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Fixed point theorems via generalized *F*-contractions with applications to functional equations occurring in dynamic programming

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Abstract. The purpose of this paper is threefold. Firstly, recognizing the concept of Piri and Kumam (Fixed Point Theor Appl 210, 2014), we define generalized F-contractive mappings in the framework of G-metric spaces and by employing this, some fixed point theorems in the structure of G-metric spaces are established that can not be obtained from the existing results in the context of allied metric spaces and do not meet the remarks of Samet et al. (Int J Anal. Article ID 917158, 2013) and Jleli et al. (Fixed Point Theor Appl 210, 2012). Infact, we utilize the pattern, mentioned in Karapinar and Agrawal (Fixed Point Theor Appl 154, 2013), a counter paper to remarks of Samet et al. (Int J Anal. Article ID 917158, 2013) and Jleli et al. (Fixed Point Theor Appl 210, 2012). Secondly, in the setting of G-metric spaces, certain fixed point results for integral inequalities under generalized F-contraction are presented. Finally, as an application, our results are utilized to establish the existence and uniqueness of solution the equations arising in Oscillation of a spring. In the sequel, another application is given to set-up the existence and uniqueness of solution of functional equations occurring in dynamic programming. Our investigations are also authenticated with the aid of some appropriate and innovative examples.

Mathematics Subject Classification. 47H10, 54H25.

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1. Introduction and preliminaries

The origination of fixed point theory on complete metric space is coupled with Banach contraction principle due to Banach [6], announced in 1922. Banach contraction principle enunciates that any contractive self-mappings on a complete metric space has a unique fixed point. This principle is one of a very ascendancy test for the existence and uniqueness of the solution of substantial problems arising in mathematics. Because of its implication in mathematical theory, Banach Contraction Principle has been extended and generalized in many directions (see [5,7,8,11,14,15,17,22,26,27,29,30]). Recently, one of the most interesting generalization of it, was placed by Wardowski [35]. He introduced a new contraction called F-contraction and established a fixed point result as a generalization of the Banach contraction principle in a dissimilar way than in the other acknowledged results from the literature (see more [23,25]). Afterward, Secelean [33] altered the condition $(F2)$ of [35] by an equivalent and more simple one. Most recently, Piri et al. [24] revealed a large class of functions by replacing condition $(F3)$ by the condition $(F3')$ in the definition of F-contraction due to Wardowski [35]. Utilizing this new idea, they established a fixed point theorem as a generalization of results of Wardowski [35].

Mustafa and Sims [20,21] initiated the notion of G-metric spaces as a generalization of metric spaces and investigated the topology of such spaces. In the last decade, the notion of G-metric spaces has concerned wide-ranging attention from researchers, more than ever from fixed point theorists [1–4,13,18,19,31,32,34]. Samet et al. [32] and Jleli et al. [13] explored that some theorems in the context of a G-metric spaces in the literature can be obtain directly by some existing results in the setting of usual metric spaces. Further E. Karapinar et al. [16] answered the approach of [13,32] with the remark that techniques used in [13,32] are inapplicable unless the contraction condition in the statement of the theorem can be reduced into two variables.

The aim of our paper is to define Ciric [10]-type generalized F-contractive mappings in G-metric spaces, recognizing the concepts of Piri et al. [24]. Employing the same, certain fixed point results are also proved under relaxed F-contraction i. e. without utilizing the condition $F2'$. In the process, the technique pointed out in [16] is exploited to obtain such type of fixed point results in G-metric spaces that cannot be obtained from the existing results in the setting of associated metric spaces. Moreover, applications of aforesaid fixed point results to real life physical problems and dynamic programming are also presented. In what follows, we collect the background material to make our presentation as self-contained as possible.

Definition 1.1. [21] Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$;
(G-2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G-3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;
(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.

The function G is called a generalized or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. [21] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$G(x, x_n, x_m) < \epsilon,$$

for all $m, n > N$. We call x, the limit of the sequence and write $x_n \to x$ or $\lim_{n,m\to+\infty} x_n = x$.

Proposition 1.1. [21] *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$;
(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$;
(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Definition 1.3. [21] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that

$$G(x_n, x_m, x_l) < \epsilon,$$

for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.2. [21] *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ is G-Cauchy;
(2) For every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.4. [21] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Lemma 1.1. [20] *By the rectangle inequality* $(G5)$ *together with the symmetry* $(G4)$*, we have*

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x).$$

Wardowski [35] initiated and considered a new contraction called F-contraction to prove a fixed point result as a generalization of the Banach contraction principle.

Definition 1.5. [35] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F1) F is strictly increasing;
(F2) for all sequence $\alpha_n \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wordowski [35], defined the class of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ by $\mathcal{F}$ and introduced the notion of F-contraction as follows.

Definition 1.6. [35] Let (X, d) be a metric space. A self-mapping T on X is called an $F-$contraction if there exists $\tau > 0$ such that for $x, y \in X$

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F \in \digamma$.

Afterward Secelean [33] established the following lemma and utilized an equivalent but a more simple condition $(F2')$ instead of condition $(F2)$.

Lemma 1.2. [33] *Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be an increasing map and α_n be a sequence of positive real numbers. Then the following assertions hold:*

(a) if $\lim_{n\rightarrow\infty} F(\alpha_n) = -\infty$ then $\lim_{n\rightarrow\infty} \alpha_n = 0$;
(b) if $inf F = -\infty$ and $\lim_{n\rightarrow\infty} \alpha_n = 0$; then $\lim_{n\rightarrow\infty} F(\alpha_n) = -\infty$.

He forwarded the following conditions.
$(F2')$ $inf F = -\infty$ or
$(F2'')$ there exists a sequence α_n be a sequence of positive real numbers such that $\lim_{n\rightarrow\infty} F(\alpha_n) = -\infty$.

Very recently Piri et al. [24] replaced the condition $(F3)$ by $(F3')$ in Definition (1.5) due to Wardowski as follows.
$(F3')$ F is continuous on $(0, \infty)$. Thus Piri and Kumam [24] established the generalization of result of Wordowski [35] using the conditions $F1, F2'$ and $F3'$. Throughout our subsequent discussion, We drop-out the condition $F2'$ and named the contraction as relaxed F- contraction. Thus we denote, the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $\Delta_\digamma$ which satisfy the following conditions.

$(F1)$ F is strictly increasing;
$(F3')$ F is continuous on $(0, \infty)$.

Example 1.1. [24] Let $F_1(\alpha) = -\frac{1}{\alpha}$, $F_2(\alpha) = -\frac{1}{\alpha} + \alpha$, $F_3(\alpha) = \frac{1}{1-e^\alpha}$, $F_4(\alpha) = \frac{1}{e^\alpha - e^{-\alpha}}$. Then $F_1, F_2, F_3, F_4 \in \Delta_\digamma$.

2. Ciric-type generalized *F*-contraction in *G*-metric spaces

In this section, Ciric-type generalized F-contractive mapping is defined in the setting of G-metric spaces. Moreover some fixed point results are established for such type of mappings.

Definition 2.1. A mapping $T : X \rightarrow X$ is said to be a Ciric-type generalized F- contractive mappings in G-metric spaces if $F \in \Delta_\digamma$ and there exists $\tau > 0$, such that

$$G(Tx, Ty, T^2y) > 0 \quad \Rightarrow \quad \tau + F(G(Tx, Ty, T^2y)) \leq F(\alpha M(x, y)). \qquad (2.1)$$

For all $x, y, \in X$, where $0 < \alpha < 1$ and

$$M(x, y) = \max\left\{G(x, y, Ty), G(x, Tx, Ty), G(x, Tx, T^2x), \frac{1}{2}[G(y, Ty, T^2y) + G(y, T^2x, T^2y)]\right\}.$$

Example 2.1. Let (X, G) be a G-metric space. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = -\frac{1}{\alpha}$ for $\alpha > 0$. Then clearly $F \in \Delta_F$. Each mapping $T : X \to X$ satisfying (2.1) is a Ciric-type generalized F-contraction such that $G(Tx, T^2x, Ty) > 0 \Rightarrow$

$$G(Tx, T^2x, Ty) \leq \frac{\alpha \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)]\}}{1+\tau\alpha \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)]\}},$$

for all $x, y \in X$.

