The proof of the next lemma can be found in [19] but we reproduce it here for the convenience of the reader.

Lemma 2.4. *Let (X, d) be a finite ultrametric space with representing tree T_X, $|X| \geq 2$. Then*

(i) *$L_{T_v} \in \mathbf{B}_X$ holds for every node $v \in V(T_X)$,*
(ii) *for every $B \in \mathbf{B}_X$ there exists a node v such that $L_{T_v} = B$.*

Proof. (i) Let $v \in V(T_X)$ and $t \in L_{T_v}$. Consider the ball

$$B_{l(v)}(t) = \big\{x \in X : d(x,t) \le l(v)\big\}.$$

Let $t_1 \in L_{T_v}$ such that $t_1 \neq t$. Since T_v contains a path joining t and t_1, according to Lemma 2.3 we have $d(t,t_1) \le l(v)$. The inclusion $L_{T_v} \subseteq B_{l(v)}(t)$ is proved. Conversely, suppose there exists $t_0 \in B_{l(v)}(t)$ such that $t_0 \notin L_{T_v}$. Let us consider the path $(t_0, v_1, \dots, v_n, t)$. From $t_0 \notin L_{T_v}$ it follows that

$$\max_{1 \le i \le n} l(v_i) > l(v),$$

i.e., $d(t_0,t) > l(v)$, we have a contradiction.

(ii) Let $t \in X$ and $B = B_r(t)$, where $r = \operatorname{diam} B$. Let $x, y \in B$ with $d(x,y) = r$. Let us consider the path $(v_1, \dots, v_n)$ with $v_1 = x$ and $v_n = y$ in the tree T_X. According to Lemma 2.3 we have

$$d(x,y) = \max_{1 \le i \le n} l(v_i).$$

Let i be an index such that max here is attained. The proof of the equality $L_{T_{v_i}} = B$ is analogous to the proof of (i). □

Let (G^i, v^i, l^i), $i = 1, 2$, be a labeled graphs with the distinguished vertices v^1, v^2 and a common set L of labels. A bijective function

$$f : V\big(G^1\big) \to V\big(G^2\big)$$

is an isomorphism of G^1 and G^2 if

$$\big(\{x,y\} \in E\big(G^1\big)\big) \Longleftrightarrow \big(\{f(x), f(y)\} \in E\big(G^2\big)\big)$$

for all x, $y \in V(G^1)$. If, in addition, we have $f(v^1) = v^2$, then f is an isomorphism of (G^1, v^1) and (G^2, v^2). The isomorphism f of (G^1, v^1) and (G^2, v^2) is called an isomorphism of (G^1, v^1, l^1) and (G^2, v^2, l^2) if

$$l^1(v) = l^2(f(v))$$

for every $v \in V(G^1)$.

Definition 2.5. Let (X,d) and (Y,ρ) be metric spaces. A bijective function $f : X \to Y$ is an *isometry* if

$$d(x,y) = \rho\big(f(x), f(y)\big)$$

holds for all $x, y \in X$.

Two metric spaces (X,d) and (Y,ρ) are *isometric* if there is an isometry $f : X \to Y$.

Theorem 2.6. *Let (X,d) and (Y,ρ) be finite nonempty ultrametric spaces. Then (X,d) and (Y,ρ) are isometric if and only if the labeled rooted trees T_X and T_Y are isomorphic.*

Proof. An isomorphism of T_X and T_Y for isometric X and Y can be inductively constructed if we use the definition of the representing trees given above. The converse statement follows from Lemma 2.3. □

For the relationships between ultrametric spaces and the leaves or the ends of certain trees see also [2, 8, 10, 11, 12, 13, 17, 18].

Let (X, d) be a finite metric space and let B_1, B_2 and B be some balls in (X, d). We shall say that B *lies between* B_1 and B_2 if

$$B_1 \subseteq B \subseteq B_2 \quad \text{or} \quad B_2 \subseteq B \subseteq B_1.$$

If B lies between B_1 and B_2, we write $B \in [B_1, B_2]$.

Definition 2.7. Let us define a graph Γ_X by the rule: $V(\Gamma_X) = \mathbf{B}_X$ and, for all B_1, $B_2 \in \mathbf{B}_X$, $\{B_1, B_2\} \in E(\Gamma_X)$ if and only if

(1) $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, and
(2) $(B \in [B_1, B_2]) \Rightarrow (B = B_1$ or $B = B_2)$ for every $B \in \mathbf{B}_X$.

For any labeled rooted tree $T_X = (T_X, X, l)$ we write $\overline{T}_X = (T_X, X)$. In the following theorem we presuppose that $X \in \mathbf{B}_X$ is a distinguished point in Γ_X.

Theorem 2.8. *Let (X, d) be a finite nonempty metric space. Then the graph Γ_X is a tree if and only if (X, d) is ultrametric. If (X, d) is ultrametric, then (Γ_X, X) is isomorphic to the rooted tree $\overline{T}_X$ with the isomorphism*

$$V(\overline{T}_X) \ni u \mapsto L_{T_u} \in V(\Gamma_X).$$

Proof. Suppose that (X, d) is not an ultrametric space. Then there are distinct $x_1, x_2, x_3 \in X$ such that

$$d(x_1, x_2) > \max\{d(x_1, x_3), d(x_3, x_2)\}. \tag{2.3}$$

Let $B^1 = B_{r_1}(x_1)$ with $r_1 = d(x_1, x_3)$ and $B^2 = B_{r_2}(x_2)$ with $r_2 = d(x_2, x_3)$. It is evident that

$$\{x_3\} \subseteq B^1 \cap B^2 \subseteq B^1 \cup B^2 \subseteq X. \tag{2.4}$$

Using (2.4) and the finiteness of $\mathbf{B}_X$ we can find $B^0 \in \mathbf{B}_X$ such that

$$B^0 \subseteq B^1 \cap B^2$$

and

$$\big(B^0 \subseteq B \subseteq B^1 \cap B^2\big) \Longrightarrow \big(B^0 = B\big)$$

for every $B \in \mathbf{B}_X$. Similarly, there exists $B^3 \in \mathbf{B}_X$ such that

$$B^3 \supseteq B^1 \cup B^2$$

and

$$\big(B^1 \cup B^2 \subseteq B \subseteq B^3\big) \Longrightarrow \big(B = B^3\big)$$

for every $B \in \mathbf{B}_X$.

