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# Viscosity approximation methods for solving fixed-point problems and split common fixed-point problems

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*Dedicated to Professor Do Hong Tan on the occasion of his 80th birthday*

**Abstract.** In this paper, we introduce the strong convergence theorem for the viscosity approximation methods for solving the split common fixed-point problem in Hilbert spaces. As a consequence, we obtain strong convergence theorems for split variational inequality problems for Lipschitz continuous and monotone operators and split common null point problems for maximal monotone operators. Our results improve and extend the corresponding results announced by many others.

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**Keywords.** Fixed-point theory, Split common fixed-point problem, Split feasibility problem, Split variational inequality problem, Split null point problem.

## 1. Introduction

Let $C$ and $Q$ be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. The split feasibility problem (SFP) is formulated as finding a point $x$ satisfying the property

$$x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_1$ is a bounded linear operator. Recently, the SFP has been widely studied by many authors (see [1,14,16,17,19]), due to its application in signal processing [2]. In particular, Byrne [1] introduced the so-called CQ algorithm. For $x_0 \in H_1$ and define the iteration $\{x_n\}$ as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \tag{1.1}$$

where $0 < \gamma < \dfrac{2}{\rho(A^*A)}$ and where $P_C$ denotes the projector onto $C$ and $\rho(A^*A)$ is the spectral radius of the operator $A^*A$. It is known that the

CQ algorithm converges weakly to a solution of the SFP if such a solution exists.

In the case, where both $C$ and $Q$ consist of fixed-point sets of some nonlinear operators, the SFP is known as the split common fixed-point problem (SCFP). More specifically, the SCFP is to find

$$x \in Fix(U) \text{ such that } Ax \in Fix(T),$$

where $Fix(U)$ and $Fix(T)$ are the fixed-point sets of $U : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively. We denote the solution set of the SCFP by

$$\Gamma := \{x \in H_1 : x \in Fix(U) \text{ and } Ax \in Fix(T)\}.$$

When $U$ and $T$ are directed operators, Censor and Segal [5] proposed and proved the convergence of the following algorithm in the setting of the finite-dimensional spaces:

$$x_{n+1} = U(I - \gamma A^*(I - T)Ax_n). \tag{1.2}$$

Note that a class of directed operators includes the metric projection. Therefore, the results of Censor and Segal recover Byrne's CQ algorithm. Moudafi [11] introduced the following algorithm:

$$\begin{cases} u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n \end{cases} \tag{1.3}$$

to solve the SCFP for demicontractive operators and he obtained the weak convergence. It is known that demicontractive operators include the directed operators. Hence, Moudafi's algorithm is an extension of the algorithm (1.2).

Recently, Moudafi [12] and Zhao and He [20] proposed the viscosity approximation methods for solving the SCFP for quasi-nonexpansive operators. Motivated by their work, in this paper, we introduce a generalized algorithm to solve the SCFP and their results are as our consequences.

## 2. Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ is written as $x_n \to x$ as $n \to \infty$.

For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|\ \forall y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive.

**Lemma 2.1** [6] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $z \in C$. Then, $z = P_C x \iff \langle x - z, z - y\rangle \geq 0\ \ \forall y \in C$.*

**Definition 2.2** [6] Assume that $T : H \to H$ is a nonlinear operator. Then, $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $H$, the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \Longrightarrow x \in Fix(T).$$

**Definition 2.3** Let $T : H \to H$ be an operator with $Fix(T) \neq \emptyset$. Then

- $T : H \to H$ is called firmly nonexpansive if
$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle,$$
or equivalently
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2;$$
- $T : H \to H$ is called directed if
$$\langle z - Tx, x - Tx\rangle \leq 0 \quad \forall z \in Fix(T), x \in H,$$
or equivalently
$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H;$$
- $T : H \to H$ is called $\alpha$-strongly quasi-nonexpansive with $\alpha > 0$ if
$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H,$$
or equivalently
$$\langle Tx - x, x - z\rangle \leq \frac{-1 - \alpha}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H;$$
- $T : H \to H$ is called quasi-nonexpansive if
$$\|Tx - z\| \leq \|x - z\| \quad \forall z \in Fix(T), x \in H;$$
- $T : H \to H$ is called $\beta$-demicontractive with $0 \leq \beta < 1$ if
$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|(I - T)x\|^2 \quad \forall z \in Fix(T), x \in H,$$
or equivalently
$$\langle x - z, Tx - x\rangle \leq \frac{\beta - 1}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H. \tag{2.1}$$

To prove its convergence, we will need the two following lemmas.

**Lemma 2.4** [9] *Let $\{a_n\}$ be a sequence of non-negative real numbers, such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$, such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$, such that $\lim_{k\to\infty} m_k = \infty$, and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*
$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$
*In fact, $m_k$ is the largest number $n$ in the set $\{1, 2, \ldots, k\}$, such that $a_n < a_{n+1}$.*

**Lemma 2.5** [13,15] *Let $\{a_n\}$ be sequences of non-negative real numbers, such that*
$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$
*where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ are a sequence, such that*
*(a)* $\sum_{n=0}^{\infty} a_n = \infty$;
*(b)* $\limsup_{n\to\infty} b_n = 0$.
*Then,* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.6** [18] *If $U : H \to H$ is $\beta_1$-strongly quasi-nonexpansive and $T : H \to H$ is $\beta_2$-strongly quasi-nonexpansive with $Fix(U) \cap Fix(T) \neq \emptyset$, then $UT$ is $\dfrac{\beta_1\beta_2}{\beta_1+\beta_2}$-strongly quasi-nonexpansive and $Fix(UT) = Fix(U) \cap Fix(T)$.*

**Lemma 2.7** *Let $U : H \to H$ be a $\beta$-demicontractive operator and $T : H \to H$ be a $\alpha_1$-strongly quasi-nonexpansive operator with $\beta < \alpha_1$. Then, the operator $UT$ is $\dfrac{\alpha_1\beta}{\alpha_1-\beta}$ demicontractive and $Fix(U) \cap Fix(T) = Fix(UT)$.*

