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Fixed point theorems in new generalized metric spaces

Erdal Karapınar, Donal O'Regan, Antonio Francisco Roldán López de Hierro and Naseer Shahzad

Abstract. The aim of our paper is to present new fixed point theorems under very general contractive conditions in generalized metric spaces which were recently introduced by Jleli and Samet in [Fixed Point Theory Appl. **2015** (2015), doi:10.1186/s13663-015-0312-7]. Although these spaces are not endowed with a triangle inequality, these spaces extend some well known abstract metric spaces (for example, b-metric spaces, Hitzler–Seda metric spaces, modular spaces with the Fatou property, etc.). We handle several types of contractive conditions. The main theorems we present involve a reflexive and transitive binary relation that is not necessarily a partial order. We give a counterexample to a recent fixed point result of Jleli and Samet. Our results extend and unify recent results in the context of partially ordered abstract metric spaces.

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1. Introduction

Metric fixed point theory was motivated by the Banach fixed point theorem [6]. Recently some researchers have focused on the existence of fixed points in metric spaces endowed with partial orders. To deduce some applications to matrix equations, Ran and Reurings [30] and, later, Nieto and Rodríguez-López [28] established fixed point results assuming the classical Banach contractive condition was only satisfied at comparable (by a partial order) points; we also refer the reader to [5, 14, 19, 25, 32, 33, 34, 35, 36] and the references therein.

Recent research in fixed point theory has also focused on generalizing the underlying metric space. For example, Czerwik [11] introduced the notion of *b-metric space* and studied contractive mappings in these spaces; see also [4, 7, 9, 12, 23, 29]. In 2000, Hitzler and Seda [17] presented the concept of *dislocated metric space*, which were later considered in [1, 3, 16, 20]. Nakano [27] proposed a different type of space, called *modular spaces*, which were later redefined and generalized by Musielak and Orlicz [26]. The main advantage of modular spaces is that, although a metric is not defined on this class of spaces, many problems in fixed point theory can be reformulated using modular spaces (for details, see [13, 15, 21, 22, 24, 31]).

Jleli and Samet [18] introduced a generalization of the notion of a metric space which they called *a generalized metric space*. They noted that the above-mentioned abstract metric spaces may be regarded as particular cases of their general definition. They also stated and proved fixed point theorems for some contractions defined on these spaces.

In this paper, we point out that one of the fixed point results in [18] is not true (we give a counterexample). In this paper, we present fixed point results under very general contractive conditions. Our main results involve a reflexive and transitive binary relation that is not necessarily a partial order so our theory extends and unify recent results in the field of partially ordered abstract metric spaces.

2. Preliminaries

Henceforth, $\mathbb{N} = \{0, 1, 2, \ldots\}$ stands for the set of all nonnegative integer numbers, and let $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. From now on, X will denote a nonempty set and $T : X \to X$ will be a self-mapping. Given a point $x_0 \in X$, the *Picard sequence of* T *based on* x_0 is the sequence $\{x_n\}_{n\geq 0}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. In particular, $x_n = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the nth iterates of T (we assume that T^0 denotes the identity mapping on X). The *orbit of* x_0 *by* T is the set

$$\mathcal{O}_T(x_0) = \{T^n x_0 : n \in \mathbb{N}\}.$$

A *binary relation on* X is a nonempty subset $\mathcal{S}$ of the Cartesian product $X \times X$. For simplicity, we denote $x\mathcal{S}y$ if $(x, y) \in \mathcal{S}$. The notions of *reflexivity*, *transitivity*, *antisymmetry*, *preorder* and *partial order* can be found in [37]. The *trivial preorder on* X is denoted by $\mathcal{S}_X$, and is given by $x\mathcal{S}_X y$ for all $x, y \in X$.

Following [2, 8], an *extended comparison function* (or, simply, a *comparison function*) is a function $\phi : [0, \infty] \to [0, \infty]$ such that

- $(\mathcal{P}_1)$ ϕ is nondecreasing;
- $(\mathcal{P}_2)$ for all $t \in (0, \infty)$, $\lim_{n\to\infty} \phi^n(t) = 0$.

Let $\mathcal{F}_{\text{com}}$ be the family of all comparison functions. From $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, it is easy to check that: (1) $\phi(t) < t$ for all $t \in (0, \infty)$; (2) $\phi(0) = 0$; (3) ϕ is continuous at $t = 0$; (4) $\phi(t) \leq t$ for all $t \in [0, \infty]$; (5) $\phi^m(t) \leq \phi^n(t) \leq t$ for all $t \in [0, \infty]$ and all $n, m \in \mathbb{N}$ such that $n \leq m$; and (6) ϕ^n is nondecreasing for all $n \in \mathbb{N}$.

Henceforth, let $\mathcal{D}: X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, define the set

$$C(\mathcal{D}, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0 \right\}.$$

Generalized metric and generalized metric space are defined as follows.

Definition 2.1 (Jleli and Samet [18, Definition 2.1]**).** Let X be a nonempty set and let $\mathcal{D} : X \times X \to [0, \infty]$ be a function which satisfies the following conditions:

($\mathcal{D}_1$) $\mathcal{D}(x, y) = 0$ implies $x = y$;
($\mathcal{D}_2$) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$ for all $x, y \in X$;
($\mathcal{D}_3$) there exists $C > 0$ such that

$$\begin{aligned} &\text{if } x, y \in X \text{ and } \{x_n\} \in C(\mathcal{D}, X, x), \\ &\qquad \text{then } \mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y). \end{aligned} \tag{2.1}$$

Then $\mathcal{D}$ is called a *generalized metric* and the pair $(X, \mathcal{D})$ is called a *generalized metric space* (in the sense of Jleli and Samet; for short, a *JS-GMS*).