For all distinct balls $C, D \in \mathbf{B}_X$ satisfying $C \subseteq D$, there is a chain

$$L(C, D) = \{A_1, \ldots, A_n\} \subseteq \mathbf{B}_X$$

such that

$$A_1 = C, \quad A_n = D, \quad A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$

and, for every $B \in \mathbf{B}_X$ and $i = 1, \ldots, n-1$,

$$\big(A_i \subseteq B \subseteq A_{i+1}\big) \Longrightarrow \big(B = A_i \text{ or } B = A_{i+1}\big). \tag{2.5}$$

Using (2.3) we see that $x_1 \notin B^2$ and $x_2 \notin B^1$. Consequently, the balls B^0, B^1, B^2, B^3 are pairwise distinct, so that there exist the chains $L(B^0, B^1)$, $L(B^0, B^2)$, $L(B^1, B^3)$ and $L(B^2, B^3)$. Let

$$L\big(B^0, B^1\big) = (A_1, \dots, A_n) \quad \text{and} \quad L\big(B^1, B^3\big) = (A_n, \dots, A_{n+m}).$$

We claim that

$$L\big(B^0, B^1, B^3\big) := (A_1, \dots, A_n, \dots, A_{n+m})$$

is a path in the graph Γ_X joining $B^0 = A_1$ and $B^3 = A_{n+m}$.

Indeed, from the definition of $L(B^0, B^1)$ and $L(B^1, B^3)$ it follows that $A_i \neq A_j$ for any distinct $i, j \in \{1, \dots, n+m\}$. Moreover, (2.5) implies that $\{A_i, A_{i+1}\} \in E(\Gamma_X)$ for every $i \in \{1, \dots, n+m-1\}$. Hence $L(B^0, B^1, B^3)$ is a path in Γ_X. Inequality (2.3) implies that $x_1 \notin B^2$ and $x_2 \notin B^1$, consequently $B^1 \nsubseteq B^2$ and $B^2 \nsubseteq B^1$. Since for every vertex $B \in V(L(B^0, B^1, B^3))$ we have

$$B \subseteq B^1 \quad \text{or} \quad B^1 \subseteq B,$$

the ball B^2 is not a vertex of the path $L(B^0, B^1, B^3)$. Similarly, we can construct a path $L(B^0, B^2, B^3)$ such that $B^1 \notin V(L(B^0, B^2, B^3))$. Consequently, the vertices B^0 and B^3 can be joined by two different paths in Γ_X. Hence Γ_X is not a tree.

Suppose now that (X, d) is an ultrametric space. We must show that Γ_X is a tree and that (Γ_X, X) is isomorphic to the rooted tree $\overline{T}_X$. By Lemma 2.4, for every $v \in V(\Gamma_X)$, we have $L_{T_v} \in \mathbf{B}_X$. Consequently, the mapping

$$V(\overline{T}_X) \ni v \mapsto L_{T_v} \in V(\Gamma_X) \tag{2.6}$$

is correctly defined. Hence it is sufficient to show that this mapping is an isomorphism of the rooted tree $\overline{T}_X$ and (Γ_X, X).

Using Lemma 2.4 again we obtain that (2.6) is bijective. It is also clear that the root of $\overline{T}_X$ corresponds to X under the mapping (2.6). It still remains to show that

$$\big(\{v_1, v_2\} \in E(\overline{T}_X)\big) \Longleftrightarrow \big(\{L_{T_{v_1}}, L_{T_{v_2}}\} \in E(\Gamma_X)\big) \tag{2.7}$$

holds for all v_1, $v_2 \in V(\overline{T}_X)$.

Write $B_1 = L_{T_{v_1}}$ and $B_2 = L_{T_{v_2}}$. Then it follows from the definition of the rooted subtrees that $B_1 \subseteq B_2$ if and only if v_1 is a node of the tree T_{v_2}. Moreover, if $B_1 \subseteq B_2$, then

$$\big(B_1 \subseteq B \subseteq B_2\big) \Longrightarrow \big(B_2 = B \text{ or } B_1 = B\big)$$

holds for all $B \in \mathbf{B}_X$ if and only if v_1 is a direct successor of v_2. Statement (2.7) follows. □

Remark 2.9. For all nonultrametric metric triangles X their graphs Γ_X are isomorphic to the graph depicted in Figure 1.

Since a connected graph G is a tree if and only if

$$|V(G)| = |E(G)| + 1$$

(see, for example, [3, Corollary 1.5.3]), Theorem 2.8 implies the following.

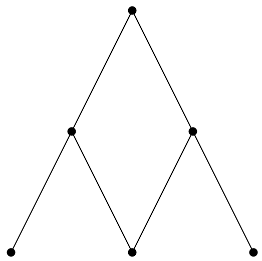

FIGURE 1

Corollary 2.10. *Let X be a finite nonempty metric space. Then X is an ultrametric space if and only if*

$$|V(\Gamma_X)| = |E(\Gamma_X)| + 1.$$

Recall that a graph H is a *spanning subgraph* of a graph G if

$$V(G) = V(H) \quad \text{and} \quad E(H) \subseteq E(G).$$

A graph is connected if and only if it has a spanning tree [1, Theorem 4.6].

Corollary 2.11. *Let X be a finite nonempty metric space and let Y be a finite nonempty ultrametric space. If $|\mathbf{B}_X| = |\mathbf{B}_Y|$, then the inequality*

$$|E(\Gamma_X)| \geq |E(\Gamma_Y)| \tag{2.8}$$

holds. Furthermore, the equality in (2.8) *occurs if and only if X is ultrametric.*

Proof. It is easy to see that Γ_X is connected. Let T be a spanning tree of Γ_X. Then we have

$$|E(\Gamma_X)| \geq |E(T)|. \tag{2.9}$$

Since

$$|V(T)| = |V(\Gamma_Y)| = |V(\Gamma_X)| = |\mathbf{B}_X| = |\mathbf{B}_Y|$$

and

$$|V(T)| = |E(T)| + 1, \quad |V(\Gamma_Y)| = |E(\Gamma_Y)| + 1,$$

inequality (2.9) implies (2.8). Now using Corollary 2.10 we obtain that

$$|E(\Gamma_X)| = |E(\Gamma_Y)|$$

holds if and only if X is an ultrametric space. □

3. Finite ultrametric spaces having maximum rigidity

In this section we describe the rigidity of finite ultrametric spaces in terms of fixed points of their isometries. The fixed point theory in ultrametric spaces was developed by mathematicians such as S. Priess-Crampe, W. A. Kirk, P. Ribenboim, N. Shahzad. (See, for example, [15, 16, 21, 22].)