*Proof* It is suffices to show that $Fix(UT) \subset Fix(U) \cap Fix(T)$. Let $p \in Fix(UT)$, it is enough to show that $p \in Fix(T)$. We take $z \in Fix(U) \cap Fix(T)$, we have

$$\begin{aligned}
\|p-z\|^2 &= \|UTp-z\|^2 \\
&\leq \|Tp-z\|^2 + \beta\|UTp-Tp\|^2 \\
&\leq \|p-z\|^2 - \alpha_1\|Tp-p\|^2 + \beta\|UTp-Tp\|^2 \\
&= \|p-z\|^2 - \alpha_1\|Tp-p\|^2 + \beta\|Tp-p\|^2 \\
&= \|p-z\|^2 - (\alpha_1-\beta)\|Tp-p\|^2.
\end{aligned}$$

This implies that $Tp = p$, that is, $p \in Fix(T)$. Therefore, $Fix(U) \cap Fix(T) = Fix(UT)$.

Take $z \in Fix(U) \cap Fix(T), x \in H$, let $a := \|x-z\|$, $b := \|Tx-z\|$, $c := \|UTx-z\|$, we have $\|a-c\| = \|UTx-x\|$, $\|a-b\| = \|Tx-x\|$, $\|b-c\| = \|UTx-Tx\|$. Since the definition of $U$ and $T$, we obtain $\|b\|^2 \leq \|a\|^2 - \alpha_1\|a-b\|^2$ and $\|c\|^2 \leq \|b\|^2 + \beta\|b-c\|^2$. This implies

$$\begin{aligned}
-2\alpha_1\langle a,b\rangle &\leq (1-\alpha_1)\|a\|^2 - (1+\alpha_1)\|b\|^2, \\
2\beta\langle b,c\rangle &\leq (1-\beta)\|b\|^2 - (1-\beta)\|c\|^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
0 &\leq \|\alpha_1 a - (\alpha_1-\beta)b - \beta c\|^2 \\
&= \alpha_1^2\|a\|^2 + (\alpha_1-\beta)^2\|b\|^2 + \beta^2\|c\|^2 - 2\alpha_1(\alpha_1-\beta)\langle a,b\rangle \\
&\quad + 2\beta(\alpha_1-\beta)\langle b,c\rangle - 2\alpha_1\beta\langle a,c\rangle \\
&= \alpha_1^2\|a\|^2 + (\alpha_1-\beta)^2\|b\|^2 + \beta^2\|c\|^2 \\
&\quad + (\alpha_1-\beta)\left[(1-\alpha_1)\|a\|^2 - (1+\alpha_1)\|b\|^2\right] \\
&\quad + (\alpha_1-\beta)\left[(1-\beta)\|b\|^2 - (1-\beta)\|c\|^2\right] - 2\alpha_1\beta\langle a,c\rangle \\
&= (\alpha_1 + \alpha_1\beta - \beta)\|a\|^2 + (-\alpha_1 + \alpha_1\beta + \beta)\|c\|^2 - 2\alpha_1\beta\langle a,c\rangle \\
&= (\alpha_1-\beta)(1 + \frac{\alpha_1\beta}{\alpha_1-\beta})\|a\|^2 - (\alpha_1-\beta)(1 - \frac{\alpha_1\beta}{\alpha_1-\beta})\|c\|^2 - 2\alpha_1\beta\langle a,c\rangle \\
&= (\alpha_1-\beta)\left(\|a\|^2 - \|c\|^2 + \frac{\alpha_1\beta}{\alpha_1-\beta}\|a-c\|^2\right).
\end{aligned}$$

Thus

$$\|c\|^2 \leq \|a\|^2 + \frac{\alpha_1\beta}{\alpha_1 - \beta}\|a - c\|^2,$$

that is, the operator $UT$ is $\frac{\alpha_1\beta}{\alpha_1 - \beta}$-demicontractive. □

**Lemma 2.8** *Let $U : H \to H$ is $\beta$ demicontractive with $F(U) \neq \emptyset$ and set $U_\lambda = (1-\lambda)I + \lambda U$, $\lambda \in (0, 1-\beta)$ then*

*(a) $Fix(U) = Fix(U_\lambda)$;*

*(b) $\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{1}{\lambda}(1-\beta-\lambda)\|(I - U_\lambda)x\|^2 \quad \forall x \in H, z \in Fix(U)$;*

*(c) $F(U)$ is a closed convex subset of $H_1$.*

*Proof* (a) It is obvious.

(b) We have

$$\begin{aligned}
\|U_\lambda x - z\|^2 &= \|(1-\lambda)x + \lambda Ux - z\|^2 \\
&= \|(x - z) + \lambda(Ux - x)\|^2 \\
&= \|x - z\|^2 + 2\lambda\langle x - z, Ux - x\rangle + \lambda^2\|Ux - x\|^2 \\
&\leq \|x - z\|^2 + \lambda(\beta - 1)\|Ux - x\|^2 + \lambda^2\|Ux - x\|^2 \\
&= \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2 \\
&= \|x - z\|^2 - \frac{1}{\lambda}(1 - \beta - \lambda)\|(I - U_\lambda)x\|^2.
\end{aligned}$$

(c) It is a consequence of Proposition 1 in [18]. □

**Lemma 2.9** *Let $T : H_2 \to H_2$ be a $\mu$-demicontractive operator, $A : H_1 \to H_2$ be a linear bounded operator with $L = \|A^*A\|$. For a positive real number $\gamma$, define the operator $V : H_1 \to H_1$ by*

$$V := I + \gamma A^*(T - I)A.$$

*Then:*

*(a) for all $x \in H_1$ and $z \in A^{-1}(Fix(T))$,*

$$\|Vx - z\|^2 \leq \|x - z\|^2 - \frac{1}{\gamma L}(1 - \mu - \gamma L)\|(I - V)x\|^2.$$