Example 2.2. Let (X, G) be a G-metric space. Consider $F(\alpha) = ln(\alpha) + \alpha$. Then obviously $F \in \Delta_F$. In this case each mapping $T : X \to X$ satisfying (2.1) is a Ciric-type generalized F-contraction such that

$$G(Tx, T^2x, Ty) > 0 \Rightarrow \frac{G(Tx, Ty, T^2y)\, e^{G(Tx,Ty,T^2y)-\{\alpha \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)]\}\}}}{\alpha \max\{G(x, y, Ty), G(x, Tx, Ty), G(x, Tx, T^2x), \frac{1}{2}[G(y, Ty, T^2y) + G(y, T^2x, T^2y)]\}} \leq e^{-\tau},$$

for all $x, y \in X$.

Next theorem is proved for the Ciric-type generalized F-contractive mappings in G-metric spaces.

Theorem 2.1. *Let (X, G) be a G-complete metric space and $T : X \to X$ be a Ciric-type generalized F-contractive mapping that is, if $F \in \Delta_F$ and there exists $\tau > 0$, such that*

$$G(Tx, Ty, T^2y) > 0 \Rightarrow \tau + F(G(Tx, Ty, T^2y)) \leq F(\alpha M(x, y)), \quad for\ all\ x, y \in X. \tag{2.2}$$

where $M(x, y) = \max\{G(x, y, Ty), G(x, Tx, Ty), G(x, Tx, T^2x), \frac{1}{2}[G(y, Ty, T^2y) + G(y, T^2x, T^2y)]\}$ and $\alpha \in (0, 1)$. Then T has a fixed point in X. Moreover, if $2\alpha \leq 1$ then fixed point of T is unique.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Now constructing the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$.

Clearly $x_n \neq x_{n+1}$, if for any n_0, we have $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$ and so T has a fixed point.

Consequently, one can obtain

$$G(Tx_{n-1}, Tx_n, T^2x_n) > 0.$$

Then from (2.2) with $x = x_{n-1}, y = x_n$, we have

$$\begin{aligned}
&\tau + F(G(x_n, x_{n+1}, x_{n+2})) \\
&\quad = \tau + F(G(Tx_{n-1}, Tx_n, T^2x_n)) \\
&\quad \leq F\Bigg(\alpha \max\Bigg\{G(x_{n-1}, x_n, Tx_n), G(x_{n-1}, Tx_{n-1}, Tx_n), \\
&\qquad G(x_{n-1}, Tx_{n-1}, T^2x_{n-1}), \\
&\qquad \frac{1}{2}[G(x_n, Tx_n, T^2x_n) + G(x_n, T^2x_{n-1}, T^2x_n)]\Bigg\}\Bigg) \\
&= F\Bigg(\alpha \max\Bigg\{G(x_{n-1}, x_n, x_{n+1}), G(x_{n-1}, x_n, x_{n+1}), \\
&\qquad G(x_{n-1}, x_n, x_{n+1}), \frac{1}{2}[G(x_n, x_{n+1}, x_{n+2}) \\
&\quad + G(x_n, x_{n+1}, x_{n+2})]\Bigg\}\Bigg). \qquad (2.3)
\end{aligned}$$

If there exist $n \in N$ such that

$$\max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\} = G(x_n, x_{n+1}, x_{n+2})$$

then from (2.3)

$$\tau + F(G(x_n, x_{n+1}, x_{n+2})) \leq F(\alpha G(x_n, x_{n+1}, x_{n+2})).$$

Which is a contradiction in view of $F1$ and $\alpha \in (0, 1)$. Therefore,

$$\max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\} = G(x_{n-1}, x_n, x_{n+1}).$$

Then (2.3) yields

$$\tau + F(G(x_n, x_{n+1}, x_{n+2})) \leq F(\alpha G(x_{n-1}, x_n, x_{n+1})). \qquad (2.4)$$

Taking $G_n = G(x_n, x_{n+1}, x_{n+2})$, (2.4) becomes

$$\tau + F(G_n) \leq F(\alpha G_{n-1}). \qquad (2.5)$$

On utilizing $(F1)$, one gets

$$G_n < \alpha G_{n-1}. \qquad (2.6)$$

Repeated applications of the same give rise to

$$G_n < \alpha^n\, G_0 \;\; or \;\; G(x_n, x_{n+1}, x_{n+2}) < \alpha^n G(x_0, x_1, x_2).$$

From $(G\text{-}3)$, one has

$$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}), \quad with \;\; x_{n+1} \neq x_{n+2}.$$

Utilizing Lemma 1.1, we get

$$\begin{aligned}
G(x_{n+1}, x_{n+1}, x_n) &\leq 2\, G(x_n, x_n, x_{n+1}) \\
&< 2\, \alpha^n\, G(x_0, x_1, x_2).
\end{aligned}$$

Furthermore, for all $m, n \in N$ $(n < m)$ using the Property (G-5), one gets

$$\begin{aligned} G(x_m, x_m, x_n) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &< 2(\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1}) G(x_0, x_1, x_2) \\ &< \frac{2\alpha^n}{1-\alpha} G(x_0, x_1, x_2). \end{aligned} \tag{2.7}$$

Which on making $m, n \to \infty$, reduces to

$$G(x_m, x_m, x_n) \to 0.$$

Thus $\{x_n\}$ is G-Cauchy sequence. Since X is G complete, there exists a point $t \in X$ such that $\{x_n\}$ converges to t.

Now from (2.2) with $x = x_n$ and $y = t$, we have

$$\begin{aligned} &\tau + F(G(Tx_n, Tt, T^2t)) \le F(\alpha M(x_n, t)) \\ &\quad = F\Big(\alpha \max\Big\{G(x_n, t, Tt), G(x_n, Tx_n, Tt), \\ &\qquad G(x_n, Tx_n, T^2x_n), \frac{1}{2}[G(t, Tt, T^2t) + G(t, T^2x_n, T^2t)]\Big\}\Big) \\ &\quad = F\Big(\alpha \max\Big\{G(x_n, t, Tt), G(x_n, Tx_{n+1}, Tt), \\ &\qquad G(x_n, Tx_{n+1}, x_{n+2}), \frac{1}{2}[G(t, Tt, T^2t) + G(t, x_{n+2}, T^2t)]\Big\}\Big). \end{aligned}$$

Letting $n \to \infty$ and using $F3'$, one finds that

$$\begin{aligned} \tau + F(G(t, Tt, T^2t)) \le F\Big(\alpha \max\Big\{&G(t, t, Tt), G(t, t, Tt), \\ &G(t, t, t), \frac{1}{2}[G(t, Tt, T^2t) + G(t, t, T^2t)]\Big\}\Big). \end{aligned} \tag{2.8}$$

To show t to be fixed point, suppose that $t \ne Tt \ne T^2t$. On utilizing the property G-3 in (2.8) , we acquire

$$\begin{aligned} \tau + F(G(t, Tt, T^2t)) &\le F(\alpha \max\{G(t, t, Tt), \tfrac{1}{2}[G(t, Tt, T^2t) + G(t, Tt, T^2t)]\}) \\ &= F(\alpha G(t, Tt, T^2t)). \end{aligned}$$

Which is a contradiction in view of $F1$ and $\alpha < 1$. Then we must have $t = Tt = T^2t$. Therefore t is a fixed point of T.

In order to show the uniqueness of fixed point. Suppose u and w be the two fixed points of T, such that $u \ne w$.