Let (X, d) be a metric space and let $\mathrm{Iso}(X)$ be the group of isometries of (X, d). We say that (X, d) is *rigid* if $|\mathrm{Iso}(X)| = 1$. It is clear that (X, d) is rigid if and only if $g(x) = x$ for every $x \in X$ and every $g \in \mathrm{Iso}(X)$.

For every self-map $f\colon X \to X$ we denote by $\operatorname{Fix}(f)$ the set of fixed points of f. Using this denotation we obtain that a finite metric space (X, d) is rigid if and only if

$$\min_{g\in \operatorname{Iso}(X)} |\operatorname{Fix}(g)| = |X|.$$

Lemma 3.1. *Let (X, d) be a nonempty finite ultrametric space. Then for every set of partial self-isometries,*

$$S = \{\psi_i : B_i \to B_i : i \in I\},$$

where I is an index set, such that

$$B_i \in \mathbf{B}_X, \quad B_i \cap B_j = \varnothing$$

for all distinct $i, j \in I$, there exists an isometry $\psi \in \operatorname{Iso}(X)$ for which the restriction $\psi|_{B_i}$ equals ψ_i for every $i \in I$.

Proof. Let us define a required ψ by the rule

$$\psi(x) = \begin{cases} \psi_i(x) & \text{if } x \in B_i,\ i \in I, \\ x & \text{if } x \in X \setminus \left(\bigcup_{i\in I} B_i\right). \end{cases}$$

It follows from Lemma 2.3 that $\psi \in \operatorname{Iso}(X)$. □

Proposition 3.2. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then (X, d) is nonrigid.*

Proof. It is sufficient to construct $g \in \operatorname{Iso}(X)$ such that

$$|\operatorname{Fix}(g)| \leq |X| - 2. \tag{3.1}$$

Since X is finite, the representing tree T_X contains an internal node v such that the direct successors of v are leaves of T_v. We have the inequality

$$|L_{T_v}| \geq 2$$

because v is internal. Hence there is a fixed-point-free bijection $\psi : L_{T_v} \to L_{T_v}$, i.e.,

$$|\operatorname{Fix}(\psi)| = 0 \tag{3.2}$$

holds. The leaves of T_X is the one-point subsets of X. Identifying the leaves with their respective points of X we can define a bijection $g : X \to X$ as

$$g(x) = \begin{cases} \psi(x) & \text{if } x \in L_{T_v}, \\ x & \text{if } x \in X \setminus L_{T_v}. \end{cases} \tag{3.3}$$

Lemma 3.1 implies that $g \in \operatorname{Iso}(X)$. From (3.2) and (3.3) we obtain the equality

$$|\operatorname{Fix}(g)| = |X \setminus L_{T_v}| = |X| - |L_{T_v}|.$$

Since $|L_{T_v}| \geq 2$, inequality (3.1) follows. □

Remark 3.3. The Fibonacci space is an interesting example of a compact infinite rigid ultrametric space. (See [13, 14] for some interesting properties of the Fibonacci space.)

If a metric space (X, d) is finite, nonempty and nonrigid, then the inequality

$$\min_{g \in \mathrm{Iso}(X)} |\,\mathrm{Fix}(g)| \leq |X| - 2 \tag{3.4}$$

holds, because the existence of $|X| - 1$ fixed points for $g \in \mathrm{Iso}(X)$ implies that g is identical.

The quantity $\min_{g \in \mathrm{Iso}(X)} |\,\mathrm{Fix}(g)|$ can be considered as a measure for "rigidness" for finite metric spaces (X, d). Thus the finite ultrametric spaces satisfying the equality

$$\min_{g \in \mathrm{Iso}(X)} |\,\mathrm{Fix}(g)| = |X| - 2 \tag{3.5}$$

are as rigid as possible. Let us denote by $\mathfrak{R}$ the class of all finite ultrametric spaces (X, d) which satisfy this equality.

Recall that a rooted tree is *strictly n-ary* if every internal node has exactly n children. In the case $n = 2$ such tree is called *strictly binary*. A level of a vertex of a rooted tree (T, v^*) can be defined by the following inductive rule: The level of the root v^* is zero and if $u \in V(T)$ has a level x, then every direct successor of u has the level $x + 1$.

Theorem 3.4. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then the following statements are equivalent:*

(i) $(X, d) \in \mathfrak{R}$*;*
(ii) $|\,\mathrm{Iso}(X)| = 2$*;*
(iii) T_X *is strictly binary with exactly one inner node at each level except the last level.*

Proof. (i)⇒(ii). Let (i) hold and

$$|\,\mathrm{Iso}(X)| \neq 2.$$

By Proposition 3.2, the ultrametric space (X, d) is nonrigid. Consequently, there exist $\psi_1, \psi_2 \in \mathrm{Iso}(X)$ such that

$$\psi_1 \neq \psi_2 \quad \text{and} \quad \psi_1 \neq \mathrm{id}_X \neq \psi_2,$$

where id_X is the identical mapping of X. Since $(X, d) \in \mathfrak{R}$, we can find the sets $\{x_1^1, x_2^1\}$ and $\{x_1^2, x_2^2\}$ such that

$$\psi_i\big(x_1^i\big) = x_2^i, \quad \psi_i\big(x_2^i\big) = x_1^i, \quad i = 1, 2,$$

and

$$\big\{x_1^1, x_2^1\big\} \neq \big\{x_1^2, x_2^2\big\}. \tag{3.6}$$

Note that if

$$\big\{x_1^1, x_2^1\big\} = \big\{x_1^2, x_2^2\big\},$$

then $\psi_1 = \psi_2$ because all points of the set $X \setminus (\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\})$ are fixed points of ψ_1 and ψ_2. A short calculation shows that the set $\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\}$ contains no fixed points of the composition $\psi_1 \circ \psi_2$ and that

$$\psi_1 \circ \psi_2(x) = x$$

for every $x \in X \setminus (\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\})$. Hence we have

$$|\operatorname{Fix}(\psi_1 \circ \psi_2)| = |X| - 3 \quad \text{or} \quad |\operatorname{Fix}(\psi_1 \circ \psi_2)| = |X| - 4,$$

that contradicts equality (3.5).