Remark 2.2. If $C > 0$ is a constant for which ($\mathcal{D}_3$) holds and $C' \geq C$, then ($\mathcal{D}_3$) also holds for C'. Then, if $(X, \mathcal{D})$ is a JS-GMS, the set of all constants for which ($\mathcal{D}_3$) holds is a nonempty, non-upper-bounded interval of nonnegative real numbers. Its infimum, which we will denote by $C_{X,\mathcal{D}}$, is the lowest (optimal) constant for which ($\mathcal{D}_3$) holds. The case $C_{X,\mathcal{D}} = 0$ leads to a trivial space. Hence, we will assume, from now on, that $C_{X,\mathcal{D}} > 0$.

Jleli and Samet presented in [18] a large list of abstract metric spaces that can be seen as particular cases of JS-GMSs: metric spaces, b-metric spaces, Hitzler–Seda metric spaces and modular spaces with the Fatou property. We add another one.

Example 2.3. Let $X = \{0, 1\}$ be endowed with the function $\mathcal{D} : X \times X \to [0, \infty]$ given by

$$\mathcal{D}(0, 0) = 0 \quad \text{and} \quad \mathcal{D}(1, 0) = \mathcal{D}(0, 1) = \mathcal{D}(1, 1) = \infty.$$

Let us show that $(X, \mathcal{D})$ is a JS-GMS. Properties ($\mathcal{D}_1$) and ($\mathcal{D}_2$) are apparent. To prove ($\mathcal{D}_3$), let $x, y \in X$ and $\{x_n\} \in C(\mathcal{D}, X, x)$. Since

$$\lim_{n\to\infty} \mathcal{D}(x_n, x) = 0,$$

there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$. If $x = y$, then $x_n = x = y$ for all $n \geq n_0$, so (2.1) holds with $C = 1$. Similarly, if $x \neq y$, then $x_n \neq y$ for all $n \geq n_0$, so

$$\mathcal{D}(x, y) = \infty = \mathcal{D}(x_n, y)$$

for all $n \geq n_0$. In any case, (2.1) holds with $C = 1$.

Given a JS-GMS $(X, \mathcal{D})$ and a point $x \in X$, a sequence $\{x_n\} \subseteq X$ is said to be:

- *$\mathcal{D}$-convergent to x* if $\{x_n\} \in C(\mathcal{D}, X, x)$, in such a case, we will write
$$\{x_n\} \xrightarrow{\mathcal{D}} x;$$
- *$\mathcal{D}$-Cauchy* if
$$\lim_{n,m\to\infty} \mathcal{D}(x_n, x_{n+m}) = 0. \tag{2.2}$$

A JS-GMS $(X, \mathcal{D})$ is *complete* if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent.

Remark 2.4. We believe that Jleli and Samet intended to define a *$\mathcal{D}$-Cauchy sequence* by
$$\lim_{n,m\to\infty} \mathcal{D}(x_n, x_m) = 0. \tag{2.3}$$
Clearly, (2.3) implies (2.2), but the converse seems to be false. Henceforth, we assume that $\mathcal{D}$-Cauchy sequences are given by (2.3).

Jleli and Samet proved that the limit of a $\mathcal{D}$-convergent sequence is unique.

Proposition 2.5 (Jleli and Samet [18, Proposition 2.4]**).** *Let $(X, \mathcal{D})$ be a JS-GMS. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ $\mathcal{D}$-converges to x and $\{x_n\}$ $\mathcal{D}$-converges to y, then $x = y$.*

Given $x_0 \in X$, the same authors denoted by $\delta(\mathcal{D}, T, x_0)$ the $\mathcal{D}$-diameter of the orbit of x_0 by T, that is,
$$\delta(\mathcal{D}, T, x_0) = \sup\left(\left\{\mathcal{D}(T^n x_0, T^m x_0) : n, m \in \mathbb{N}\right\}\right).$$

Theorem 2.6 (Jleli and Samet [18, Theorem 3.3]**).** *Suppose that the following conditions hold:*

(i) *$(X, \mathcal{D})$ is complete;*

(ii) *f is a k-contraction for some $k \in (0, 1)$, that is,*
$$\mathcal{D}\big(f(x), f(y)\big) \leq k\, \mathcal{D}(x, y)$$
for all $(x, y) \in X \times X$;

(iii) *there exists $x_0 \in X$ such that $\delta(\mathcal{D}, f, x_0) < \infty$.*

Then $\{f^n(x_0)\}$ converges to $\omega \in X$, a fixed point of f. Moreover, if $\omega' \in X$ is another fixed point of f such that $\mathcal{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.

Theorem 2.7 (Jleli and Samet [18, Theorem 4.3]**).** *Suppose that the following conditions hold:*

(i) *$(X, \mathcal{D})$ is complete;*

(ii) *f is a k-quasicontraction for some $k \in (0, 1)$, that is,*
$$\mathcal{D}\big(f(x), f(y)\big) \leq k \max\left\{\mathcal{D}(x, y), \mathcal{D}\big(x, f(x)\big), \mathcal{D}\big(y, f(y)\big), \mathcal{D}\big(x, f(y)\big), \mathcal{D}\big(y, f(x)\big)\right\}$$
for all $(x, y) \in X \times X$;

(iii) *there exists $x_0 \in X$ such that $\delta(\mathcal{D}, f, x_0) < \infty$.*

Then $\{f^n(x_0)\}$ *converges to some* $\omega \in X$*. If* $\mathcal{D}(x_0, f(\omega)) < \infty$ *and* $\mathcal{D}(\omega, f(\omega)) < \infty$*, then* ω *is a fixed point of* f*. Moreover, if* $\omega' \in X$ *is another fixed point of* f *such that* $\mathcal{D}(\omega, \omega') < \infty$ *and* $\mathcal{D}(\omega', \omega') < \infty$*, then* $\omega = \omega'$*.*

Definition 2.8 (Jleli and Samet [18, Definition 5.1]**).** A mapping $f : X \to X$ is *weak continuous* if the following condition holds: if $\{x_n\} \subset X$ is $\mathcal{D}$-convergent to $x \in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{f(x_{n_q})\}$ is $\mathcal{D}$-convergent to $f(x)$ (as $q \to \infty$).

Given a partial order $\preceq$ on X, let $E_{\preceq} = \{(x, y) \in X \times X : x \preceq y\}$.