*(b) for all $x \in H_1$ and $z \in A^{-1}(Fix(T))$,*

$$\|Vx - z\|^2 \leq \|x - z\|^2 - \gamma(1 - \mu - \gamma L)\|(I - T)Ax\|^2.$$

*(c) $x \in Fix(V)$ if $Ax \in Fix(T)$ provided that $\gamma \in (0, \frac{1-\mu}{L})$.*

*Proof* (a) Given $x \in H_1$ and $z \in A^{-1}(Fix(T))$, we have

$$\begin{aligned}
\langle A^*(I - T)Ax, x - z\rangle &= \langle (I - T)Ax, Ax - Az\rangle \\
&\geq \frac{1-\mu}{2}\|(I - T)Ax\|^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}\|A^*(I-T)Ax\|^2 &= \langle A^*(I-T)Ax, A^*(I-T)Ax\rangle \\ &= \langle (I-T)Ax, AA^*(I-T)Ax\rangle \\ &\le L\|(I-T)Ax\|^2.\end{aligned}$$

Thus

$$\langle A^*(I-T)Ax, x-z\rangle \ge \frac{1-\mu}{2L}\|A^*(I-T)Ax\|^2.$$

We have

$$\begin{aligned}\|Vx-z\|^2 &= \|x-\gamma A^*(I-T)Ax-z\|^2 \\ &= \|x-z\|^2 - 2\gamma\langle x-z, A^*(I-T)Ax\rangle + \gamma^2\|A^*(I-T)Ax\|^2 \\ &\le \|x-z\|^2 - 2\gamma\frac{1-\mu}{2L}\|A^*(I-T)Ax\|^2 + \gamma^2\|A^*(I-T)Ax\|^2 \\ &= \|x-z\|^2 - \frac{\gamma}{L}(1-\mu-\gamma L)\|A^*(I-T)Ax\|^2 \\ &= \|x-z\|^2 - \frac{1}{\gamma L}(1-\mu-\gamma L)\|\gamma A^*(I-T)Ax\|^2 \\ &= \|x-z\|^2 - \frac{1}{\gamma L}(1-\mu-\gamma L)\|(I-V)x\|^2.\end{aligned}$$

(b) Given $x \in H_1$ and $z \in A^{-1}(Fix(T))$, we have

$$\begin{aligned}\|Vx-z\|^2 &= \|x+\gamma A^*(T-I)Ax-z\|^2 \\ &= \|x-z\|^2 + 2\gamma\langle x-z, A^*(T-I)Ax\rangle + \gamma^2\|A^*(T-I)Ax\|^2 \\ &= \|x-z\|^2 + 2\gamma\langle Ax-Az, (T-I)Ax\rangle \\ &+ \gamma^2\langle A^*(T-I)Ax, A^*(T-I)Ax\rangle \\ &\le \|x-z\|^2 + \gamma(-1+\mu)\|(T-I)Ax\|^2 \\ &+ \gamma^2\langle AA^*(T-I)Ax, (T-I)Ax\rangle \\ &\le \|x-z\|^2 + \gamma(-1+\mu)\|(T-I)Ax\|^2 + \gamma^2\|AA^*\|\|(T-I)Ax\|^2 \\ &= \|x-z\|^2 - \gamma(1-\mu-\gamma L)\|(T-I)Ax\|^2.\end{aligned}$$

(c) It is obvious that $Ax \in Fix(T)$ then $x \in Fix(V)$. We show the converse, let $x \in Fix(V)$ and $z \in A^{-1}(Fix(T))$, we have

$$\|x-z\|^2 = \|Vx-z\|^2 \le \|x-z\|^2 - \gamma(1-\mu-\gamma L)\|(T-I)Ax\|^2.$$

Since $\gamma \in (0, \frac{1-\mu}{L})$, we obtain $(T-I)Ax = 0$, that is, $Ax \in Fix(T)$.

## 3. Main results

**Theorem 3.1** *Let $U : H \to H$ be a $\alpha$-strongly quasi-nonexpansive operator such that $I-U$ is demiclosed at zero. Suppose that $f : H \to H$ is a contraction with constant $\rho \in (0, 1)$. Let $\{x_n\}$ be a sequence in $H$ defined by*

$$x_0 \in H, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ux_n, \tag{3.1}$$

*where the sequence* $\{\alpha_n\}$ *satisfies the following conditions:*

$$\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

*Then* $\{x_n\}$ *strongly converges to an element* $q \in Fix(U)$*, where* $q = P_{Fix(U)} \circ f(q)$.

*Proof* First, we note that $Fix(U)$ is a closed convex subset by Lemma 2.8. Thus, the mapping $P_{Fix(U)} \circ f : H \to H$ is a contraction. By Banach' s contraction principle that there exists a unique element $q \in H$, such that $q = P_{Fix(U)} \circ f(q)$. In particular, $q \in Fix(U)$ and

$$\langle (I - f)(q), q - z \rangle \le 0 \ \ \forall z \in Fix(U). \tag{3.2}$$

Now, we show that $\{x_n\}$ is bounded. Indeed, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n f(x_n) + (1-\alpha_n)Ux_n - q\| \\ &= \|\alpha_n(f(x_n) - x_n) + (1-\alpha_n)(Ux_n - q)\| \\ &\le \alpha_n\|f(x_n) - q\| + (1-\alpha_n)\|Ux_n - q\| \\ &\le \alpha_n\|f(x_n) - f(q)\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|Ux_n - q\| \\ &\le \alpha_n\rho\|x_n - q\| + \alpha_n\|f(q) - q\| + (1-\alpha_n)\|x_n - q\| \\ &= [1 - \alpha_n(1-\rho)]\|x_n - q\| + \alpha_n\|f(q) - q\| \\ &= [1 - \alpha_n(1-\rho)]\|x_n - q\| + \alpha_n\frac{\|f(q) - q\|}{1-\rho} \\ &\le \max\left\{\|x_n - q\|, \frac{\|f(q) - q\|}{1-\rho}\right\} \\ &\le \cdots \le \max\left\{\|x_0 - q\|, \frac{\|f(q) - q\|}{1-\rho}\right\}. \end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded and $\{f(x_n)\}, \{Ux_n\}$ are bounded.