(ii)⇒(iii). Let $|\operatorname{Iso}(X)| = 2$. We must prove (iii). First, we prove that

(s_1) For every inner node v of T_X, the set $Ch(v)$ of children of v contains at most one inner node and at most two leaves.

Let v be an inner node of T_X and let $v_1, v_2 \in Ch(v)$ be distinct inner nodes. Then the balls $B_1 = L_{T_{v_1}}$ and $B_2 = L_{T_{v_2}}$ are disjoint and the inequalities

$$|B_1| \geq 2, \quad |B_2| \geq 2$$

hold. By Proposition 3.2 the metric spaces (B_1, d) and (B_2, d) are nonrigid. Hence there are $\psi_1 \in \operatorname{Iso}(B_1)$ and $\psi_2 \in \operatorname{Iso}(B_2)$ such that

$$\psi_1 \neq \operatorname{id}_{B_1} \quad \text{and} \quad \psi_2 \neq \operatorname{id}_{B_2},$$

where id_{B_i} is the identical mapping of the set B_i, $i = 1, 2$. By Lemma 3.1 there are $\psi^1, \psi^2 \in \operatorname{Iso}(X)$ such that

$$\psi^1|_{B_1} = \psi_1, \quad \psi^1|_{B_2} = \operatorname{id}_{B_2}, \quad \psi^2|_{B_1} = \operatorname{id}_{B_1}, \quad \psi^2|_{B_2} = \psi_2.$$

Since $\psi^1, \psi^2 \in \operatorname{Iso}(X)$ and $\psi^1 \neq \psi^2$ and $\psi^1 \neq \operatorname{id}_X \neq \psi^2$, we have

$$|\operatorname{Iso}(X)| \geq 3,$$

contrary to $|\operatorname{Iso}(X)| = 2$. The first part of ($s_1$) is proved. In what follows we write $B = L_{T_v}$ for short. Let $G_B = G_B[X_1, \ldots, X_k]$ be the diametrical graph of the ball B. A leaf $\{x\}$ of T_v is a child of v if and only if there is $i \in \{1, \ldots, k\}$ such that $X_i = \{x\}$. Suppose that there are some three distinct leaves among the children of v. For certainty, we can assume that

$$X_1 = \{x_1\}, \quad X_2 = \{x_2\}, \quad X_3 = \{x_3\}.$$

Let $S = \{x_1, x_2, x_3\}$. Using the definition of the diametrical graphs we can easily prove that every bijection $\alpha \colon S \to S$ can be extended to an isometry of B. Consequently, by Lemma 3.1, there is an extension of α to an isometry of S. Since $\operatorname{Sym}(S)$ (the group of symmetries of S) has the order 6, we have $|\operatorname{Iso}(X)| \geq 6$, contrary to $|\operatorname{Iso}(X)| = 2$. Statement ($s_1$) follows.

To finish the proof of (iii), it suffices to show that T_X does not contain any node with exactly three children. Suppose to the contrary that v is a node of T_X with three children. Write $B = L_{T_v}$. Then we have

$$G_B = G_B[X_1, X_2, X_3].$$

Using (s_1) we can suppose that

$$|X_1| = |X_2| = 1 \quad \text{and} \quad |X_3| \geq 2.$$

Let $\{x_1\} = X_1$ and $\{x_2\} = X_2$, x_1, $x_2 \in X$, and let $S = \{x_1, x_2\}$. Let us consider $\alpha \in \operatorname{Sym}(S)$ and $\beta \in \operatorname{Iso}(X_3)$. Define $\psi : B \to B$ as

$$\psi(x) = \begin{cases} \alpha(x) & \text{if } x \in S, \\ \beta(x) & \text{if } x \in X_3. \end{cases}$$

It follows directly from Definition 2.1 that $\psi \in \mathrm{Iso}(B)$. Hence, $|\,\mathrm{Iso}(B)| \geq 4$. This inequality and Lemma 3.1 imply the inequality $|\,\mathrm{Iso}(X)| \geq 4$, contrary to $|\,\mathrm{Iso}(X)| = 2$.

(iii)⇒(i). Let (iii) hold. By Theorem 2.6, to prove (i) it suffices to show that every self-isomorphism

$$f : V(T_X) \to V(T_X)$$

of representing tree T_X has at most two points which are not fixed points. Let us consider a self-isomorphism $f\colon V(T_X) \to V(T_X)$. The root X is evidently a fixed point of f. If v_1' and v_2' are the nodes of the second level and v_2' is inner, then according to (iii), v_2' is a leaf and $f(v_1') = v_1'$ and $f(v_2') = v_2'$ because f preserves the levels, the inner nodes and the labels. Similarly, we can prove that all vertices on an arbitrary level are fixed if it is not the last level of T_X. Now it is sufficient to note that the last level contains exactly two vertices, because T_X is strictly binary and the previous level contains exactly one inner node of T_X. □

Remark 3.5. An example of the representing tree T_X for $X \in \mathfrak{R}$ with $|X| = 4$ is depicted in Figure 2.

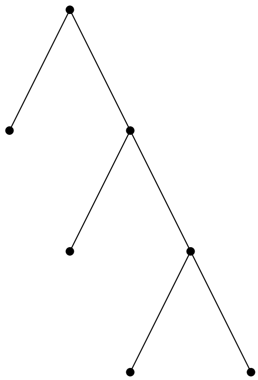

Figure 2

Using Theorem 3.4 we can obtain the following extremal property of ultrametric spaces belonging to $\mathfrak{R}$.

Corollary 3.6. *Let X and Y be finite ultrametric spaces with $|X| = |Y|$. If $Y \in \mathfrak{R}$, then the inequality*

$$|\,\mathrm{Sp}(X)| \leq |\,\mathrm{Sp}(Y)| \tag{3.7}$$

holds.