Definition 2.9 (Jleli and Samet [18, Definition 5.3]**).** We say that the pair $(X, \mathcal{D})$ is $\mathcal{D}$*-regular* if the following condition holds: for every sequence $\{x_n\} \subset X$ satisfying $(x_n, x_{n+1}) \in E_{\preceq}$, for every n large enough, if $\{x_n\}$ is $\mathcal{D}$-convergent to $x \in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $(x_{n_q}, x) \in E_{\preceq}$, for every q large enough.

Definition 2.10 (Jleli and Samet [18, Definition 5.4]**).** We say that $f : X \to X$ is a *weak* k*-contraction* for some $k \in (0, 1)$ if the following condition holds:

$$(x, y) \in E_{\preceq} \Longrightarrow \mathcal{D}\big(f(x), f(y)\big) \leq k\, \mathcal{D}(x, y).$$

Theorem 2.11 (Jleli and Samet [18, Theorem 5.5]**).** *Suppose that the following conditions hold:*

- (i) $(X, \mathcal{D})$ *is complete;*
- (ii) f *is weak continuous;*
- (iii) f *is a weak* k*-contraction for some* $k \in (0, 1)$*, that is,*

$$\mathcal{D}\big(f(x), f(y)\big) \leq k\, \mathcal{D}(x, y) \quad \textit{for all } (x, y) \in E_{\preceq};$$

- (iv) *there exists* $x_0 \in X$ *such that* $\delta(\mathcal{D}, f, x_0) < \infty$ *and* $(x_0, f(x_0)) \in E_{\preceq}$*;*
- (v) f *is* $\preceq$*-monotone.*

Then $\{f^n(x_0)\}$ *converges to* $\omega \in X$ *such that* ω *is a fixed point of* f*. Moreover, if* $\mathcal{D}(\omega, \omega) < \infty$*, then* $\mathcal{D}(\omega, \omega) = 0$*.*

Theorem 2.12 (Jleli and Samet [18, Theorem 5.7]**).** *Suppose that the following conditions hold:*

- (i) $(X, \mathcal{D})$ *is complete;*
- (ii) $(X, \mathcal{D})$ *is* $\mathcal{D}$*-regular;*
- (iii) f *is a weak* k*-contraction for some* $k \in (0, 1)$*, that is,*

$$\mathcal{D}\big(f(x), f(y)\big) \leq k\, \mathcal{D}(x, y) \quad \textit{for all } (x, y) \in E_{\preceq};$$

- (iv) *there exists* $x_0 \in X$ *such that* $\delta(\mathcal{D}, f, x_0) < \infty$ *and* $(x_0, f(x_0)) \in E_{\preceq}$*;*
- (v) f *is* $\preceq$*-monotone.*

Then $\{f^n(x_0)\}$ *converges to* $\omega \in X$ *such that* ω *is a fixed point of* f*. Moreover, if* $\mathcal{D}(\omega, \omega) < \infty$*, then* $\mathcal{D}(\omega, \omega) = 0$*.*

3. Some considerations concerning the previous results

Basically, Jleli and Samet considered three kinds of contractivity conditions:

(C_1) $\mathcal{D}\big(f(x),f(y)\big)\leq k\,\mathcal{D}(x,y)$ for all $x,y\in X$;

(C_2) $\mathcal{D}\big(f(x),f(y)\big)\leq k\,\max\big\{\mathcal{D}(x,y),\,\mathcal{D}\big(x,f(x)\big),\,\mathcal{D}\big(y,f(y)\big),\,\mathcal{D}\big(x,f(y)\big),\,\mathcal{D}\big(y,f(x)\big)\big\}$ for all $x,y\in X$.

(C_3) $\mathcal{D}\big(f(x),f(y)\big)\leq k\,\mathcal{D}(x,y)$ for all $(x,y)\in E_{\preceq}$.

Conditions (C_1) and (C_3) are natural extensions of Banach's original contractivity condition, and assumption (C_2) was inspired by Ćirić quasicontraction-type mappings (see [10]). Although one can imagine that the techniques employed with each contractivity condition are similar, we must highlight that condition (C_2) is more complex than (C_1) and (C_3), especially in the case of generalized metric spaces in the sense of Jleli and Samet. To illustrate this, we present an example in which Theorem 2.7 fails (and Theorem 2.12 would also be incorrect if we had assumed a Ćirić-type contractivity condition).

Let X be the subset of real numbers given by $[0,1]\cup\{2\}$ and let

$$\mathcal{D}:X\times X\to[0,\infty]$$

be the function

$$\mathcal{D}\,(x,y)=\begin{cases}10 & \text{if either } (x,y)=(0,2) \text{ or } (x,y)=(2,0),\\ |x-y| & \text{otherwise.}\end{cases}$$

Notice that

$$|\,x-y\,|\leq\mathcal{D}\,(x,y)\leq 10\quad\text{for all } x,y\in X. \tag{3.1}$$

Let us show that $(X,\mathcal{D})$ is a complete JS-GMS. Indeed, properties $(\mathcal{D}_1)$ and $(\mathcal{D}_2)$ are apparent. Let us show that $(\mathcal{D}_3)$ also holds using $C=5$. Let $x,y\in X$ and let $\{x_n\}\in C(\mathcal{D},X,x)$. We have to prove that

$$\mathcal{D}(x,y)\leq 5\,\limsup_{n\to\infty}\mathcal{D}(x_n,y). \tag{3.2}$$

If $x=y$, then $\mathcal{D}(x,y)=0$, so (3.2) trivially holds. Assume that $x\neq y$. Since $\{\mathcal{D}(x_n,x)\}\to 0$, there exists $n_0\in\mathbb{N}$ such that

$$\mathcal{D}(x_n,x)\leq 1\quad\text{for all } n\geq n_0.$$

Hence $\mathcal{D}(x_n,x)=|x_n-x|\leq 1$ for all $n\geq n_0$ and $\{|x_n-x|\}\to 0$. We consider the following three cases.