On the other hand, we get

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(Ux_n - x_n).$$

This implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \alpha_n^2\|f(x_n) - x_n\|^2 + (1-\alpha_n)^2\|Ux_n - x_n\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - x_n, Ux_n - x_n\rangle \\ &\le \alpha_n^2\|f(x_n) - x_n\|^2 + (1-\alpha_n)\|Ux_n - x_n\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - x_n, Ux_n - x_n\rangle \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|x_{n+1}-q\|^2-\|x_n-q\|^2-\|x_{n+1}-x_n\|^2 &= 2\langle x_{n+1}-x_n, x_n-q\rangle\\ &= 2\alpha_n\langle f(x_n)-x_n, x_n-q\rangle\\ &\quad +2(1-\alpha_n)\langle Ux_n-x_n, x_n-q\rangle\\ &\le 2\alpha_n\langle f(x_n)-x_n, x_n-q\rangle\\ &\quad -(1-\alpha_n)(1+\alpha)\|x_n-Ux_n\|^2.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2-\|x_n-q\|^2 \le\ & \alpha_n^2\|f(x_n)-x_n\|^2+2\alpha_n\langle f(x_n)-x_n, x_n-q\rangle\\ &+2\alpha_n(1-\alpha_n)\langle f(x_n)-x_n, Ux_n-x_n\rangle\\ &-\alpha(1-\alpha_n)\|x_n-Ux_n\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha(1-\alpha_n)\|x_n-Ux_n\|^2 \le\ & \alpha_n^2\|f(x_n)-x_n\|^2+2\alpha_n\langle f(x_n)-x_n, x_n-q\rangle\\ &+\|x_n-q\|^2-\|x_{n+1}-q\|^2.\end{aligned} \tag{3.5}$$

Let us consider the following two cases.

**Case 1** There exists $N\in\mathbb{N}$, such that $\|x_{n+1}-q\|^2\le\|x_n-q\|^2$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\|x_n-q\|^2$ exists. Since (3.5), we have

$$\lim_{n\to\infty}\|x_n-Ux_n\|=0. \tag{3.6}$$

Now, we show that

$$\limsup_{n\to\infty}\langle x_n-q, f(q)-q\rangle\le 0. \tag{3.7}$$

Indeed, we take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle x_n-p, f(p)-p\rangle=\lim_{j\to\infty}\langle x_{n_j}-q, f(q)-q\rangle.$$

We may assume that $x_{n_j}\rightharpoonup x^*$. By (3.6), we have $x^*\in Fix(U)$. Thus

$$\limsup_{n\to\infty}\langle x_n-q, f(q)-q\rangle=\lim_{j\to\infty}\langle x_{n_j}-q, f(q)-q\rangle=\langle x^*-q, f(q)-q\rangle\le 0.$$

Next, we will show that $x_n\to q$. we get

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q)+(1-\alpha_n)(Ux_n-q)\|^2\\ &= (1-\alpha_n)^2\|Ux_n-q)\|^2+\alpha_n^2\|f(x_n)-q\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\langle Ux_n-q, f(x_n)-q\rangle\\ &\le (1-\alpha_n)^2\|x_n-q\|^2+\alpha_n^2\|f(x_n)-q\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\langle Ux_n-q, f(x_n)-f(q)\rangle\\ &\quad +2\alpha_n(1-\alpha_n)\langle Ux_n-q, f(q)-q\rangle\\ &\le (1-\alpha_n)^2\|x_n-q\|^2+\alpha_n^2\|f(x_n)-q\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\rho\|x_n-q\|^2+2\alpha_n(1-\alpha_n)\langle Ux_n-p, f(q)-q\rangle\\ &= [1-\alpha_n(2-\alpha_n-2\rho(1-\alpha_n))]\|x_n-q\|^2+\alpha_n^2\|f(x_n)-q\|^2\end{aligned}$$

$$+ 2\alpha_n(1-\alpha_n)\langle Ux_n - q, f(q) - q\rangle$$
$$= (1-\gamma_n)\|x_n - q\|^2 + \gamma_n\delta_n,$$

where

$$\gamma_n = \alpha_n(2-\alpha_n - 2\rho(1-\alpha_n)),$$
$$\delta_n = \frac{\alpha_n\|f(x_n) - q\|^2 + 2(1-\alpha_n)\langle Ux_n - q, f(q) - q\rangle}{2-\alpha_n-2\rho(1-\alpha_n)}.$$

We have $\gamma_n \to 0, \sum_{n=1}^{\infty}\gamma_n = \infty$, and by (3.7), we get $\limsup_{n\to\infty}\delta_n \leq 0$. By Lemma 2.5, we conclude that $x_n \to q$.

**Case 2** There exists a subsequence $\{\|x_{n_j} - q\|^2\}$ of $\{\|x_n - q\|^2\}$, such that $\|x_{n_j} - q\|^2 < \|x_{n_j+1} - q\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.4 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$, such that $\lim_{k\to\infty} m_k = \infty$, and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - q\|^2 \leq \|x_{m_k+1} - q\|^2 \quad \text{and} \quad \|x_k - q\|^2 \leq \|x_{m_k} - q\|^2. \tag{3.8}$$

Similarly, we get

$$\lim_{n\to\infty}\|x_{m_k} - Ux_{m_k}\| = 0, \tag{3.9}$$

$$\limsup_{k\to\infty}\langle x_{m_k} - q, f(q) - q\rangle \leq 0, \tag{3.10}$$

and

$$\|x_{m_k+1} - q\|^2 \leq (1-\gamma_{m_k})\|x_{m_k} - q\|^2 + \gamma_{m_k}\delta_{m_k}, \tag{3.11}$$

where

$$\gamma_{m_k} = \alpha_{m_k}(2-\alpha_{m_k} - 2\rho(1-\alpha_{m_k})),$$
$$\delta_{m_k} = \frac{\alpha_{m_k}\|f(x_{m_k}) - q\|^2 + 2(1-\alpha_{m_k})\langle Ux_{m_k} - q, f(q) - q\rangle}{2-\alpha_{m_k}-2\rho(1-\alpha_{m_k})}.$$