Proof. Indeed, it was proved by Gomory and Hu [9] that for every finite ultrametric space X we have the inequality

$$|\,\mathrm{Sp}(X)| \leq |X|.$$

As was shown in [20] the equality $|\operatorname{Sp}(X)| = |X|$ holds if and only if T_X is strictly binary and the labels of different internal nodes are different. Note that if Z is a finite ultrametric space with $u, v \in V(T_Z)$ and u is a child of v, then Definition 2.1, Lemma 2.2 and the definition of T_Z imply the strict inequality

$$l(u) < l(v). \tag{3.8}$$

Since, for every $Y \in \mathfrak{R}$, the representing tree T_Y has exactly one inner node at each level except the last level, inequality (3.8) shows that the labels of different internal nodes are different. Hence (3.7) holds if $Y \in \mathfrak{R}$. □

Remark 3.7. If X is a finite ultrametric space and $|X| = |\operatorname{Sp}(X)|$, then X is generally not an element of $\mathfrak{R}$ (see Figure 3).

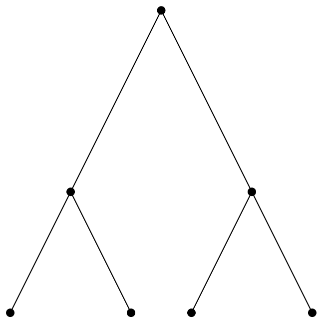

FIGURE 3. The representing tree for an ultrametric space X with $|X| = |\operatorname{Sp}(X)| = 4$.

4. Rigidness, stars and weak similarities

Following [1] denote by $K_{m,n} = K_{m,n}[X, Y]$ a complete bipartite graph such that $|X| = m$ and $|Y| = n$. In the case when $m = 1$ or $n = 1$ such graphs are called *stars*.

The following proposition gives us the first characterization of the class $\mathfrak{R}$ by stars.

Proposition 4.1. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then $X \in \mathfrak{R}$ if and only if the diametrical graphs G_B are stars for all $B \in \mathbf{B}_X$ with* $\operatorname{diam} B > 0$.

Proof. From Lemma 2.4 and the definition of strictly binary trees it follows that the proposition is a reformulation of the logic equivalence (i)⇔(iii) of Theorem 3.4. □

Definition 4.2. Let (X, d) be a metric space with a spectrum $\operatorname{Sp}(X)$ and let $r \in \operatorname{Sp}(X)$ be positive. Denote by $G_{r,X}$ a graph for which $V(G_{r,X}) = X$ and

$$\big(\{u, v\} \in E(G_{r,X})\big) \Longleftrightarrow \big(d(u, v) = r\big), \quad u, v \in X.$$

For $r = \operatorname{diam} X$ it is clear that $G_{r,X}$ is the diametrical graph of X.

Let $G = (V, E)$ be a nonempty graph, and let V_0 be the set (possibly empty) of all isolated vertices of the graph G. Denote by G' the subgraph of the graph G, generated by the set $V \backslash V_0$.

Proposition 4.3. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then $(X, d) \in \mathfrak{R}$ if and only if for every positive $r \in \mathrm{Sp}(X)$, the graph $G'_{r,X}$ is isometric to the star $K_{1,n-p}$, where p is the level of a node of T_X labeled by r and $n = |X| - 1$.*

Proof. Let $(X, d) \in \mathfrak{R}$ and let $r \in \mathrm{Sp}(X)$ be positive. Let x_n be a leaf of the last level n. Consider the path $(x_n, x_{n-1}, \ldots, x_0)$ from x_n to the root $x_0 = X$ of T_X. Using statement (iii) of Theorem 3.4 and the properties of representing trees we conclude that $x_{n-1}, \ldots, x_0$ are the only possible inner nodes of T_X and all labels of $x_n, x_{n-1}, \ldots, x_0$ are different. Hence there is a unique node, say x_p, $0 \leq p \leq n-1$, labeled by r. Let x' be a direct successor of x_p which is leaf and let x'' be another direct successor of x_p. According to Lemma 2.3 the equality $d(x, y) = r$ is possible only if $x = x'$ and $y \in L_{T_{x''}}$. Since $|L_{T_{x''}}| = n - p$, the graph $G'_{r,X}$ is isomorphic to $K_{1,n-p}$.

The converse follows from Definition 4.2, statement (iii) of Theorem 3.4 and the definition of representing tree T_X. □

The following lemma is a reformulation of Theorem 4.1 from [4].

Lemma 4.4. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$ and let G_X be the diametrical graph of X. Then the inequality*

$$|E(G_X)| \geq |X| - 1 \tag{4.1}$$

holds. The equality in (4.1) occurs if and only if G_X is isomorphic to a star.

Lemma 4.5. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. If $(X, d) \in \mathfrak{R}$, then for every $Y \subseteq X$, $|Y| \geq 2$, we have $(Y, d) \in \mathfrak{R}$.*

Proof. Let $n = |X|$. It is sufficient to prove that $(Y, d) \in \mathfrak{R}$ for the case

$$Y = X \setminus \{x_i\}, \quad x_i \in X.$$

Taking into consideration Lemma 2.4, the space $X = \{x_1, \ldots, x_n\}$ for which T_X fulfils statement (iii) of Theorem 3.4 can be the uniquely presented by sequence of nested balls

$$B_1 \subset B_2 \subset \cdots \subset B_{n-1} \subset B_n,$$

where $B_1 = \{x_1\}$, $B_i = B_{i-1} \cup \{x_i\}$, $i = 2, \ldots, n$. Let us consider the set $\mathbf{B}_Y$. It is clear that the following relations hold:

$$\overline{B}_1 \subset \overline{B}_2 \subset \cdots \subset \overline{B}_{n-2} \subset \overline{B}_{n-1}, \tag{4.2}$$

where $\overline{B}_1 = B_1, \ldots, \overline{B}_{i-1} = B_{i-1}$, $\overline{B}_i = B_{i+1} \setminus \{x_i\}, \ldots, \overline{B}_{n-1} = B_n \setminus \{x_i\}$.

Relations (4.2) and Lemma 2.4 imply that T_Y fulfils statement (iii) of Theorem 3.4. □

The next proposition gives us a new characteristic extremal property of spaces $X \in \mathfrak{R}$.