- If $\mathcal{D}\,(x,y)=|x-y|$, it follows from (3.1) that

$$\mathcal{D}\,(x,y)=|x-y|=\lim_{n\to\infty}\,|x_n-y| = \limsup_{n\to\infty}\,|x_n-y|\leq\limsup_{n\to\infty}\mathcal{D}(x_n,y),$$

so (3.2) holds (even for $C=1$). Notice that the limit superior is always finite because $\mathcal{D}$ is bounded.

If $\mathcal{D}\,(x,y)=10$, we have two possibilities.

- If $x=0$ and $y=2$, then $\mathcal{D}(x,y)=\mathcal{D}(0,2)=10$ and, on the other hand, by (3.1),

$$\begin{aligned}5\limsup_{n\to\infty}\mathcal{D}(x_n,y) &\geq 5\limsup_{n\to\infty}|x_n-y|\\ &=5\lim_{n\to\infty}(2-x_n)=5\cdot 2=10.\end{aligned}$$

  Hence, the equality in (3.2) is reached with $C=5$ (for instance, by using the sequence $\{x_n\}$ given by $x_n=1/2^n$ for all $n\in\mathbb{N}$).
- If $x=2$ and $y=0$, taking into account that $\{|x_n-x|\}\to 0$, necessarily there exists $m_0\in\mathbb{N}$ such that $x_n=x=2$ for all $n\geq m_0$. Then

$$\mathcal{D}(x,y)=\mathcal{D}(0,2)=10=\mathcal{D}(x_n,y)$$

  for all $n\geq m_0$, and (3.2) holds (even with $C=1$).

As a result, property $(\mathcal{D}_3)$ holds and $(X,\mathcal{D})$ is a JS-GMS. Let us show that it is complete. Let $\{x_n\}\subseteq X$ be a $\mathcal{D}$-Cauchy sequence. As

$$|x_n-x_m|\leq\mathcal{D}(x_n,x_m)\quad\text{for all } n,m\in\mathbb{N},$$

then $\{x_n\}$ is a Cauchy sequence in X endowed with the Euclidean metric $d_E(x,y)=|x-y|$ for all $x,y\in X$. Since X is closed in $(\mathbb{R},d_E)$, then it is also d_E-complete. Hence, there exists $\omega\in X$ such that $\{|\,x_n-\omega\,|\}\to 0$. If $\omega\in[0,1]$, then there is $n_0\in\mathbb{N}$ such that $x_n\in[0,1]$ for all $n\geq n_0$. In this case,

$$\lim_{n\to\infty}\mathcal{D}(x_n,\omega)=\lim_{n\to\infty}|x_n-\omega|=0,$$

so $\{x_n\}$ $\mathcal{D}$-converges in X. On the contrary case, if $\omega=2$, there is an $n_0\in\mathbb{N}$ such that $x_n=2$ for all $n\geq n_0$. Therefore,

$$\lim_{n\to\infty}\mathcal{D}(x_n,\omega)=\lim_{n\to\infty}|\,\omega-\omega\,|=0,$$

so $\{x_n\}$ is also $\mathcal{D}$-convergent in X. In any case, $\{x_n\}$ is $\mathcal{D}$-convergent in X, and $(X,\mathcal{D})$ is a complete JS-GMS.

Next, consider $T:X\to X$ defined by

$$Tx=\begin{cases}2 & \text{if } x=0,\\ x/2 & \text{if } x\in(0,1]\cup\{2\}\ \text{(that is, if } x>0).\end{cases}$$

We claim that

$$\mathcal{D}(Tx,Ty)\leq\frac{1}{2}\max\left\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\right\}\tag{3.3}$$

for all $x,y\in X$. Indeed, let $x,y\in X$ be arbitrary points. If $x=y$, then $\mathcal{D}(Tx,Ty)=0$ and (3.3) trivially holds. Assume that $x\neq y$. Since inequality (3.3) is symmetric in x and y, we can assume that $x<y$. Necessarily $y>0$ and $Ty=y/2$. We consider the following cases.

- If $x=0$, then $\mathcal{D}(Tx,Ty)=\mathcal{D}(2,y/2)=2-y/2\leq 2$ and $\mathcal{D}(x,Tx)=\mathcal{D}(0,2)=10$, so (3.3) holds.
- If $x>0$, then $\mathcal{D}(Tx,Ty)=\mathcal{D}(x/2,y/2)=|x/2-y/2|=|x-y|/2=\mathcal{D}(x,y)/2$, so (3.3) also holds.

In any case, T satisfies the contractivity condition (3.3). If we take $x_0 = 1$, then $T^n x_0 = 1/2^n$ for all $n \in \mathbb{N}$, which converges to the point $\omega = 0$. Furthermore, $\delta(\mathcal{D}, T, x_0) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ and $\mathcal{D}(\omega, T\omega) < \infty$ because $\mathcal{D}$ is bounded. However, ω is not a fixed point of T. In fact, T is fixed point free. Therefore, Theorem 2.7 fails.

In addition to this, if we consider on X the partial order $\leq$, then $(X, \mathcal{D})$ is $\leq$-regular (to prove it, take into account that

$$\{x_n\} \xrightarrow{\mathcal{D}} x$$

implies that

$$\{x_n\} \xrightarrow{d_E} x$$

because $|x_n - x| \leq \mathcal{D}(x_n, x)$). However, this property is not strong enough to guarantee that T has a fixed point.

The main aim of our paper is to present some fixed point theorems in the context of Jleli and Samet's generalized metric spaces using the Ćirić-type contractivity condition (C_2).

4. Some notions in generalized metric spaces

In this section, we introduce some preliminaries on Jleli and Samet's generalized metric spaces endowed with arbitrary binary relations. Notice that the following notions are given involving nondecreasing sequences. Similar concepts can be introduced for nonincreasing sequences (we leave this task to the reader).

From now on, let $(X, \mathcal{D})$ be a JS-GMS and let $\mathcal{S}$ be a binary relation on X.