So that

$$G(Tu, Tw, T^2w) > 0.$$

Then on utilizing the condition (2.2) with $x = u$ and $y = w$ also on using $(F1)$, Lemma (1.1), one can obtain

$$\begin{aligned}
\tau + F(G(u,w,w)) &= \tau + F(G(Tu,Tw,T^2w)) \\
&\leq F\Bigg(\alpha \max\Bigg\{G(u,w,Tw), G(u,Tu,Tw) \\
&\quad G(u,Tu,T^2u), \frac{1}{2}[G(w,Tw,T^2w) + G(w,T^2u,T^2w)]\Bigg\}\Bigg) \\
&= F\Bigg(\alpha \max\Bigg\{G(u,w,w), G(u,u,w) \\
&\quad \times G(u,u,u), \frac{1}{2}[G(w,w,w) + G(w,u,w)]\Bigg\}\Bigg) \\
&= F(\alpha \max\{G(u,w,w), G(u,u,w)\}) \\
&\leq F(\alpha \max\{G(u,w,w), 2G(u,w,w)\} \\
&= F(2\alpha G(u,w,w)).
\end{aligned}$$

This is a contradiction, in perception of $F1$ and $2\alpha \leq 1$, therefore $u = w$.

This concludes the proof. □

Next, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of Theorem 2.1.

Example 2.3. Consider the sequence $\{S_n\}_{n\in N}$ as
$S_1 = 3.1$,
$S_2 = 3.1 + 3^2.3$,
$S_3 = 3.1 + 3^2.3 + 3^3.5$,
$\vdots$
$S_n = 3.1 + 3^2.3 + 3^3.5 + \cdots + 3^n.(2n-1) = (n-1)3^{n+1} + 3.$
Let $X = \{S_n : n \in X\}$ and

$$G(x,y,z) = \begin{cases} 0, & \textit{if and only if } x = y = z, \\ \max\{x,y,z\}, & \textit{otherwise}. \end{cases}$$

Then (X, G) is a complete G-metric space.

Define the mapping $T : X \to X$ by $T(S_1) = S_1$ and $T(S_n) = S_{n-1}$ for every $n > 1$.

Now we claim that T is a Ciric-type generalized F-contraction in the framework of G-metric spaces with $F(\alpha) = \ln\alpha + \alpha$, then clearly $F \in \Delta_F$.

We notice following

$$\begin{aligned}
&G(TS_n, TS_m, T^2S_m) > 0 \\
&\quad \Leftrightarrow\ (n = 1 \wedge m > 2) \vee (m = 1 \wedge n > 2) \vee (1 < n < m) \vee (1 < m < n).
\end{aligned}$$

For all the possible cases, we will show that

$$\frac{G(Tx,Ty,T^2y)e^{G(Tx,Ty,T^2y) - \alpha\ max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}}}{\alpha\ \ max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}} \leq e^{-\tau}, \tag{2.9}$$

for $x = S_n, y = S_m, \quad n, m \in N$ and for $\tau = 24 > 0$ with $\alpha = \frac{1}{3}$. Clearly $2\alpha \leq 1$.

In view of structure of $\{S_n\}$, one can easily notice that

$$\begin{aligned} \tfrac{S_n}{3} &= (n-1)3^n + 1 \\ &= (n-2)3^n + 3 + (3^n - 2) \\ &= S_{n-1} + 3^n - 2. \end{aligned}$$

This yields

$$S_{n-1} < \frac{S_n}{3}.$$

Case I When $n = 1, m > 2$.

Then

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(TS_n, TS_m, T^2S_m) = S_{m-1}, \\ G(x, y, Ty) &= G(S_1, S_m, TS_m) = S_m, \\ G(x, Tx, Ty) &= G(S_1, TS_1, TS_m) = S_{m-1}, \\ G(x, Tx, T^2x) &= G(S_1, TS_1, T^2S_1) = 0, \\ G(y, Ty, T^2y) &= G(S_m, TS_m, T^2S_m) = S_m, \\ G(y, T^2x, T^2y) &= G(S_m, T^2S_1, T^2S_m) = S_m. \end{aligned}$$

Utilizing (2.9) and the fact that $S_{n-1} < \alpha S_n, \forall n > 1$ with $\alpha = \frac{1}{3}$, we have

$$\begin{aligned} \tfrac{S_{m-1}.e^{S_{m-1}-\alpha S_m}}{\alpha S_m} &\leq e^{S_{m-1}-\frac{1}{3}S_m} \\ &= e^{-(\frac{1}{3}S_m - S_{m-1})} \\ &< e^{-24}, \quad since \quad \tfrac{1}{3}S_m - S_{m-1} > 24 \; (for\ m > 2). \end{aligned}$$

Thus in this case T is a Ciric-type generalized F-contractive mapping with $\tau = 24$.

Case II When $m = 1, n > 2$.

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(TS_n, TS_1, T^2S_1) = S_{n-1}, \\ G(x, y, Ty) &= G(S_n, S_1, TS_1) = S_n, \\ G(x, Tx, Ty) &= G(S_n, TS_n, TS_1) = S_n, \\ G(x, Tx, T^2x) &= G(S_n, TS_n, T^2S_n) = S_n, \\ G(y, Ty, T^2y) &= G(S_1, TS_1, T^2S_1) = 0, \\ G(y, T^2x, T^2y) &= G(S_1, T^2S_n, T^2S_1) = 0 \;\; or \;\; S_{n-2}. \end{aligned}$$

Thus from (2.9), we have

$$\begin{aligned} \frac{S_{n-1}.e^{S_{n-1}-\alpha S_n}}{\alpha S_n} &\leq e^{S_{n-1}-\frac{1}{3}S_n} \\ &= e^{-(\frac{1}{3}S_n - S_{n-1})} \\ &< e^{-24}, \quad since \; \frac{1}{3}S_n - S_{n-1} > 24 \; (for\ n > 2). \end{aligned}$$

Thus we get desired.

Case III When $1 < n < m$. Then consider

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(TS_n, TS_m, T^2S_m) = S_{m-1}, \\ G(x, y, Ty) &= G(S_n, S_m, TS_m) = S_m, \\ G(x, Tx, Ty) &= G(S_n, TS_n, TS_m) = S_{m-1}, \\ G(x, Tx, T^2x) &= G(S_n, TS_n, T^2S_n) = S_n, \\ G(y, Ty, T^2y) &= G(S_m, TS_m, T^2S_m) = S_m, \\ G(y, T^2x, T^2y) &= G(S_m, T^2S_n, T^2S_m) = S_m. \end{aligned}$$

Using (2.9), we have

$$\begin{aligned} \frac{S_{m-1}.e^{S_{m-1}-\alpha S_m}}{\alpha S_m} &\leq e^{S_{m-1}-\frac{1}{3}S_m} \\ &= e^{-(\frac{1}{3}S_m - S_{m-1})} \\ &< e^{-24}, \quad since\ \frac{1}{3}S_m - S_{m-1} > 24\ (for\ m > 2). \end{aligned}$$

This follows T is a Ciric-type generalized F contractive mapping with $\tau = 24$.
Case IV When $1 < m < n$. Then

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(TS_n, TS_m, T^2S_m) = S_{n-1}, \\ G(x, y, Ty) &= G(S_n, S_m, TS_m) = S_n, \\ G(x, Tx, Ty) &= G(S_n, TS_n, TS_m) = S_n, \\ G(x, Tx, T^2x) &= G(S_n, TS_n, T^2S_n) = S_n \\ G(y, Ty, T^2y) &= G(S_m, TS_m, T^2S_m) = S_m, \\ G(y, T^2x, T^2y) &= G(S_m, T^2S_n, T^2S_m) = S_m\ \ or\ \ S_{n-2}. \end{aligned}$$

Employing (2.9), one gets

$$\begin{aligned} \frac{S_{n-1}.e^{S_{n-1}-\alpha S_n}}{\alpha S_n} &\leq e^{S_{n-1}-\frac{1}{3}S_n} \\ &= e^{-(\frac{1}{3}S_n - S_{n-1})} \\ &< e^{-24}, \quad since\ \frac{1}{3}S_n - S_{n-1} > 24\ (for\ n > 2). \end{aligned}$$

As required.
Thus T is a Ciric-type F-contraction mapping and S_1 is a fixed point which is indeed unique.

Next, we present one more example which demonstrates the validity of Theorem 2.1, pictorially.

Example 2.4. Let $X = [0, 0.9]$ and $G(x, y, z) =$

$$\begin{cases} 0, & if\ and\ only\ if\ x = y = z, \\ \max\{x, y\} + \max\{y, z\} + \max\{x, z\}, & otherwise. \end{cases}$$

Then (X, G) is a complete G-metric space.