Proposition 4.6. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then the following statements are equivalent:*

(i) $X \in \mathfrak{R}$;

(ii) *the inequality*

$$|E(G_Y)| \leq |E(G_Z)| \tag{4.3}$$

holds for all $Y \subseteq X$ and all ultrametric spaces Z which satisfy the condition $|Y| = |Z| \geq 2$.

Proof. Let $X \in \mathfrak{R}$. Then by Lemma 4.5 we have $Y \in \mathfrak{R}$ for every $Y \subseteq X$ with $|Y| \geq 2$. Hence G_Y is a star, so that

$$|E(G_Y)| = |Y| - 1. \tag{4.4}$$

Lemma 4.4 implies

$$|E(G_Z)| \geq |Z| - 1 \tag{4.5}$$

if Z is a finite ultrametric space with $|Z| \geq 2$. Now (4.3) follows from (4.4), (4.5) and the equality $|Y| = |Z|$.

Let (ii) hold and let $B \in \mathbf{B}_X$. Condition (ii) and Lemma 4.4 imply that the diametrical graph G_B is a star. Hence $X \in \mathfrak{R}$ by Proposition 4.1. □

A graph $G = (V, E)$ together with a function $\omega : E \to \mathbb{R}^+$ is called a *weighted graph* with the *weight* ω. The weighted graph will be denoted by (G, ω). In the following we identify a finite ultrametric space (X, d) with a complete weighted graph (G, ω_d) such that $V(G) = X$ and

$$\omega_d(\{x, y\}) = d(x, y)$$

for all distinct $x, y \in X$.

Lemma 4.7. *Let (X, d) be an ultrametric space. Then for any cycle C in (G, ω_d) there exist at least two different edges $e_1, e_2 \in E(C)$ such that*

$$\omega_d(e_1) = \omega_d(e_2) = \max_{e \in E(C)} \omega_d(e).$$

If $|E(C)| = 3$, then Lemma 4.7 is a reformulation of the strong triangle inequality. For $|E(C)| \geq 4$, it can be proved by induction on $|E(C)|$. (For details see [5, Lemma 1].)

For a graph $G = (V, E)$ a *Hamiltonian path* is a path in G that visits every vertex of G exactly once. It is clear that a path $P \subseteq G$ is Hamiltonian if and only if P is a spanning tree of G. The following theorem gives us some characterizations of ultrametric spaces $(X, d) \in \mathfrak{R}$ via Hamiltonian paths and spanning stars of (G, ω_d).

Theorem 4.8. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then the following statements are equivalent:*

(i) $(X, d) \in \mathfrak{R}$;

(ii) *the graph (G, ω_d) has a Hamiltonian path $P = (x_1, \ldots, x_n)$ such that*

$$\omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\}) \tag{4.6}$$

for $k = 1, \ldots, n-2$, where $n = |X|$;

(iii) *the graph* (G, ω_d) *has a spanning star* S *with*

$$E(S) = \{\{y_0, y_1\}, \dots, \{y_0, y_{n-1}\}\}$$

such that

$$\omega_d(\{y_0, y_i\}) \neq \omega_d(\{y_0, y_j\}) \tag{4.7}$$

for distinct $i, j \in \{1, \dots, n-1\}$, *where* $n = |X|$.

Proof. (i)⇒(ii). Let $(X, d) \in \mathfrak{R}$ and let $n = |X|$. By statement (iii) of Theorem 3.4 the representing tree T_X has exactly one inner node at each level expect the last level and, moreover, the last level contains exactly two leaves. Hence the number of the levels of T_X is n and we can enumerate the points of X in the sequence $(x_1, \dots, x_n)$ such that, for $k = 1, \dots, n-2$, $\{x_k\}$ is the leaf on the kth level and $\{x_{n-1}\}$, $\{x_n\}$ are the leaves of the last level. From the definition of the diametrical graphs it follows that

$$\omega_d(\{x_1, x_2\}) = d(x_1, x_2) = l(X) = \operatorname{diam} X.$$

(Recall that X is the root of T_X.)

Similarly, for every $k \in \{2, \dots, n-1\}$, we have

$$\omega_d(\{x_k, x_{k+1}\}) = d(x_k, x_{k+1}) = l(v_k),$$

where v_k is the unique inner node at the kth level. Note now that if v, u are the nodes of T_X and v is a child of u, then the inequality

$$l(v) < l(u)$$

holds. It follows directly from statement (iii) of Theorem 3.4 that v_{k+1} is a child of v_k for $k = 1, \dots, n-2$ and that v_1 is a child of X. Inequality (4.6) follows.

(ii)⇒(iii). Let $P = (x_1, \dots, x_n)$ be a Hamiltonian path in (G, ω_d) such that (4.6) holds for $k = 1, \dots, n-2$. Let us define $y_i = x_{n-i}$ for $i = 0, \dots, n-1$. Using Lemma 4.7 with the cycles $C = C_i$ and $C = C_{i+1}$ such that

$$E(C_i) = \{\{y_0, y_1\}, \{y_1, y_2\}, \dots, \{y_{i-1}, y_i\}, \{y_i, y_0\}\}$$

and

$$E(C_{i+1}) = \{\{y_0, y_1\}, \{y_1, y_2\}, \dots, \{y_{i-1}, y_i\}, \{y_i, y_{i+1}\}, \{y_{i+1}, y_0\}\},$$

we obtain

$$\omega_d(\{y_0, y_i\}) < \omega_d(\{y_0, y_{i+1}\})$$

for $i = 0, \dots, n-1$. Statement (iii) follows.

(iii)⇒(i). Let (iii) hold. Then, without loss of generality, we can set

$$X = \{x_1, x_2, \dots, x_n\}$$

and

$$d(x_n, x_1) > d(x_n, x_2) > \cdots > d(x_n, x_{n-1}).$$

See Figure 4 for the case $n = 8$.