Definition 4.1. A sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-*nondecreasing* if $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 4.2. The JS-GMS $(X, \mathcal{D})$ is $\mathcal{S}$-*nondecreasing-regular* if

$$\left.\begin{array}{l} \{x_n\} \in C(\mathcal{D}, X, z) \\ \{x_n\}\ \mathcal{S}\text{-nondecreasing} \end{array}\right\} \Longrightarrow x_n \mathcal{S} z \quad \text{for all } n \in \mathbb{N}.$$

Every regular JS-GMS is also $\mathcal{S}$-nondecreasing-regular, whatever the binary relation $\mathcal{S}$, but the converse is false.

Example 4.3. Let X be the real interval $[0, 1]$ endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\mathcal{S}$ be the binary relation on X given by $x \mathcal{S} y$ if $0 < x \leq y \leq 1$. Then (X, d) is $\leq$-nondecreasing-regular. However, the sequence $\{1/n\}_{n \geq 1}$ shows that (X, d) is not regular.

Definition 4.4. The JS-GMS $(X, \mathcal{D})$ is $\mathcal{S}$-*nondecreasing-complete* if

$$\left.\begin{array}{l} \{x_n\} \subseteq X \text{ is } \mathcal{D}\text{-Cauchy} \\ \{x_n\}\ \mathcal{S}\text{-nondecreasing} \end{array}\right\} \Longrightarrow \{x_n\} \text{ is } \mathcal{D}\text{-convergent in } X.$$

Every complete JS-GMS is also $\mathcal{S}$-nondecreasing-complete, whatever the binary relation $\mathcal{S}$. However, Example 4.3 shows that the converse is false.

Definition 4.5. A mapping T is *$\mathcal{S}$-nondecreasing-continuous at $z \in X$* if $\{Tx_n\} \in C(\mathcal{D}, X, Tz)$ for all $\mathcal{S}$-nondecreasing sequence $\{x_n\} \in C(\mathcal{D}, X, z)$. The mapping T is *$\mathcal{S}$-nondecreasing-continuous* if it is $\mathcal{S}$-nondecreasing-continuous at each point of X.

Every continuous mapping is also $\mathcal{S}$-nondecreasing-continuous, whatever the binary relation $\mathcal{S}$, but the converse is false.

Example 4.6. Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d_E(x, y) = |x - y|$ for all $x, y \in X$. If $\leq$ denotes the canonical order in $\mathbb{R}$ and $T : X \to X$ is given by

$$Tx = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

then T is $\leq$-nondecreasing-continuous at $x = 0$, but it is not continuous at $x = 0$.

Definition 4.7. Given a nonempty subset A of X, we will say that $\mathcal{S}$ is *transitive on A* if

$$x, y, z \in A,\ x\mathcal{S}y,\ y\mathcal{S}z \implies x\mathcal{S}z.$$

Given $n_0 \in \mathbb{N}$, we will use the notation

$$\delta_{n_0}(\mathcal{D}, T, x_0) = \sup\left(\left\{\mathcal{D}\big(T^n x_0, T^m x_0\big) : n, m \in \mathbb{N},\ n, m \geq n_0\right\}\right)$$

and

$$\delta(\mathcal{D}, T, x_0) = \delta_0\big(\mathcal{D}, T, x_0\big) = \sup\left(\left\{\mathcal{D}\big(T^n x_0, T^m x_0\big) : n, m \in \mathbb{N}\right\}\right).$$

By the symmetry of $\mathcal{D}$, we can alternatively express

$$\delta_{n_0}(\mathcal{D}, T, x_0) = \sup\left(\left\{\mathcal{D}\big(T^n x_0, T^m x_0\big) : n, m \in \mathbb{N},\ m \geq n \geq n_0\right\}\right).$$

Notice that if $n, m \in \mathbb{N}$ verify $n \leq m$, then

$$\delta_m(\mathcal{D}, T, x_0) \leq \delta_n(\mathcal{D}, T, x_0) \leq \delta(\mathcal{D}, T, x_0). \tag{4.1}$$

Example 4.8. Let $X = \{0, 1\}$ and let $\mathcal{D}$ be given by

$$\mathcal{D}(0, 0) = 0, \quad \mathcal{D}(1, 0) = \mathcal{D}(0, 1) = \mathcal{D}(1, 1) = \infty.$$

In Example 2.3 we showed that $(X, \mathcal{D})$ is a JS-GMS. Let $T : X \to X$ be defined by $Tx = 0$ for all $x \in X$. If we take $x_0 = 1 \in X$, then

$$\delta(\mathcal{D}, T, x_0) = \delta_0(\mathcal{D}, T, x_0) \geq \mathcal{D}(x_0, x_0) = \mathcal{D}(1, 1) = \infty,$$

but $\delta_1(\mathcal{D}, T, x_0) = 0$.

5. Fixed point theorems in the context of generalized metric spaces

In this section we will introduce the main results of this paper. We divide this section into four parts: in the first one, we present general results in order to deduce that a Picard sequence is $\mathcal{D}$-Cauchy and, when the space is appropriately complete, we describe an upper bound concerning how the sequence approximates its limit; in the second part, we prove that its limit is a fixed point of the operator by assuming some kind of continuity; in the third part, we show that the limit of the sequence is a fixed point if we replace the continuity with regularity and other appropriate conditions; finally, in the last part, we explain that, when the contractivity condition is the simplest one and is assumed for all pairs of points of the space, the operator is continuous.

Remark 5.1. For simplicity, we will always consider a preorder $\mathcal{S}$ on X. However, the reader may notice that the same results can also be obtained by using a binary relation $\mathcal{S}'$ on X that has only to be reflexive and transitive on the orbit $\mathcal{O}_T(x_0)$ (or in $\mathcal{O}'_T(x_0)$ or in $E_{\mathcal{S}} = \{(x,y) \in X \times X : x\mathcal{S}y\}$, when these sets are considered).

5.1. General scheme and an upper bound for convergence

From now on, let $(X,\mathcal{D})$ be a JS-GMS and let $\mathcal{S}$ be a binary relation on X. We begin this section by explaining why, although our main results will involve a binary relation, we will only assume the contractivity condition over pairs of points in the orbit of an initial condition.