Define a mapping $T : X \to X$ by $Tx = \frac{x^4}{1+x}$.

In order to verify the Condition (2.2) with $\tau = 0.35$, $\alpha = 0.9 < 1$ and $F(\beta) = \frac{1}{1-e^{\beta}}$, then clearly $F \in \Delta_F$, we notice that,

$$G(Tx, Ty, T^2y) \Leftrightarrow (x = 0 \wedge y > 0) \vee (x > 0 \wedge y = 0) \vee (x > 0 \wedge y > 0).$$

Now various cases are discussed.

Case I When $x = 0, y > 0$. Then

$$\begin{aligned} G(Tx, Ty, T^2y) &= \frac{y^4}{1+y} + \frac{\left(\frac{y^4}{1+y}\right)^4}{1+\frac{y^4}{1+y}}, \\ G(x, y, Ty) &= 2y + \frac{y^4}{1+y}, \\ G(x, Tx, Ty) &= \frac{2y^4}{1+y}, \\ G(x, Tx, T^2x) &= 0, \\ G(y, Ty, T^2y) &= 2y + \frac{y^4}{1+y}, \\ G(y, T^2x, T^2y) &= 2y + \frac{\left(\frac{y^4}{1+y}\right)^4}{1+\frac{y^4}{1+y}}. \end{aligned}$$

Utilizing aforementioned values, consider the L.H.S. of (2.2)

$$\tau + F(G(Tx, Ty, T^2y)) = 0.35 + \frac{1}{1 - e^{\frac{2y^4}{1+y} + \frac{\left(\frac{y^4}{1+y}\right)^4}{1+\frac{y^4}{1+y}}}}$$

and R.H.S. is

$$F(\alpha M(x, y)) = \frac{1}{1 - e^{0.9\left(2y + \frac{2y^4}{1+y}\right)}}.$$

Following figures (Figs. 1, 2) show that R.H.S. expression dominates the L.H.S. expression for $x = 0$ and $y > 0$ in $[0, 0.9]$, which validates our inequality in this case.

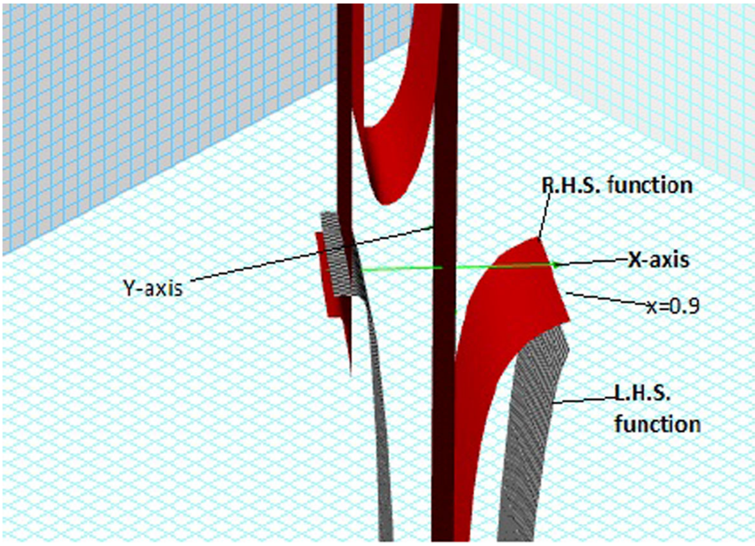

FIGURE 1. *Plot* of inequality for Case I , 3D view

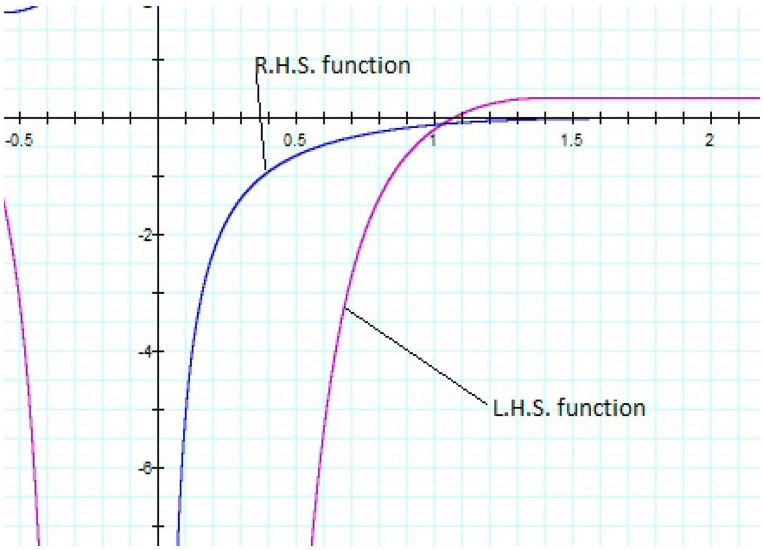

FIGURE 2. *Plot* of inequality for Case I , 2D view

Case II When $y = 0, x > 0$. Then

$$\begin{array}{ll} G(Tx, Ty, T^2y) & = \frac{2x^4}{1+x}, \\ G(x, y, Ty) & = 2x, \\ G(x, Tx, Ty) & = 2x + \frac{2x^4}{1+x}, \\ G(x, Tx, T^2x) & = 2x + \frac{2x^4}{1+x}, \\ G(y, Ty, T^2y) & = 0, \\ G(y, T^2x, T^2y) & = \frac{\left(2\frac{x^4}{1+x}\right)^4}{1+\frac{x^4}{1+x}}. \end{array}$$

L.H.S. of (2.2) corresponding to above values is

$$\tau + F(G(Tx, Ty, T^2y)) = 0.35 + \frac{1}{1 - e^{\frac{2x^4}{1+x}}}$$

and R.H.S. is

$$F(\alpha M(x, y)) = \frac{1}{1 - e^{0.9\left(2x + \frac{2x^4}{1+x}\right)}}.$$

Subsequent figures (Figs. 3, 4) demonstrate that R.H.S. with green surface dominates the black doted surface i.e. L.H.S and they interchange their domination after $x = 1.04$, this amounts to say that for $x, y \in [0, 0.9]$, inequality (2.2) is satisfied.

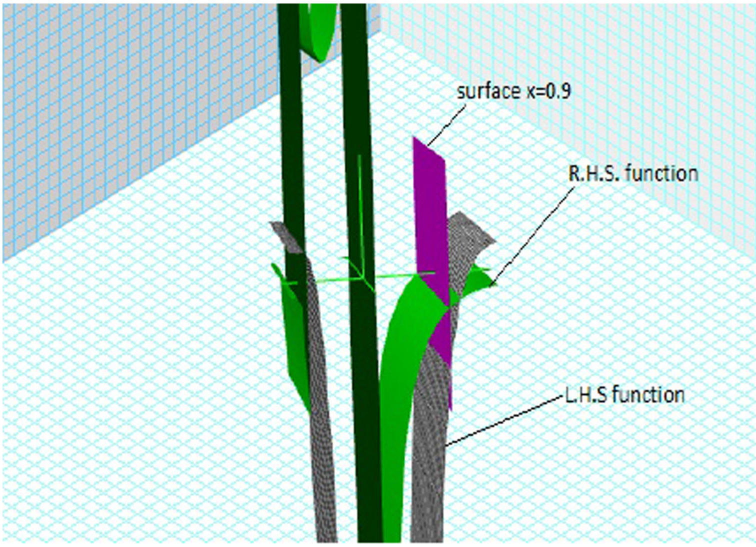

FIGURE 3. *Plot* of inequality for Case II , 3D view

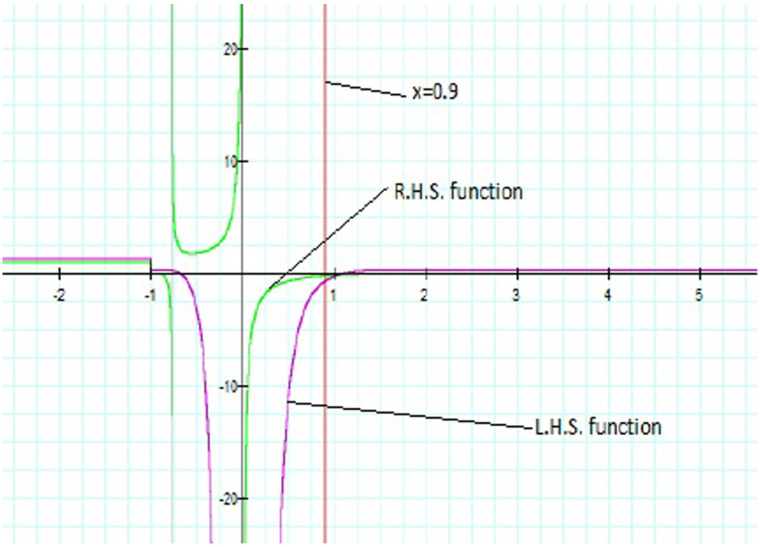

FIGURE 4. *Plot* of inequality for Case I , 2D view

Case III When $x > 0, y > 0$. Now following sub-cases arise.