Using the strong triangle inequality we obtain that

$$d(x_1, x_i) = \operatorname{diam} X \quad \text{and} \quad d(x_i, x_j) < \operatorname{diam} X$$

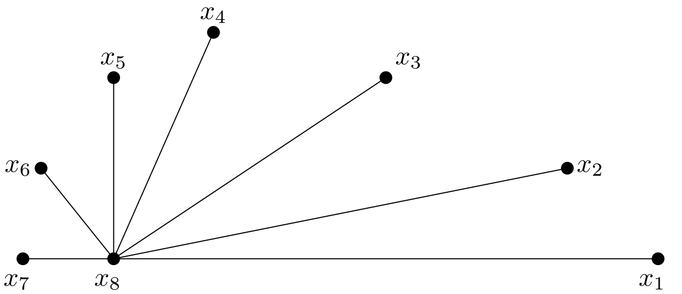

Figure 4. A spanning star in (G, ω_d) for $(X, d) \in \mathfrak{R}$ with $|X| = 8$.

for all distinct $i, j \in \{2, \ldots, n\}$. Similarly, we see that

$$d(x_k, x_i) = \operatorname{diam}\{x_k, x_{k+1}, \ldots, x_n\} \quad \text{and} \quad d(x_i, x_j) < d(x_k, x_i)$$

if $i, j \in \{k+1, \ldots, n\}$.

Hence statement (iii) of Theorem 3.4 holds. The implication (iii)⇒(i) follows. □

Recall that a cycle C in a graph G is Hamiltonian if $V(C) = V(G)$.

Corollary 4.9. *Let (X, d) be a finite ultrametric space with $|X| \geq 3$. Then $(X, d) \in \mathfrak{R}$ if and only if the weighted graph (G, ω_d) contains a Hamiltonian cycle $(x_1, \ldots, x_n)$ such that*

$$\omega_d(\{x_1, x_2\}) = \omega_d(\{x_n, x_1\}) = \max_{e \in E(C)} \omega_d(e)$$

and

$$\omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\})$$

for $k = 1, \ldots, n-2$.

The proof is immediate from statement (ii) of Theorem 4.8 and Lemma 4.7. The next lemma is a particular case of Theorem 7 from [5].

Lemma 4.10. *Let (S, ω) be a weighted star with $\omega(e) > 0$ for every $e \in E(S)$. Then the following conditions are equivalent:*

1. *there is a unique ultrametric space (X, d) such that $X = V(S)$ and $d(x, y) = \omega(\{x, y\})$ for every $\{x, y\} \in E(S)$;*
2. *the weight $\omega \colon E(S) \to \mathbb{R}^+$ is an injective function.*

This lemma and statement (iii) of Theorem 4.8 give us the following.

Corollary 4.11. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then $(X, d) \in \mathfrak{R}$ if and only if (G, ω_d) contains a spanning star S such that, for every ultrametric $\rho : X \times X \to \mathbb{R}$, we have*

$$\big(\forall e \in E(C) : \omega_d(e) = \omega_\rho(e)\big) \Longrightarrow (\rho = d).$$

Remark 4.12. Using Theorem 2.5 from [7] and Theorem 7 from [5] we can show that the statement

"There is a path P in (G, ω_d) such that

$$(\forall e \in E(P) : \omega_d(e) = \omega_\rho(e)) \Rightarrow (\rho = d)$$

holds for every ultrametric $\rho : X \times X \to \mathbb{R}$"

is equivalent to

$$|X| = |\operatorname{Sp}(X)|.$$

Hence we cannot use any Hamiltonian path instead of the star in Corollary 4.11.

Recall that a function Φ from a metric space (X, d) to a metric space (Y, ρ) is a similarity if there is $\lambda > 0$ such that

$$\lambda\big(d(x, y)\big) = \rho\big(\Phi(x), \Phi(y)\big)$$

for all $x, y \in X$.

Definition 4.13. Let (X, d) and (Y, ρ) be metric spaces. A bijective mapping $\Phi\colon X \to Y$ is a weak similarity if there is a strictly increasing bijection $f\colon \operatorname{Sp}(X) \to \operatorname{Sp}(Y)$ such that the equality

$$f\big(d(x, y)\big) = \rho\big(\Phi(x), \Phi(y)\big) \tag{4.8}$$

holds for all $x, y \in X$.

If $\Phi\colon X \to Y$ is a weak similarity, we say that X and Y are *weakly similar*. If (X, d) is a finite metric space, then every weak similarity $\Phi\colon X \to X$ is an isometry. The notion of weak similarity was introduced in [6] for more general case of semimetric spaces in a slightly different form.

Proposition 4.14. *Let $(X, d) \in \mathfrak{R}$ and let (Y, ρ) be a metric space. If (X, d) and (Y, ρ) are weakly similar, then $(Y, \rho) \in \mathfrak{R}$.*

Proof. If $\Phi : X \to Y$ is a weak similarity, then $\mathbf{B}_Y = \{\Phi(B) : B \in \mathbf{B}_X\}$ and the mapping

$$\mathbf{B}_X \ni B \mapsto \Phi(B) \in \mathbf{B}_Y$$

is a bijection. It is clear also that

$$(A \subseteq C) \Longleftrightarrow \big(\Phi(A) \subseteq \Phi(C)\big)$$

holds for all $A \subseteq X$ and $C \subseteq X$. Hence the graphs (Γ_X, X) and (Γ_Y, Y) are isomorphic. Using Theorem 2.8 we obtain that Y is ultrametric and $\overline{T}_Y$ is isomorphic to $\overline{T}_X$. Now $Y \in \mathfrak{R}$ follows from statement (iii) of Theorem 3.4. $\square$

Proposition 4.15. *Let $X, Y \in \mathfrak{R}$. Then the following statements are equivalent:*

(i) *the trees $\overline{T}_X$ and $\overline{T}_Y$ are isomorphic as rooted trees;*
(ii) *X and Y are weakly similar;*
(iii) *the equality $|X| = |Y|$ holds.*

Proof. The implication (i)⇒(iii) is immediate. Analysis similar to that in the proof of Proposition 4.14 shows that (ii)⇒(i) holds.

Let us prove (iii)⇒(ii). Let $|X| = |Y|$. Write $n = |X| = |Y|$. Statement (ii) of Theorem 4.8 implies that there are Hamiltonian paths

$$(x_1, \ldots, x_n) \subseteq (G, \omega_d) \quad \text{and} \quad (y_1, \ldots, y_n) \subseteq (G, \omega_\rho)$$

such that

$$\omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\}) \tag{4.9}$$

and

$$\omega_\rho(\{x_k, x_{k+1}\}) > \omega_\rho(\{x_{k+1}, x_{k+2}\}) \tag{4.10}$$

for $k = 1, \ldots, n-2$. The Gomory–Hu inequality implies that

$$\mathrm{Sp}(X) = \big\{d(x_k, x_{k+1}) : k = 1, \ldots, n-1\big\} \cup \{0\}$$

and

$$\mathrm{Sp}(Y) = \big\{\rho(y_k, y_{k+1}) : k = 1, \ldots, n-1\big\} \cup \{0\}.$$

Let us define the functions $\Phi : X \to Y$ and $f : \mathrm{Sp}(X) \to \mathrm{Sp}(Y)$ such that

$$\Phi(x_i) = y_i, \quad f(0) = 0, \quad f\big(d(x_k, x_{k+1})\big) = \rho\big(x_k, x_{k+1}\big)$$

for $k = 1, \ldots, n-1$.