Lemma 5.2. *Let $(X,\mathcal{D})$ be a JS-GMS endowed with a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that x_0 and Tx_0 are $\mathcal{S}$-comparable. Suppose that there exists a function $\phi : [0,\infty] \to [0,\infty]$ such that*

$$\mathcal{D}(Tx,Ty) \leq \phi\Big(\max\big\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\big\}\Big) \tag{5.1}$$

for $x,y \in X$ such that $x\mathcal{S}y$. Then (5.1) holds for all $x,y \in \mathcal{O}_T(x_0)$.

Proof. Let us consider the Picard sequence $\{x_{n+1} = Tx_n = T^{n+1}x_0\}_{n\in\mathbb{N}}$ of T based on x_0. If we suppose that $x_0\mathcal{S}Tx_0 = x_1$ (the case $Tx_0\mathcal{S}x_0$ is similar), as T is $\mathcal{S}$-nondecreasing, then $x_1 = Tx_0\mathcal{S}Tx_1 = x_2$. Repeating this argument, $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, and as $\mathcal{S}$ is a preorder, then $x_n\mathcal{S}x_m$ for all $n,m \in \mathbb{N}$ such that $n \leq m$. Furthermore, as condition (5.1) is symmetric on x and y (because $\mathcal{D}$ is symmetric), then (5.1) holds for all x_n and x_m (being $n,m \in \mathbb{N}$ arbitrary), so it holds for all $x,y \in \mathcal{O}_T(x_0)$. □

In the following result we establish a relationship between $\delta_k(\mathcal{D},T,x_0)$ and $\delta_{k+1}(\mathcal{D},T,x_0)$ under a contractivity condition that must be only satisfied by pairs of points in an orbit.

Lemma 5.3. *Let $(X,\mathcal{D})$ be a JS-GMS, let $T : X \to X$ be a self-mapping and let $x_0 \in X$ be a point for which there exists $n_0 \in \mathbb{N}$ such that*

$$\delta_{n_0}(\mathcal{D},T,x_0) < \infty.$$

Suppose that there exists a nondecreasing function $\phi\colon[0,\infty]\to[0,\infty]$ *such that*

$$\mathcal{D}(Tx,Ty)\le\phi\big(\max\left\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\right\}\big) \tag{5.2}$$

for $x,y\in\mathcal{O}_T(x_0)$. *Then*

$$\delta_{k+1}(\mathcal{D},T,x_0)\le\phi\big(\delta_k(\mathcal{D},T,x_0)\big)\quad \textit{for all } k\in\mathbb{N},\ k\ge n_0.$$

In particular,

$$\delta_{n_0+k}(\mathcal{D},T,x_0)\le\phi^k\big(\delta_{n_0}(\mathcal{D},T,x_0)\big)\quad \textit{for all } k\in\mathbb{N}.$$

Proof. Let $k\in\mathbb{N}$ be an arbitrary integer number such that $k\ge n_0$. By (4.1),

$$\delta_{k+1}(\mathcal{D},T,x_0)\le\delta_k(\mathcal{D},T,x_0)\le\delta_{n_0}(\mathcal{D},T,x_0)<\infty.$$

Let $n,m\in\mathbb{N}$ be such that $m\ge n\ge k+1$, and let us define $m'=m-1$ and $n'=n-1$. Then $m'\ge n'\ge k$. By (5.2),

$$\begin{aligned}\mathcal{D}(T^nx_0,T^mx_0)&=\mathcal{D}\big(T^{n'+1}x_0,T^{m'+1}x_0\big)=\mathcal{D}\big(TT^{n'}x_0,TT^{m'}x_0\big)\\&\le\phi\Big(\max\Big\{\mathcal{D}\big(T^{n'}x_0,T^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,TT^{n'}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,TT^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,TT^{m'}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,TT^{n'}x_0\big)\Big\}\Big)\\&=\phi\Big(\max\Big\{\mathcal{D}\big(T^{n'}x_0,T^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{n'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,T^{m'+1}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{m'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,T^{n'+1}x_0\big)\Big\}\Big).\end{aligned}$$

If we denote

$$\Omega_k=\left\{\mathcal{D}(T^px_0,T^qx_0):p,q\in\mathbb{N},\ p,q\ge k\right\},$$

then

$$\begin{aligned}&\mathcal{D}\big(T^{n'}x_0,T^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{n'+1}x_0\big),\mathcal{D}\big(T^{m'}x_0,T^{m'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{n'}x_0,T^{m'+1}x_0\big),\mathcal{D}\big(T^{m'}x_0,T^{n'+1}x_0\big)\in\Omega_k.\end{aligned}$$

Hence,

$$\begin{aligned}&\max\Big\{\mathcal{D}\big(T^{n'}x_0,T^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{n'+1}x_0\big),\mathcal{D}\big(T^{m'}x_0,T^{m'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{n'}x_0,T^{m'+1}x_0\big),\mathcal{D}\big(T^{m'}x_0,T^{n'+1}x_0\big)\Big\}\le\sup\Omega_k=\delta_k(\mathcal{D},T,x_0).\end{aligned}$$

As a result, as ϕ is nondecreasing, for all $m\ge n\ge k+1$,

$$\begin{aligned}\mathcal{D}(T^nx_0,T^mx_0)&\le\phi\Big(\max\Big\{\mathcal{D}\big(T^{n'}x_0,T^{m'}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{n'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,T^{m'+1}x_0\big),\mathcal{D}\big(T^{n'}x_0,T^{m'+1}x_0\big),\\&\qquad\mathcal{D}\big(T^{m'}x_0,T^{n'+1}x_0\big)\Big\}\Big)\\&\le\phi\big(\delta_k(\mathcal{D},T,x_0)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\delta_{k+1}(\mathcal{D},T,x_0) &= \sup\left(\left\{\mathcal{D}(T^n x_0, T^m x_0) : n,m\in\mathbb{N},\ m\geq n\geq k+1\right\}\right)\\ &\leq \phi\big(\delta_k(\mathcal{D},T,x_0)\big).\end{aligned}$$

Repeating this argument and taking into account that ϕ is nondecreasing, it follows that for all $k\in\mathbb{N}$,

$$\begin{aligned}\delta_{n_0+k}(\mathcal{D},T,x_0) &\leq \phi\big(\delta_{n_0+k-1}(\mathcal{D},T,x_0)\big) \leq \phi^2\left(\delta_{n_0+k-2}(\mathcal{D},T,x_0)\right)\\ &\leq \cdots \leq \phi^k\left(\delta_{n_0}(\mathcal{D},T,x_0)\right).\end{aligned}$$

□

The following statement will be useful to prove that a Picard sequence is $\mathcal{D}$-Cauchy.