By Lemma 2.5, we obtain $x_{m_k} \to q$. By (3.8), we get $\|x_k - q\| \leq \|x_{m_k} - q\|\ \forall k \in \mathbb{N}$. Therefore, $x_k \to q$. □

**Corollary 3.2** *Let $U : H \to H$ be a $\beta$ demicontractive, such that $I - U$ is demiclosed at zero. Suppose that $f : H \to H$ is a contraction with constant $\rho \in (0,1)$. Let $\{x_n\}$ be a sequence in $H$ defined by*

$$x_0 \in H, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda x_n, \tag{3.12}$$

*where the parameter $\lambda$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\lambda \in (0, 1-\beta)$*;*
*(b)* $\alpha_n \in (0,1), \lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$.

*Then, $\{x_n\}$ strongly converges to an element $q \in Fix(U)$, where $q = P_{Fix(U)} \circ f(q)$.*

*Proof* By Lemma 2.8, we have $U_\lambda$ is $\alpha$-strongly quasi- nonexpansive with $\alpha = \dfrac{1}{\lambda}(1-\beta-\lambda)$. Since $\lambda \in (0, 1-\beta)$, we get $\alpha > 0$. On the other hand, $Fix(U) = Fix(U_\lambda)$ and $\lambda(I-U) = I - U_\lambda$, and thus, $I - U_\lambda$ is demiclosed at zero. The remaining of the proof is followed from Theorem 3.1. □

**Corollary 3.3** *Let $U : H \to H$ be a quasi-nonexpansive operator, such that $I - U$ is demiclosed at zero. Suppose that $f : H \to H$ is a contraction with constant $\rho \in (0,1)$. Let $\{x_n\}$ be a sequence in $H$ defined by*

$$x_0 \in H, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda x_n, \tag{3.13}$$

*where the parameter $\lambda \in (0,1)$ and the sequence $\{\alpha_n\}$ satisfy the following conditions: $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} = \infty$. Then, $\{x_n\}$ strongly converges to an element $q \in Fix(U)$, where $q = P_{Fix(U)} \circ f(q)$.*

Corollary 3.3 extends Maingé's result in [10] from $\lambda \in (0, \frac{1}{2})$ to $\lambda \in (0,1)$.

**Theorem 3.4** *Let $U : H_1 \to H_1$ be a $\alpha_2$-strongly quasi-nonexpansive operator and $T : H_2 \to H_2$ be a $\mu$-demicontractive operator that both $I - U$ and $I - T$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$, and $f : H_1 \to H_1$ be a contraction with constant $\rho \in (0,1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U(I + \gamma A^*(T-I)A)x_n, \end{cases} \tag{3.14}$$

*where the parameters $\gamma$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a) $\gamma \in (0, \frac{1-\mu}{L})$;*
*(b) $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.*

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

*Proof* We will show as follows:

(a) The operator $UV$ is $\alpha$-strongly quasi-nonexpansive, where $V := I + \gamma A^*(T-I)A$;
(b) The operator $I - UV$ is demiclosed at zero.

By Lemma 2.9, then $V := I + \gamma A^*(T-I)A$ is $\alpha_1$-strongly quasi-nonexpansive with $\alpha_1 = \frac{1}{\gamma L}(1 - \mu - \gamma L)$. By Lemma 2.6, then $UV$ is $\alpha$-strongly quasi-nonexpansive and $Fix(U) \cap Fix(V) = Fix(UV)$, where $\alpha = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$.

First, we show that $\Gamma = Fix(U) \cap Fix(V) = Fix(UV)$. Indeed, it follows from Lemma 2.9 that

$$\begin{aligned} \Gamma &= \{x \in H_1 : x \in Fix(U) \text{ and } Ax \in Fix(T)\} \\ &= \{x \in H_1 : x \in Fix(U) \text{ and } x \in Fix(V)\} \\ &= Fix(U) \cap Fix(V) \\ &= Fix(UV). \end{aligned}$$

Let $\{x_n\}$ be a sequence such that $x_n - UVx_n \to 0$ and $x_n \rightharpoonup x$. We have $\|x_n - q\| \leq \|x_n - UVx_n\| + \|UVx_n - q\|$, that is $\|x_n - q\| - \|UVx_n - q\| \leq \|x_n - UVx_n\| \to 0$. This implies that

$$\|x_n - q\|^2 - \|UVx_n - q\|^2 \to 0.$$

We have

$$\begin{aligned}\|UVx_n - q\|^2 &\le \|Vx_n - q\|^2 - \alpha_2\|UVx_n - Vx_n\|^2\\ &\le \|x_n - q\|^2 - \alpha_1\|Vx_n - x_n\|^2 - \alpha_2\|UVx_n - Vx_n\|^2.\end{aligned}$$

It follows

$$Vx_n - x_n \to 0 \quad \text{and} \quad UVx_n - Vx_n \to 0.$$

This implies that $Vx_n \rightharpoonup x$, and by the demiclosedness of $I - U$, we get $x \in Fix(U)$.