(i) When $x < T^2y < Ty < y$

This sub-case can easily be verified as in Case I

(ii) When $T^2y < Tx < Ty < x < y$,

after calculating various terms appearing in the inequality (2.2) we come to L.H.S. as

$$\tau + F(G(Tx, Ty, T^2y)) = 0.35 + \frac{1}{1 - e^{\left(\frac{x^4}{1+x} + \frac{2y^4}{1+y}\right)}}$$

and R.H.S. is

$$F(\alpha M(x, y)) = \frac{1}{1 - e^{0.9\left(2y + \frac{2y^4}{1+y}\right)}}.$$

By the following figure (Fig. 5) it is verified that $L.H.S. \leq R.H.S.$

(iii) When $Ty < Tx < y < x$,

following the above calculation approach it is inferred as $L.H.S. < R.H.S.$

Which is verified by following figure (Fig. 6).

Similarly one can verify that inequality (2.2) is verified for all other sub-cases and cases.

Thus all the conditions of Theorem 2.1 are satisfied and T has a unique fixed point $x = 0 \in X$, which is shown by following figure (Fig. 7).

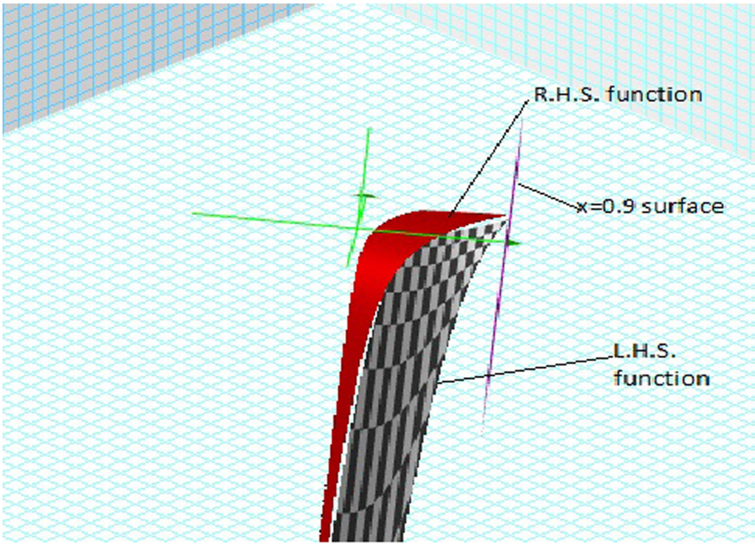

FIGURE 5. *Plot* of inequality for Case III(ii)

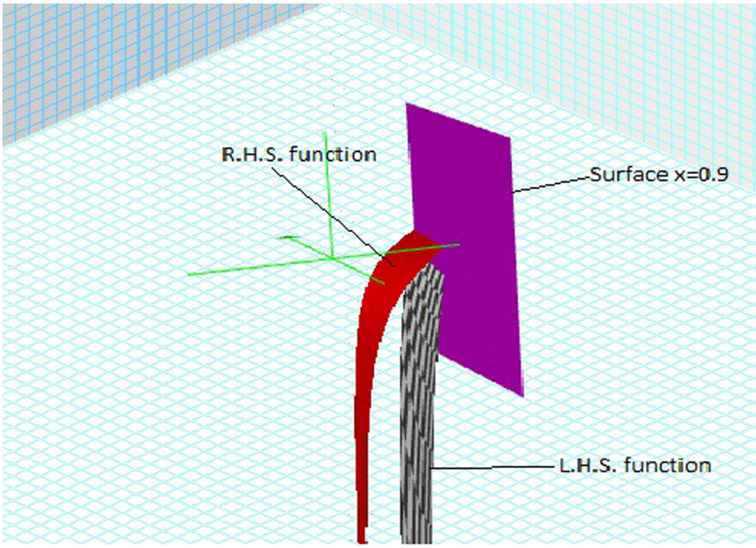

Figure 6. *Plot* of inequality for Case III(iii), 3D view

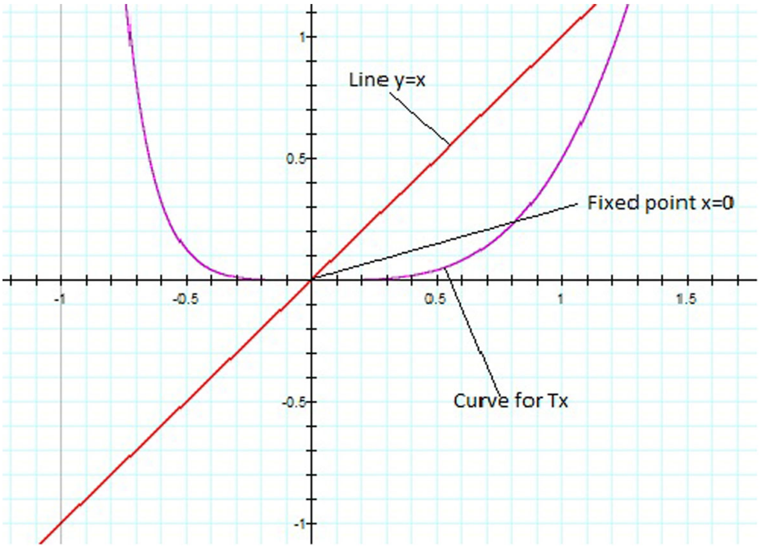

Figure 7. *Plot* showing fixed point of T

Utilizing $(F1)$, resulting subsequent corollary which is new in case of G-metric spaces.

Corollary 2.1. *Let (X, G) be a G-complete metric space and T be a self-mapping on X such that*

$$G(Tx,Ty,T^2y) \le \alpha \max\Big\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)]\Big\}. \tag{2.10}$$

For all $x, y, \in X$ and $\alpha \in (0,1)$. Then T has a fixed point. Moreover if $2\alpha \le 1$ then fixed point is unique.

With a view to demonstrate the validity of Corollary (2.1), following example is adopted.

Example 2.5. Let $X = [0,\infty)$ and

$$G(x,y,z) = \begin{cases} 0, & if\ and\ only\ if\ x = y = z, \\ \max\{x,y,z\}, & otherwise. \end{cases}$$

Then (X, G) is a complete G-metric space.

Define a self-mapping $T : X \to X$ by $Tx = \frac{x}{3}$. In view of verification of condition (2.10), subsequent terms are evaluated and accordingly various possible cases are discussed with $\alpha = \frac{2}{5}$.

$$\begin{aligned} G(Tx,Ty,T^2y) &= \tfrac{1}{3}max\{x,\ y\}, \\ G(x,y,Ty) &= max\{x,\ y\}, \\ G(x,Tx,Ty) &= max\{x,\ \tfrac{y}{3}\}, \\ G(x,Tx,T^2x) &= x, \\ \tfrac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)] &= \tfrac{1}{2}[y + max\{y,\ \tfrac{x}{9}\}]. \end{aligned}$$

Case I When $\frac{y}{3} \le x \le y$, immediately we have

$G(Tx,Ty,T^2y) = \frac{y}{3}$ and

$max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}$ $= y$.

Clearly

$\frac{1}{3}\ y \le \alpha\ y$ with $\alpha = \frac{2}{5}$.