Lemma 5.4. *Let $(X,\mathcal{D})$ be a JS-GMS, let $T:X\to X$ be a self-mapping and let $x_0\in X$ be a point for which there exists $n_0\in\mathbb{N}$ such that*

$$\delta_{n_0}(\mathcal{D},T,x_0)<\infty.$$

Suppose that there exists $\phi\in\mathcal{F}_{\mathrm{com}}$ such that

$$\mathcal{D}(Tx,Ty)\leq\phi\big(\max\left\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\right\}\big) \tag{5.3}$$

for all $x,y\in\mathcal{O}_T(x_0)$. Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 is a $\mathcal{D}$-Cauchy sequence.

Proof. Let us consider the Picard sequence $\{x_{n+1}=Tx_n=T^{n+1}x_0\}_{n\in\mathbb{N}}$ of T based on x_0. Taking into account that $\phi(t)\leq t$ (in particular, $\phi^k(t)\leq t<\infty$) for all $t\in(0,\infty)$, Lemma 5.3 guarantees that

$$\delta_{n_0+k}(\mathcal{D},T,x_0)\leq\phi^k\left(\delta_{n_0}(\mathcal{D},T,x_0)\right)<\infty \quad \text{for all } k\in\mathbb{N}.$$

Let us show that the sequence $\{x_n\}$ is $\mathcal{D}$-Cauchy. Let $t_0=\delta_{n_0}(\mathcal{D},T,x_0)$. If $t_0=0$, then $\mathcal{D}(x_n,x_m)=0$ for all $n,m\geq n_0$. In particular,

$$\lim_{n,m\to\infty}\mathcal{D}(x_n,x_m)=0,$$

so $\{x_n\}$ is $\mathcal{D}$-Cauchy. In this case, the proof is finished. Next, assume that $t_0=\delta_{n_0}(\mathcal{D},T,x_0)\in(0,\infty)$. Let $\varepsilon>0$ be arbitrary. Since $\lim_{n\to\infty}\phi^n(t_0)=0$, then there exists $k_0\in\mathbb{N}$ such that $\phi^k(t_0)<\varepsilon$ for all $k\geq k_0$. In such a case, by the symmetry of $\mathcal{D}$,

$$\begin{aligned}&\sup\left(\left\{\mathcal{D}(T^n x_0,T^m x_0) : n,m\in\mathbb{N},\ n,m\geq n_0+k_0\right\}\right)\\ &\qquad=\delta_{n_0+k_0}(\mathcal{D},T,x_0)\leq\phi^{k_0}\left(\delta_{n_0}(\mathcal{D},T,x_0)\right)=\phi^{k_0}(t_0)<\varepsilon.\end{aligned}$$

Hence, $\lim_{n,m\to\infty}\mathcal{D}(x_n,x_m)=0$ and $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence. □

In the following result, a preorder $\mathcal{S}$ and an $\mathcal{S}$-nondecreasing operator are considered. Under the appropriate class of completeness, the Picard sequence is convergent and its limit satisfies some properties. In fact, we describe upper bounds for the estimated error depending on the constant $C=C_{X,\mathcal{D}}$, which is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3)$.

Theorem 5.5. *Let $(X, \mathcal{D})$ be a JS-GMS with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\big(\max\{\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty), \mathcal{D}(y, Tx)\}\big) \quad \text{for all } x, y \in \mathcal{O}_T(x_0).$$

Then the Picard sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ of T based on x_0 is an $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy sequence.

Furthermore, if $(X, \mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete, then $\{T^n x_0\}_{n \in \mathbb{N}}$ $\mathcal{D}$-converges to a point $\omega \in X$ that verifies the following conditions:

$$\mathcal{D}(\omega, \omega) = 0 \tag{5.4}$$

and

$$\mathcal{D}(T^n x_0, \omega) \leq C\, \phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D}, T, x_0)\big) \quad \text{for all } n \in \mathbb{N}, n \geq n_0, \tag{5.5}$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X, \mathcal{D})$ satisfies property $(\mathcal{D}_3)$.

Proof. As shown in the proof of Lemma 5.2, the Picard sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is $\mathcal{S}$-nondecreasing, and Lemma 5.4 guarantees that it is a $\mathcal{D}$-Cauchy sequence. Additionally, assume that $(X, \mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete. Hence the sequence $\{T^n x_0\}$ is $\mathcal{D}$-convergent, that is, there exists $\omega \in X$ such that

$$\{x_n\} \xrightarrow{\mathcal{D}} \omega.$$

By using $(\mathcal{D}_3)$,

$$\mathcal{D}(\omega, \omega) \leq C \limsup_{m \to \infty} \mathcal{D}(T^m x_0, \omega) = 0,$$

so $\mathcal{D}(\omega, \omega) = 0$. Moreover, it follows from $(\mathcal{D}_3)$ and Lemma 5.3 that, for all $n \in \mathbb{N}$ such that $n \geq n_0$,

$$\begin{aligned} \mathcal{D}(\omega, T^n x_0) &\leq C \limsup_{m \to \infty} \mathcal{D}\left(T^{m+n_0} x_0, T^n x_0\right) \\ &\leq C\, \delta_n(\mathcal{D}, T, x_0) \\ &\leq C\, \phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D}, T, x_0)\big). \end{aligned}$$

□

In the next result, we slightly change the points for which the contractivity condition must hold.