On the other hand, by Lemma 2.9, we get

$$\begin{aligned}\|UVx_n - q\|^2 &\le \|Vx_n - q\|^2 - \alpha_2\|U_\lambda Vx_n - Vx_n\|^2\\ &\le \|x_n - q\|^2 - \gamma(1-\mu-\gamma L)\|(T-I)Ax_n\|^2\\ &\quad - \alpha_2\|U_\lambda Vx_n - Vx_n\|^2.\end{aligned}$$

It follows

$$\begin{aligned}\gamma(1-\mu-\gamma L)\|(T-I)Ax_n\|^2 &\le \|x_n - q\|^2 - \|UVx_n - q\|^2\\ &-\alpha_2\|UVx_n - Vx_n\|^2 \to 0.\end{aligned}$$

Since $Ax_n \rightharpoonup Ax$ and the demiclosedness of $I-T$, we get $Ax \in Fix(T)$, that is $x \in Fix(V)$. Therefore, $x \in Fix(U) \cap Fix(V) = Fix(UV)$. That is, $I - UV$ is demiclosed at zero. □

**Corollary 3.5** *Let $U : H_1 \to H_1$ be a $\beta$-demicontractive operator and $T : H_2 \to H_2$ be a $\mu$-demicontractive operator that both $I - U$ and $I - T$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$ and $f : H_1 \to H_1$ be a contraction with constant $\rho \in (0, 1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1,\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(I + \gamma A^*(T-I)A)x_n,\end{cases} \tag{3.15}$$

*where the parameters $\lambda$, $\gamma$, and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\lambda \in (0, 1-\beta)$;
*(b)* $\gamma \in (0, \frac{1-\mu}{L})$;
*(c)* $\alpha_n \in (0,1), \lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$.

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

*Proof* By Lemma 2.9, then $V := I + \gamma A^*(T - I)A$ is $\alpha_1$-strongly quasi-nonexpansive with $\alpha_1 = \dfrac{1}{\gamma L}(1-\mu-\gamma L)$, and by Lemma 2.8, then $U_\lambda$ is $\alpha_2$-strongly quasi-nonexpansive with $\alpha_2 = \dfrac{1}{\lambda}(1-\beta-\lambda)$. The remaining of the proof is followed from Theorem 3.4. □

**Corollary 3.6** *Let $U : H_1 \to H_1$ be a quasi-nonexpansive operator and $T : H_2 \to H_2$ be a quasi-nonexpansive operator that both $I - U$ and $I - T$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with*

*$L = \|A^*A\|$ and $f : H_1 \to H_1$ be a contraction with constant $\rho \in (0,1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(I + \gamma A^*(T-I)A)x_n, \end{cases} \tag{3.16}$$

*where the parameters $\lambda$, $\gamma$, and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\lambda \in (0,1)$;
*(b)* $\gamma \in (0, \frac{1}{L})$;
*(c)* $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

**Theorem 3.7** *Let $S : H_1 \to H_1$ be a $\beta$-demicontractive operator and $T : H_2 \to H_2$ be a $\mu$-demicontractive operator. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$ and $f : H_1 \to H_1$ be a contraction with constant $\rho \in (0,1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda x_n, \end{cases} \tag{3.17}$$

*where $U := S(I+\gamma A^*(T-I)A)$. Assume that $I-U$ is demiclosed at zero and the parameters $\beta, \lambda, \gamma$, and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\gamma \in (0, \frac{1-\mu}{L})$;
*(b)* $\beta < \alpha_1$, *where* $\alpha_1 := \frac{1}{\gamma L}(1-\mu-\gamma L)$;
*(c)* $\lambda \in (0, 1 - \frac{\alpha_1\beta}{\alpha_1+\beta})$;
*(d)* $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

*Proof* Let $V := I + \gamma A^*(T-I)A$, by Lemma 2.9, then the operator $V$ is $\alpha_1$-strongly quasi-nonexpansive. Therefore, by Lemma 2.7, then the operator $SV$ is $\frac{\alpha_1\beta}{\alpha_1-\beta}$ demicontractive and $Fix(S) \cap Fix(V) = Fix(SV)$. We show that $\Gamma = Fix(S) \cap Fix(V) = Fix(SV)$. Indeed, it follows from Lemma 2.9 that

$$\begin{aligned} \Gamma &= \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\} \\ &= \{x \in H_1 : x \in Fix(S) \text{ and } x \in Fix(V)\} \\ &= Fix(S) \cap Fix(V). \end{aligned}$$

The remaining of the proof is followed by Corollary 3.2. □

**Corollary 3.8** *Let $S : H_1 \to H_1$ be a quasi-nonexpansive operator $T : H_2 \to H_2$ be a $\mu$-demicontractive operator that both $I-S$ and $I-T$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$ and*

*$f : H_1 \to H_1$ be a contraction with constant $\rho \in (0,1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) U_\lambda x_n, \end{cases} \tag{3.18}$$

*where $U := S(I + \gamma A^*(T-I)A)$ and the parameters $\lambda$, $\gamma$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\lambda \in (0,1)$;
*(b)* $\gamma \in (0, \frac{1-\mu}{L})$;
*(c)* $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

*Proof* We will show that $I - SV$ is demiclosed at zero.

To prove (b), let $\{x_n\}$ be a sequence, such that $x_n - SVx_n \to 0$ and $x_n \rightharpoonup x$. We have $\|x_n - q\| \leq \|x_n - SVx_n\| + \|SVx_n - q\|$, that is, $\|x_n - q\| - \|SVx_n - q\| \leq \|x_n - SVx_n\| \to 0$. This implies that

$$\|x_n - q\|^2 - \|SVx_n - q\|^2 \to 0.$$

On the other hand

$$\begin{aligned} \|SVx_n - q\|^2 &\leq \|Vx_n - q\|^2 \\ &\leq \|x_n - q\|^2 - \alpha_1 \|Vx_n - x_n\|^2. \end{aligned}$$

It follows

$$Vx_n - x_n \to 0.$$

This implies that $Vx_n \rightharpoonup x$. Since $\|SVx_n - Vx_n\| \leq \|SVx_n - x_n\| + \|x_n - Vx_n\| \to 0$ and by the demiclosedness of $I - S$ we get $x \in Fix(S)$. On the other hand, by Lemma 2.9, we have

$$\begin{aligned} \|SVx_n - q\|^2 &\leq \|Vx_n - q\|^2 \\ &\leq \|x_n - q\|^2 - \gamma(1 - \mu - \gamma L)\|(T-I)Ax_n\|^2. \end{aligned}$$

It follows $\gamma(1-\mu-\gamma L)\|(T-I)Ax_n\|^2 \leq \|x_n - q\|^2 - \|SVx_n - q\|^2 \to 0$. Since $Ax_n \rightharpoonup Ax$ and$(I-T)Ax_n \to 0$, by the demiclosedness of $I - T$, we get $Ax \in Fix(T)$, that is $x \in Fix(V)$. Therefore, $x \in Fix(S) \cap Fix(V) = Fix(SV)$. □

Corollary 3.8 extends Zhao's and He's result in [20] from $\lambda \in (0, \frac{1}{2})$ to $\lambda \in (0,1)$ and Corollary 3.8 answers the question's Moudafi in [12].