Case II When $x \le \frac{y}{3}$, then one can obtain

$G(Tx,Ty,T^2y) = \frac{y}{3}$ and

$max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}$ $= y$.

Therefore

$\frac{1}{3}\, y \le \alpha\, y$ with $\alpha = \frac{2}{5}$.

Case III When $\frac{x}{9} \le y \le x$, then one can obtain

$G(Tx,Ty,T^2y) = \frac{x}{3}$ and

$max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}$ $= x$.

Therefore

$\frac{1}{3}\, x \le \alpha\, x$ with $\alpha = \frac{2}{5}$.

Arguing the same with $y \le \frac{x}{9}$, condition (2.10) remains true.

Thus all the conditions of Corollary 2.1 are satisfied and $x = 0$ is the unique fixed point of T.

Theorem 2.2. *Let (X,G) be a G-complete metric space and T be a self-mapping on X. Assume that there exist $F \in \Delta_F$ and $\tau > 0$ such that*

$$\begin{aligned} &G(Tx,Ty,T^2y) > 0 \\ &\quad\Rightarrow \tau + F(G(Tx,Ty,T^2y)) \leq F(aG(x,y,Ty) + bG(x,Tx,Ty) \\ &\qquad + cG(x,Tx,T^2x) + d[G(y,Ty,T^2y) + G(y,T^2x,T^2y)]), \end{aligned} \tag{2.11}$$

for all $x, y, \in X$, where $a,b,c,d \geq 0$ and $a+b+c+2d < 1$. Then T has a fixed point in X. Moreover if $2(a+b+c+2d) \leq 1$, then fixed point of T is unique.

Proof. Utilizing $F1$ and for all $x, y \in X$, we have

$$\begin{aligned} &G(Tx,Ty,T^2y) > 0 \\ &\quad\Rightarrow \tau + F(G(Tx,Ty,T^2y)) \leq F(aG(x,y,Ty) + bG(x,Tx,Ty) \\ &\qquad + cG(x,Tx,T^2x) + d[G(y,Ty,T^2y) + G(y,T^2x,T^2y)]) \\ &\quad\leq F\Bigg((a+b+c+2d)\max\Bigg\{G(x,y,Ty), G(x,Tx,Ty), \\ &\qquad G\left(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y) + G(y,T^2x,T^2y)]\right)\Bigg\}\Bigg) \\ &\quad= F\Bigg(\alpha\max\Bigg\{G(x,y,Ty), G(x,Tx,Ty), \\ &\qquad G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y) + G(y,T^2x,T^2y)])\Bigg\}\Bigg). \end{aligned}$$

Remaining proof follows on the similar lines as done in Theorem (2.1) with $a+b+c+2d = \alpha < 1$. □

Remark 2.1. On choosing suitable values of the constants $a, b, c,$ *and* d in Theorem 2.2, a multitude of the corollaries can be obtained which comprise new versions of Chatterjea-type result [9], Kannan-type theorem [15], Reich-type result [28] and Hardy–Rogers-type theorem [12] in the context of G-metric spaces; e.g. if we set $a = b = c = 0$ then Chatterjea[9]-type fixed point result in the context of G-metric space is obtained.

3. Results with integral inequalities under generalized F-contraction

Subsequent fixed point results for integral inequalities are inferred in the setting of G-metric spaces.

Theorem 3.1. *Let (X,G) be a G-complete metric space and $T : X \to X$ be a continuous self-mapping such that for $x, y \in X$ with*

$$\begin{aligned} &\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t > 0 \\ &\quad\Rightarrow \tau + F\left(\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t\right) \leq F\left(\alpha\int_0^{M(x,y)} \varphi(t)\mathrm{d}t\right), \end{aligned}$$

where $F \in \Delta_F$, $\alpha \in (0,1)$ and $\varphi : [0,\infty] \to [0,\infty]$ is a Lebesgue-integrable mapping satisfying

$$\int_0^{\epsilon} \varphi(t)\mathrm{d}t > 0$$

for $\epsilon > 0$ and $M(x,y) = \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), \frac{1}{2}[G(y,Ty,T^2y) + G(y,T^2x,T^2y)]\}$. Then T has a fixed point. Furthermore if $2\alpha \le 1$ then fixed point is unique.

Proof. Proof of the theorem is an easy consequence of Theorem 2.1. □

Following example demonstrates the validity of hypotheses of Theorem (3.1).

Example 3.1. Consider the following sequence $\{S_n\}_{n\in N}$ defined by
$S_1 = 4,$
$S_2 = 4 + 4^2,$
$S_3 = 4 + 4^2 + 4^3,$
$\vdots$
$S_n = 4 + 4^2 + 4^3 + \cdots + 4^n = \frac{4}{3}(4^n - 1).$
Let $X = \{S_n : n \in X\}$ and

$$G(x,y,z) = \begin{cases} 0, & if\ and\ only\ if\ x = y = z, \\ \max\{x,y,z\}, & otherwise. \end{cases}$$

Then (X, G) is a complete G-metric space.

Define the mapping $T : X \to X$ by $T(S_1) = S_1$ and $T(S_n) = S_{n-1}$ for every $n > 1$.

Now we assert that T is a Ciric-type generalized F-contraction for integral inequality of Theorem 3.1 in the framework of G-metric spaces with $F(\alpha) = \ln\alpha + \alpha$, clearly $F \in \Delta_F$ and for Lebesgue-integrable function $\varphi(t) = 2t$.

First of all, we examine that

$$\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t > 0$$
$$\Leftrightarrow\ (n = 1 \wedge m > 2) \vee (m = 1 \wedge n > 2) \vee (1 < n < m) \vee (1 < m < n).$$

For all the possible cases, we claim that

$$\frac{\int_0^{G(Tx,Ty,T^2y)} \varphi(t)dt e^{\int_0^{G(Tx,Ty,T^2y)} \varphi(t)dt - \alpha \int_0^{max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}} \varphi(t)dt}}{\alpha \int_0^{max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}} \varphi(t)dt}$$
$$\le e^{-\tau}, \tag{3.1}$$

for $x = S_n, y = S_m$, $n, m \in N$ and for $\tau = 50 > 0$ with $\alpha = \frac{1}{4}$. Clearly $2\alpha \le 1$.
In view of structure of $\{S_n\}$, one can conclude that

$$S_{n-1} < \frac{S_n}{4}.$$

Case I When $n = 1, m > 2$.

Then

$$\begin{aligned} G(Tx,Ty,T^2y) &= G(TS_n,TS_m,T^2S_m) = S_{m-1},\\ G(x,y,Ty) &= G(S_1,S_m,TS_m) = S_m,\\ G(x,Tx,Ty) &= G(S_1,TS_1,TS_m) = S_{m-1},\\ G(x,Tx,T^2x) &= G(S_1,TS_1,T^2S_1) = 0,\\ G(y,Ty,T^2y) &= G(S_m,TS_m,T^2S_m) = S_m,\\ G(y,T^2x,T^2y) &= G(S_m,T^2S_1,T^2S_m) = S_m. \end{aligned}$$

Then on utilizing (3.1) with $\alpha = \frac{1}{4}$, one gets

$$\begin{aligned} \frac{\{S_{m-1}\}^2 \cdot e^{\{S_{m-1}\}^2 - \alpha\{S_m\}^2}}{\alpha {S_m}^2} &\leq e^{{S_{m-1}}^2 - \frac{1}{4}{S_m}^2}\\ &= e^{-(\frac{1}{4}{S_m}^2 - {S_{m-1}}^2)}\\ &< e^{-50}, \quad since\ \tfrac{1}{4}{S_m}^2 - {S_{m-1}}^2 > 50\ (\ for\ m > 2). \end{aligned}$$

Thus in this case T is a Ciric-type generalized F-contractive mapping under integral inequality with $\tau = 50$.
Applying the routine calculations as done in Example 2.3, one can easily verify that T satisfies all the conditions of Theorem 3.1. Whereas S_1 is the unique fixed point of T.