Inequalities (4.9) and (4.10) imply that f is strictly increasing. Moreover, it is clear that Φ and f are bijective. Now using Lemma 4.7 we obtain that equality (4.8) holds for all $x, y \in X$. The implication (iii)⇒(ii) follows. □

Remark 4.16. Let $a, b > 0$. If (X, d) and (Y, ρ) are ultrametric spaces for which

$$d(x_1, x_2) = a \quad \text{and} \quad \rho(y_1, y_2) = b$$

for all distinct $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then (X, d) and (Y, ρ) are weakly similar if and only if $|X| = |Y|$. For these spaces we have also $(X, d) \notin \mathfrak{R}$ and $(Y, \rho) \notin \mathfrak{R}$ if $|X|, |Y| \geq 3$.

It seems to be interesting to find a "representing tree description" of the classes of finite ultrametric spaces X, Y for which the condition $|X| = |Y|$ implies that X and Y are weakly similar.

Acknowledgments

The research of the first and second authors was supported as a part of EUMLS project with grant agreement PIRSES – GA – 2011 – 295164. Publications are based on the research provided by the grant support of the State Fund for Fundamental Research (project F71/20570).

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory.* Grad. Texts Math. 244, Springer, New York, 2008.

[2] E. D. Demaine, G. M. Landau and O. Weimann, *On Cartesian trees and range minimum queries.* In: Proceedings of the 36th International Colloquium, ICALP 2009, Rhodes, Greece, 2009, Part I, Lecture Notes in Comput. Sci. 5555, Springer, Berlin, 2009, 341–353.

[3] R. Diestel, *Graph Theory.* 3rd ed., Grad. Texts in Math. 173, Springer, Berlin, 2005.

[4] D. Dordovskyi, O. Dovgoshey and E. Petrov, *Diameter and diametrical pairs of points in ultrametric spaces.* p-Adic Numbers Ultrametric Anal. Appl. **3** (2011), 253–262.

[5] O. Dovgoshey and E. Petrov, *Subdominant pseudoultrametric on graphs.* Sb. Math. **204** (2013), 1131–1151 [Translated from Mat. Sb. **204** (2013), 51–72].

[6] O. Dovgoshey and E. Petrov, *Weak similarities of metric and semimetric spaces.* Acta Math. Hungar. **141** (2013), 301–319.

[7] O. Dovgoshey, E. Petrov and H.-M. Teichert, *On spaces extremal for the Gomory-Hu inequality.* p-Adic Numbers Ultrametric Anal. Appl. **7** (2015), 133–142.

[8] M. Fiedler, *Ultrametric sets in Euclidean point spaces.* Electron. J. Linear Algebra **3** (1998), 23–30.

[9] R. E. Gomory and T. C. Hu, *Multi-terminal network flows.* J. Soc. Indust. Appl. Math. **9** (1961), 551–570.

[10] V. Gurvich and M. Vyalyi, *Characterizing (quasi-)ultrametric finite spaces in terms of (directed) graphs.* Discrete Appl. Math. **160** (2012), 1742–1756.

[11] V. Gurvich and M. Vyalyi, *Ultrametrics, trees, flows, and bottleneck arcs.* Mathematical Education, series 3, vol. 16, MCCME, Moscow, 2012 (In Russian).

[12] J. E. Holly, *Pictures of Ultrametric Spaces, the p-Adic Numbers, and Valued Fields.* Amer. Math. Monthly **108** (2001), 721–728.

[13] B. Hughes, *Trees and ultrametric spaces: A categorical equivalence.* Adv. Math. **189** (2004), 148–191.

[14] B. Hughes, *Trees, ultrametrics, and noncommutative geometry.* Pure Appl. Math. Q. **8** (2012), 221–312.

[15] W. A. Kirk and N. Shahzad, *Some fixed point results in ultrametric spaces.* Topology Appl. **159** (2012), 3327–3334.

[16] W. A. Kirk and N. Shahzad, *Fixed point theory in distance spaces.* Springer, Cham, 2014.

[17] A. J. Lemin, *On Gelgfand's problem concerning graphs, lattices, and ultrametric spaces.* AAA62 – Workshop on General Algebra, 2001, 12–13.

[18] A. J. Lemin, *The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices* $\mathbf{LAT}^*$. Algebra Universalis **50** (2003), 35–49.

[19] E. A. Petrov, *Ball-preserving mappings of finite ulrametric spaces.* Proceedings of IAMM **26** (2013), 150–158 (In Russian).

[20] E. Petrov and A. Dovgoshey, *On the Gomory-Hu inequality*. J. Math. Sci. **198** (2014), 392–411 [Translated from Ukr. Mat. Visn. **10** (2013), 469–496].

[21] S. Priess-Crampe and P. Ribenboim, *Fixed point and attractor theorems for ultrametric spaces*. Forum Math. **12** (2000), 53–64.

[22] S. Priess-Crampe and P. Ribenboim, *The approximation to a fixed point*. J. Fixed Point Theory Appl. **14** (2013), 41–53.

O. Dovgoshey
Division of Applied Problems in Contemporary Analysis
Institute of Mathematics of NASU
Tereshenkivska str. 3
Kyiv 01601
Ukraine
e-mail: aleksdov@mail.ru

E. Petrov
Division of Applied Problems in Contemporary Analysis
Institute of Mathematics of NASU
Tereshenkivska str. 3
Kyiv 01601
Ukraine
e-mail: eugeniy.petrov@gmail.com

H.-M. Teichert
Institute of Mathematics
University of Lübeck
Ratzeburger Allee 160
23562 Lübeck
Germany
e-mail: teichert@math.uni-luebeck.de