## 4. The split variational inequality problem

Given operators $f : H_1 \to H_1$, $g : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$ and nonempty closed convex subsets $C \subset H_1$ and $Q \subset H_2$, the split variational inequality problem (SVIP) is the problem of finding a point $x^* \in VIP(C,f)$, such that $Ax^* \in VIP(Q,g)$, that is

$$\begin{cases} x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 & \text{for all } x \in C, \\ Ax^* \in Q \text{ such that } \langle g(Ax^*), y - Ax^* \rangle \geq 0 & \text{for all } y \in Q. \end{cases}$$

This is equivalent to the problem of finding $x^* \in Fix(P_C(I - \eta f))$, such that $Ax^* \in Fix(P_Q(I - \eta g))$, where $\eta > 0$. We denote the set of solutions by $SVIP(A,C,Q,f,g)$. Therefore, SVIP can be viewed as SCFP. Under appropriate conditions of the operators $f$ and $g$, we can apply our results for SVIP.

**Lemma 4.1** [4,8] *Let $f : H_1 \to H_1$ be a monotone and $k$-Lipschitz continuous on $C$. Let $S := P_C(I - \eta f)$, where $\eta > 0$. If $x_n$ is a sequence in $C$ satisfying $x_n \rightharpoonup x^*$ and $x_n - Sx_n \to 0$, then $x^* \in VIP(C,f)$.*

**Lemma 4.2** [9] *Let $f : H_1 \to H_1$ be a monotone and $k$-Lipschitz operator on $C$ and $\eta > 0$. Let $W := P_C(I - \eta f)$ and $S := P_C(I - \eta f W)$. Then, for all $z \in VIP(C,f)$, we have*

$$\|Sx - z\|^2 \leq \|x - z\|^2 - (1 - k^2\eta^2)\|x - Wx\|^2.$$

*In particular, if $k\eta < 1$, $S$ is a quasi-nonexpansive operator and $Fix(S) = Fix(W) = VIP(C,f)$.*

**Corollary 4.3** *Let $C$ and $Q$ be nonempty closed convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be monotone and $k$-Lipschitz continuous operators on $C$ and $Q$, respectively, and $A : H_1 \to H_2$ a bounded linear operator with $\|A^*A\| = L$. Suppose $SVIP(A,C,Q,f,g) \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) U_\lambda x_n, \end{cases} \tag{4.1}$$

*where $U = ST, S := P_C(I - \eta f P_C(I - \eta f)), T := P_Q(I - \eta g P_Q(I - \eta g))$, and the parameters $\lambda$, $\gamma, \eta$, and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\eta \in (0, \frac{1}{k})$;
*(b)* $\lambda \in (0,1)$;
*(c)* $\gamma \in (0, \frac{1}{L})$;
*(d)* $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, $\{x_n\}$ converges strongly to $x^* \in SVIP(A,C,Q,f,g)$.*

*Proof* Since Lemma 4.2, we obtain that both operators $S$ and $T$ are two quasi-nonexpansive operators. Next, we show that $I - S$ is demiclosed at zero. Let $\{x_n\}$ be a sequence in $H_1$, such that $x_n - Sx_n \to 0$ and $x_n \rightharpoonup x$ as $n \to \infty$. For some $q \in VIP(C,f)$ we have $\|x_n - q\|^2 - \|Sx_n - q\|^2 \to 0$ as $n \to \infty$. By Lemma 4.2, we get

$$(1 - \eta^2k^2)\|x_n - P_C(I - \eta f)x_n\|^2 \leq \|x_n - q\|^2 - \|Sx_n - q\|^2.$$

This implies $x_n - P_C(I - \eta f)x_n \to 0$. By Lemma 4.1, we obtain $x \in VIP(C, f) = Fix(S)$. Similarly, $I - T$ is also demiclosed at zero. The result implies from Corollary 3.8. □

## 5. The split common null point problem

Given two set-valued operators $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ and a bounded linear operator $A : H_1 \to H_2$, the split common null point problem (SCNP) is the problem of finding

$$x \in H_1 \text{ such that } 0 \in B_1(x) \text{ and } 0 \in B_2(Ax). \tag{5.1}$$

Recently, Byrne et al.[3] and Kazmi et al. [7] proposed a strong convergence theorem for finding such a solution $x$ when $B_1$ and $B_2$ are maximal monotone. Recall that $B : H \to 2^H$ is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0 \ \ \forall x, y \in D(B), u \in Bu, v \in By,$$

where $B(D) := \{x \in H, Bx \neq \emptyset\}$.

A monotone operator is said to be maximal if its graph is not properly contained in the graph of any other monotone operator.

For a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, we can define a single-valued operator:

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to H.$$

It is known that $J_\lambda^B$ is firmly nonexpansive and $0 \in B(x)$ iff $x \in Fix(J_\lambda^B)$.

Therefore, the problem (5.1) is equivalently to the problem of finding

$$x \in H_1 \text{ such that } x \in Fix(J_\lambda^{B_1}) \text{ and } Ax \in Fix(J_\lambda^{B_2}),$$

where $\lambda > 0$, that is, the SCNP reduces to the SCFP.