Theorem 3.2. *Let (X,G) be a G-complete metric space and $T : X \to X$ be a continuous self-mapping such that for $x, y \in X$ with*

$$\begin{aligned} &\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t > 0\\ &\qquad \Rightarrow \tau + F\left(\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t\right)\\ &\qquad \leq F\left(a\int_0^{G(x,y,Ty)} \varphi(t)\mathrm{d}t + b\int_0^{G(x,Tx,Ty)} \varphi(t)\mathrm{d}t\right.\\ &\qquad\quad + c\int_0^{G(x,Tx,T^2x)} \varphi(t)\mathrm{d}t\\ &\qquad\quad \left. + d\int_0^{\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}} \varphi(t)\mathrm{d}t\right), \end{aligned}$$

for all $x, y \in X$. Where $F \in \Delta_F$, $a, b, c, d \geq 0$ with $a + b + c + 2d < 1$ and $\varphi : [0,\infty] \to [0,\infty]$ is a Lebesgue-integrable mapping satisfying

$$\int_0^{\epsilon} \varphi(t)\mathrm{d}t > 0$$

for $\epsilon > 0$. Then T has a fixed point.

Proof. Proof can easily be obtained in view of Theorem 2.2. □

Theorem 3.3. *Let (X, G) be a G-complete metric space and T be a self-mapping on X such that*

$$\int_0^{G(Tx,Ty,T^2y)} \varphi(t)\mathrm{d}t > 0$$
$$\leq \alpha \int_0^{max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{1}{2}[G(y,Ty,T^2y)+G(y,T^2x,T^2y)]\}} \varphi(t)\mathrm{d}t,$$

for all $x, y, \in X$. Where $\alpha \in (0, 1)$ and $\varphi : [0, \infty] \to [0, \infty]$ is a Lebesgue-integrable mapping satisfying

$$\int_0^{\epsilon} \varphi(t)\mathrm{d}t > 0.$$

Then T has a fixed point. Moreover if $2\alpha \leq 1$ then fixed point is unique.

Proof. Proof can easily be obtained in view of Corollary 2.1. □

4. Some applications

(1) Application to spring mass system

Considering the motion of a spring that is subject to a frictional force (in the case of horizontal spring) or a damping force (in the case of a vertical spring moves through a fluid, an example is the damping force supplied by a shock absorber in a car or a bicycle). In addition to these, the motion of spring is affected by an external force. Then such type of system for critical damped motion is represented by

$$\begin{cases} \frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{c}{m}\frac{\mathrm{d}u}{\mathrm{d}t} = K(t, u(t)); \\ u(0) = 0, u'(0) = a, \end{cases} \tag{4.1}$$

where $K : [0, I] \times R^+ \to R$ is a continuous function and $I > 0$.
Above problem is equivalent to the integral equation

$$u(t) = \int_0^t G(t, s)K(s, u(s))\mathrm{d}s, \quad t \in [0, I], \tag{4.2}$$

where $G(t, s)$ is the Green's function, given by

$$G(t, s) = \begin{cases} (t - s)e^{\tau(t-s)}, & 0 \leq s \leq t \leq I; \\ 0, & 0 \leq t \leq s \leq I. \end{cases} \tag{4.3}$$

where $\tau > 0$ is a constant, calculated in terms of c and m, mentioned in (4.1). Let $X = C([0, I], R^+)$ be the set of all non negative continuous real functions defined on $[0, I]$. For an arbitrary $u \in X$, we define

$$\|u\|_\tau = \sup_{t\in[0,I]} \{|u(t)|e^{-2\tau t}\}, \quad \textit{where } \tau > 0. \tag{4.4}$$

Define $G : X \times X \times X \to R^+$ by

$$G(u, v, w) = \max\{\|u - v\|_\tau, \|v - w\|_\tau, \|w - u\|_\tau\}, \tag{4.5}$$

where $\|u\|_\tau$ is defined by (4.4). Then clearly (X, G) is a complete G-metric space.

Consider the self-map $T : X \times X$, defined by

$$Tu(t) = \int_0^t G(t,s)K(s,u(s))\mathrm{d}s, \quad t \in [0,I]. \tag{4.6}$$

Then clearly u^* is a solution of (4.2), if and only if u^* is a fixed point of T.

Now we prove the subsequent theorem to guarantee the existence of fixed point of T.

Theorem 4.1. *Suppose the following hypotheses hold:*

(i) *K is increasing function ,*
(ii) *there exists $\tau > 0$ such that*

$$|K(s,u)-K(s,v)| \le \tau^2 e^{-\tau}\left[a|u-v| + c|u-Tu| + d\left(|v-Tv| + |v-T^2u|\right)\right],$$

for all $s \in [0,I]$, $u,v \in R^+$. Where $a \ge 0, c \ge 0$ and $d \ge 0$ such that $a+c+2d < 1$.

Then the integral equation (4.2) has a solution.

Proof. Already noted that $(C([0,I],R^+),G)$ is a complete G-metric space, where $G(u,v,w)$ is given by (4.5).
From assumption (i), T is increasing. Next, for all $u,v \in X$ such that $Tu(t) \neq Tv(t)$, we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\le \int_0^t G(t,s)|K(s,u(s)) - K(s,v(s))|\mathrm{d}s \\
&\le \int_0^t G(t,s)\tau^2 e^{-\tau}[a|u(s)-v(s)| + c|u(s)-Tu(s)| \\
&\quad + d(|v(s)-Tv(s)| + |v(s)-T^2u(s)|]\mathrm{d}s \\
&= \int_0^t \tau^2 e^{-\tau} e^{2\tau s} e^{-2\tau s}[a|u(s)-v(s)| + c|u(s)-Tu(s)| \\
&\quad + d(|v(s)-Tv(s)| + |v(s)-T^2u(s)|)]\,\mathrm{G(t,s)d}s \\
&\le \tau^2 e^{-\tau}[a\|u-v\|_\tau + c\|u-Tu\|_\tau + d(\|v-Tv\|_\tau + \|v-T^2u\|_\tau)] \\
&\quad \times \int_0^t e^{2\tau s}(t-s)e^{\tau(t-s)}\mathrm{d}s \\
&= \tau^2 e^{-\tau}\Big[a\|u-v\|_\tau + c\|u-Tu\|_\tau \\
&\quad + d(\|v-Tv\|_\tau + \|v-T^2u\|_\tau)]e^{\tau t}\int_0^t e^{\tau s}(t-s)\mathrm{d}s \\
&= \tau^2 e^{-\tau}[a\|u-v\|_\tau + c\|u-Tu\|_\tau + d(\|v-Tv\|_\tau + \|v-T^2u\|_\tau)] \\
&\quad \times e^{\tau t}\frac{e^{\tau t}}{\tau^2}(1-\tau t e^{-\tau t} - e^{-\tau t}) \\
&\le e^{-\tau}[a\|u-v\|_\tau + c\|u-Tu\|_\tau + d\left(\|v-Tv\|_\tau + \|v-T^2u\|_\tau\right)]e^{2\tau t}.
\end{aligned}$$

Clearly, in above expression $(1-\tau t e^{-\tau t} - e^{-\tau t}) \le 1$.