**Theorem 5.1** *Let $B_1 : H_1 \to 2^{H_1}$ and $B : H_2 \to B_2^{H_2}$ be two set-valued maximal monotone operators. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$ and $f : H_1 \to H_1$ be a contraction with constant $\rho \in (0, 1)$. Suppose $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n, \end{cases} \tag{5.2}$$

*where the parameters $\gamma$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:*

*(a)* $\gamma \in (0, \frac{1}{L})$;

*(b)* $\alpha_n \in (0, 1), \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$.

*Then, $x_n \to q$, where $q = P_\Gamma \circ f(q)$.*

*Proof* We have $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are two firmly nonexpansive operators and hence nonexpansive. Therefore, $I - J_\lambda^{B_1}$ and $I - J_\lambda^{B_2}$ are demiclosed at zero. $J_\lambda^{B_1}$ is 1-strongly quasi-nonexpansive and $J_\lambda^{B_2}$ is 0 demicontractive. Therefore, the remaining of the proof is followed from Theorem 3.4. □

TABLE 1. With $\alpha_n = \dfrac{1}{n+1}$

| n | $t_n$ | $z_n$ | $\|\|x_n - x^*\|\|$ |
|---|---|---|---|
| 1 | 0.3417 | 0.8333 | 2.23606797749 |
| 2 | 0.1177 | 0.3704 | 0.90065562539 |
| 3 | 0.0407 | 0.1698 | 0.38861808251 |
| 4 | 0.0141 | 0.0792 | 0.17456390343 |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| 34 | 0.0000 | 0.0000 | 0.00000000007 |

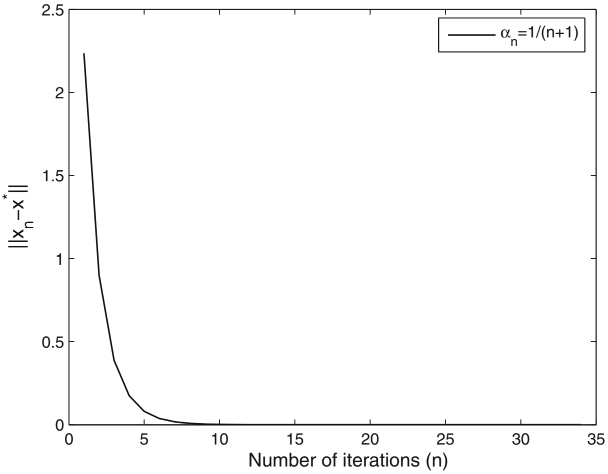


FIGURE 1. Figure for Case 1

The result of Byrne et al. [3] is a consequence of our Theorem 5.1.

**Corollary 5.2** *Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B : H_2 \to B_2^{H_2}$ *be two set-valued maximal monotone operators. Let* $A : H_1 \to H_2$ *be a bounded linear operator with* $L = \|A^*A\|$. *Suppose* $\Gamma \neq \emptyset$. *Let* $\{x_n\} \subset H_1$ *be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n, \end{cases} \tag{5.3}$$

*where the parameters* $\gamma$ *and the sequence* $\{\alpha_n\}$ *satisfy the following conditions:*

*(a)* $\gamma \in (0, \frac{1}{L})$;

*(b)* $\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then,* $x_n \to x_0$, *where* $x_0 = P_\Gamma x_0$.

Table 2. With $\alpha_n = \dfrac{1}{(n+1)^{0.1}}$

| n | $t_n$ | $z_n$ | $\|\|x_n - x^*\|\|$ |
|---|---|---|---|
| 1 | 0.1121 | 0.2416 | 0.76587354093 |
| 2 | 0.0376 | 0.0857 | 0.26633371649 |
| 3 | 0.0126 | 0.0307 | 0.09362927367 |
| 4 | 0.0126 | 0.0307 | 0.03320182345 |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| 25 | 0.0000 | 0.0000 | 0.00000000005 |

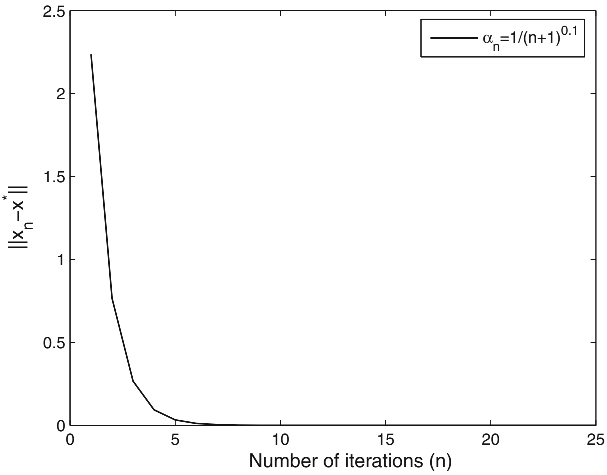


Figure 2. Figure for Case 2

## 6. Numerical example

In this section, let us show numerical example to demonstrate the convergence of our algorithm.

Let $H_1 = \mathbb{R}^2$ and $H_2 = \mathbb{R}^2$. Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined $Ux = (\frac{1}{2}t, \frac{1}{2}z)^t$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Tx = (0, z)^t$, where $x = (t, z)^t$. It is easy to see that $U$ is 1-strongly quasi-nonexpansive mapping and $T$ is 0-demicontractive mapping.

Choose $\gamma = 0.3$ and $x_0 = (1, 2)^t$. The stopping criterion for our testing method is taken as: $\|x_{n+1} - x^*\| < 10^{-10}$, where $x_n = (t_n, z_n)^t$ and $x^*$ are a solution problem. Let assume that $f(x) = \frac{1}{3}x$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. We set $L = \|A^*A\|_2$ and $\gamma \in (0, \frac{1}{L})$, where $\|.\|_2$ is the matrix 2-norm.

Case 1: Take $\alpha_n = \dfrac{1}{n+1}$. Then, using (3.14), we have Table 1 and Fig. 1.

Case 2: Take $\alpha_n = \dfrac{1}{(n+1)^{0.1}}$. Then, using (3.14), we have Table 2 and Fig. 2.

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