This implies that

$$\begin{aligned}
|Tu(t) - Tv(t)|e^{-2\tau t} &\le e^{-\tau}[a\|u-v\|_\tau \\
&\quad + c\|u-Tu\|_\tau + d(\|v-Tv\|_\tau + \|v-T^2u\|_\tau)],
\end{aligned}$$

or equivalently,

$$\|Tu-Tv\|_\tau \leq e^{-\tau}[a\|u-v\|_\tau+c\|u-Tu\|_\tau+d(\|v-Tv\|_\tau+\|v-T^2u\|_\tau)]. \quad (4.7)$$

Similarly, one can derive that

$$\|Tv-T^2v\|_\tau\leq e^{-\tau}[a\|v-Tv\|_\tau+c\|Tu-T^2u\|_\tau+d(\|Tv-T^2v\|_\tau+\|T^2v-T^2u\|_\tau)] \quad (4.8)$$

and

$$\|T^2v-Tu\|_\tau \leq e^{-\tau}[a\|Tv-u\|_\tau+c\|T^2u-u\|_\tau+d(\|v-T^2v\|_\tau+\|v-T^2v\|_\tau)]. \quad (4.9)$$

Utilizing (4.7), (4.8) and (4.9), one can get

$$\begin{aligned} &\max\{\|Tu-Tv\|_\tau, \|Tv-T^2v\|_\tau, \|T^2-Tu\|_\tau\} \\ &\quad \leq e^{-\tau}[a\max\{\|u-v\|_\tau, \|v-Tv\|_\tau, \|Tv-u\|_\tau\} \\ &\quad + c\max\{\|u-Tu\|_\tau, \|Tu-T^2u\|_\tau, \|T^2u-u\|_\tau\} \\ &\quad + d\max\{\|v-Tv\|_\tau, \|Tv-T^2v\|_\tau, \|T^2v-v\|_\tau\} \\ &\quad + d\max\{\|v-T^2u\|_\tau, \|T^2u-T^2v\|_\tau, \|T^2v-v\|_\tau\}. \end{aligned}$$

This leads to say that

$$\begin{aligned} G(Tu,Tv,T^2v) \leq e^{-\tau}[&aG(u,v,Tv)+cG(u,Tu,T^2u) \\ &+d\left(G(v,Tv,T^2v)+G(v,T^2u,T^2v)\right)]. \end{aligned}$$

Consequently, by passing to logarithms, one can obtain

$$\begin{aligned} ln(G(Tu,Tv,T^2v)) \leq ln[e^{-\tau}(&aG(u,v,Tv)+cG(u,Tu,T^2u) \\ &+d(G(v,Tv,T^2v)+G(v,T^2u,T^2v)))]. \end{aligned}$$

or

$$\begin{aligned} \tau + ln(G(Tu,Tv,T^2v)) \leq ln[&aG(u,v,Tv)+cG(u,Tu,T^2u) \\ &+d(G(v,Tv,T^2v)+G(v,T^2u,T^2v))]. \end{aligned}$$

Here, we notice that the function $F: R^+ \to R$ defined by $F(\alpha) = ln(\alpha)$, for each $\alpha \in C([0,I], R^+)$ and for $\tau > 0$, is in Δ_F. Consequently all the conditions of Theorem 2.2 are satisfied by operator T with $a \geq 0, c \geq 0, d \geq 0$ and $b = 0$ such that $a + c + 2d < 1$. Consequently T has a fixed point which is the solution of integral equation (4.2) and hence spring mass system has a solution. □

Remark 4.1. Moreover our Theorem 2.2 can be utilized to find the solution of following real time problems:

(i) Solution of electrical circuit equation.
(ii) Solution of equation generating by the motion of pendulum.
(iii) Problems related simple harmonic motion etc.

(2) Existence and uniqueness of bounded solutions of functional equations in dynamic programming:

In this section, the existence of solution for a class of functional equations through generalized F-contraction in G-metric spaces is established.

Let U and V be Banach spaces, and R is the field of real numbers. Let $X = B(W)$ denotes the set of all bounded real valued function on W. We define $G : X \times X \times X \to R^+$ by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

where $d(x, y) = \sup_{t\in W}\{|h_1(t) - h_2(t)|\}$. Then (X, G) is $G-$ complete metric space.

Consider the following functional equation

$$q(x) = \sup_{y\in D}\{g(x, y) + H(x, y, q(\rho(x, y)))\}, \quad x \in W. \tag{4.10}$$

where $g : W \times D \to R$ and $H : W \times D \times R \to R$ are bounded function. We consider W and D as the state and the decision spaces, respectively, $\rho : W \times D \to W$ represents transformation of the process and $q(x)$ represents the optimal return function with initial state x. We also define $T : B(W) \to B(W)$ by

$$T(h(x)) = \sup_{y\in D}\{g(x, y) + H(x, y, h(\rho(x, y)))\}, \quad for\ all\ x \in W\ and\ h \in X. \tag{4.11}$$

Naturally, if functions g and H are bounded then T is well-defined.

Let

$$M(h, k) = \max\Big\{G(h, k, Tk), G(h, Th, Tk), G(h, Th, T^2h), \frac{1}{2}[G(k, Tk, T^2k) + G(k, T^2h, T^2k)]\Big\}.$$

Now, we prove the existence and uniqueness of the solution of functional Eq. (4.10).

Theorem 4.2. *Let $T : X \to X$ be an upper-semi-continuous operator defined by (4.11) and assume that the following conditions are satisfied.*

(i) *$H : W \times D \times R \to R$ and $g : W \times D \times R \to R$ are continuous and bounded,*

(ii) *There exists $\tau \in R^+$ such that*

$$|H(x, y, h(x)) - H(x, y, k(x))| \le e^{-3\lambda}M(h, k), \quad \forall h, k, \in B(W),$$

where $x \in W$ and $y \in D$.

Then the functional Eq. 4.10 has a bounded solution.

Proof. Let λ be any arbitrary positive number, $x \in W$ and $h, k \in B(W)$, we select $y_1, y_2 \in D$ so that

$$T(h(x)) < g(x, y_1) + H(x, y_1, h(\rho(x, y_1))) + \lambda, \tag{4.12}$$

$$T(k(x)) < g(x, y_2) + H(x, y_2, k(\rho(x, y_2))) + \lambda. \tag{4.13}$$

On the other hand, by the definition of T, we have

$$T(h(x)) \ge g(x, y_2) + H(x, y_2, h(\rho(x, y_2))), \tag{4.14}$$

$$T(k(x)) \ge g(x, y_1) + H(x, y_1, k(\rho(x, y_1))). \tag{4.15}$$

Utilizing (4.12) and (4.15) with $T(h(x)) \neq T(k(x))$,one can get

$$\begin{aligned} T(h(x)) - T(k(x)) &< H(x, y_1, h(\rho(x, y_1))) - H(x, y_1, k(\rho(x, y_1))) + \lambda \\ &\leq |H(x, y_1, h(\rho(x, y_1))) - H(x, y_1, k(\rho(x, y_1)))| + \lambda. \end{aligned}$$

That is

$$T(h(x)) - T(k(x)) \leq e^{-3\tau} M(h, k) + \lambda. \tag{4.16}$$

Similarly from (4.13) and (4.14)

$$T(k(x)) - T(h(x)) \leq e^{-3\tau} M(h, k) + \lambda. \tag{4.17}$$

From (4.16) and (4.17),

$$|T(h(x)) - T(k(x))| \leq e^{-3\tau} M(h, k) + \lambda$$

or equivalently

$$d(T(h), T(k)) \leq e^{-3\tau} M(h, k) + \lambda \tag{4.18}$$

on the similar lines, one can obtain that

$$d(T(k), T^2(k)) \leq e^{-3\tau} M(h, k) + \lambda \tag{4.19}$$

and

$$d(T^2(k), T(h)) \leq e^{-3\tau} M(h, k) + \lambda. \tag{4.20}$$

From (4.18),(4.19) and (4.20), we have

$$\max\{d(T(h), T(k)), d(T(k), T^2(k)), d(T^2(k), T(h))\} \leq e^{-3\tau} M(h, k) + \lambda$$

this implies that

$$G(Th, Tk, T^2k) \leq e^{-3\tau} M(h, k) + \lambda.$$

Since above inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrary, then we conclude that

$$G(Th, Tk, T^2k) \leq e^{-3\tau} M(h, k), \quad \forall x \in W.$$

By passing to logarithms, we have

$$\ln(G(Th, Tk, T^2k)) \leq \ln(e^{-3\tau} M(h, k))$$

or

$$\tau + \ln((Th, Tk, T^2k)) \leq \ln(e^{-2\tau} M(h, k)).$$

We notice that the function $F : R^+ \to R$ defined by $F(x) = ln(x)$, for each $x \in W$, is in Δ_F. This amounts to say that the operator T is a Ciric-type generalized F-contraction. Thus, in view of continuity of T, Theorem 2.1 applies to the operator T with $\alpha = e^{-2\tau} < 1, \quad \tau > 0$. Indeed, T has a fixed point $h^* \in B(W)$ that is, h^* is a bounded solution of the functional equation (4.10). Moreover, if $\tau \geq 0.35$ then this solution is unique. □

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