Corollary 5.6. *Let $(X, \mathcal{D})$ be a JS-GMS with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\big(\max\{\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty), \mathcal{D}(y, Tx)\}\big) \quad \text{for all } x, y \in X \text{ such that } x \mathcal{S} y.$$

Then the Picard sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ of T based on x_0 is an $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy sequence.

Furthermore, if $(X,\mathcal{D})$ *is* $\mathcal{S}$*-nondecreasing-complete, then* $\{T^n x_0\}_{n\in\mathbb{N}}$ $\mathcal{D}$*-converges to a point* $\omega \in X$ *that verifies* $\mathcal{D}(\omega,\omega)=0$ *and*

$$\mathcal{D}(T^n x_0,\omega) \leq C\,\phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D},T,x_0)\big) \quad \textit{for all } n\in\mathbb{N}, n\geq n_0,$$

where $C = C_{X,\mathcal{D}}$ *is the (lowest) constant for which* $(X,\mathcal{D})$ *satisfies property* $(\mathcal{D}_3)$.

Proof. If the contractivity condition holds for all $x,y\in X$ such that $x\mathcal{S}y$, then Lemma 5.2 guarantees that it also holds for all $x,y\in\mathcal{O}_T(x_0)$, so Theorem 5.5 is applicable. □

5.2. Some fixed point theorems under $\mathcal{S}$-nondecreasing-continuity

In this section, we introduce some fixed point results by showing that the limit ω of the Picard sequence is a fixed point of T. To do this, we will assume that T is $\mathcal{S}$-nondecreasing-continuous. Hence, we obtain the following statement.

Theorem 5.7. *Let* $(X,\mathcal{D})$ *be an* $\mathcal{S}$*-nondecreasing-complete JS-GMS with respect to a preorder* $\mathcal{S}$ *and let* $T: X\to X$ *be an* $\mathcal{S}$*-nondecreasing self-mapping. Let* $x_0\in X$ *be a point such that* $x_0\mathcal{S}Tx_0$ *and* $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ *for some* $n_0\in\mathbb{N}$. *Suppose that there exists* $\phi\in\mathcal{F}_{\text{com}}$ *such that*

$$\begin{aligned}\mathcal{D}(Tx,Ty)\leq \phi\big(\max\{&\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\\ &\mathcal{D}(y,Tx)\}\big) \quad \textit{for all } x,y\in\mathcal{O}_T(x_0).\end{aligned} \tag{5.6}$$

Additionally, assume that

(a) T *is* $\mathcal{S}$*-nondecreasing-continuous.*

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 $\mathcal{D}$*-converges to a fixed point* ω *of* T*. Furthermore,* $\mathcal{D}(\omega,\omega)=0$ *and*

$$\mathcal{D}(T^n x_0,\omega) \leq C\,\phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D},T,x_0)\big) \quad \textit{for all } n\in\mathbb{N}, n\geq n_0,$$

where $C = C_{X,\mathcal{D}}$ *is the (lowest) constant for which* $(X,\mathcal{D})$ *satisfies property* $(\mathcal{D}_3)$.

In addition to this, if condition (5.6) *holds for all* $x,y\in X$ *such that* $x\mathcal{S}y$*, and* ω' *is another fixed point of* T *such that* $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ *and* $\mathcal{D}(\omega',\omega')<\infty$*, then* $\omega=\omega'$.

The main advantages of Theorem 5.7 over Theorems 2.6 and 2.11 are the following ones.

- $\mathcal{S}$ is a preorder, but it has not to be a partial order (as a consequence, an interesting particular case is the binary relation $\mathcal{S}_X$ given by $x\mathcal{S}_X y$ for all $x,y\in X$).
- The class of auxiliary function $\mathcal{F}_{\text{com}}$ is wider than the subclass
$$\{\phi_k(t)=kt\}_{k\in[0,1]}.$$
- The contractivity condition (5.6) must only be satisfied over points x and y in the orbit of x_0, but we do not have to prove it for all $x,y\in X$.
- The generalized metric space $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete, but it does not have to be complete (see Example 4.3).

- The mapping T is $\mathcal{S}$-nondecreasing-continuous, but it does not have to be continuous (see Example 4.6).
- We assume that
$$\delta_{n_0}(\mathcal{D},T,x_0)<\infty$$
for some $n_0\in\mathbb{N}$ rather than $\delta(\mathcal{D},T,x_0)<\infty$ (see Example 4.8).

Proof. By Theorem 5.5, the Picard sequence $\{T^nx_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing and it converges to a point $\omega\in X$ verifying (5.4) and (5.5). Moreover, as we additionally assume that T is $\mathcal{S}$-nondecreasing-continuous, then
$$\{x_{n+1}=Tx_n\}\xrightarrow{\mathcal{D}}T\omega.$$
Proposition 2.5 guarantees that $T\omega=\omega$, so ω is a fixed point of T.

Next suppose that condition (5.6) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and assume that ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$. As $\mathcal{S}$ is reflexive, then $\omega'\mathcal{S}\omega'$, and condition (5.6) yields
$$\begin{aligned}\mathcal{D}(\omega',\omega')&=\mathcal{D}(T\omega',T\omega')\\&\leq\phi\big(\max\big\{\mathcal{D}(\omega',\omega'),\mathcal{D}(\omega',T\omega'),\mathcal{D}(\omega',T\omega'),\\&\qquad\qquad\mathcal{D}(\omega',T\omega'),\mathcal{D}(\omega',T\omega')\big\}\big)\\&=\phi\big(\mathcal{D}(\omega',\omega')\big).\end{aligned}$$
As $\mathcal{D}(\omega',\omega')<\infty$, then necessarily $\mathcal{D}(\omega',\omega')=0$. As a consequence,
$$\begin{aligned}\mathcal{D}(\omega,\omega')&=\mathcal{D}(T\omega,T\omega')\\&\leq\phi\big(\max\big\{\mathcal{D}(\omega,\omega'),\mathcal{D}(\omega,T\omega),\mathcal{D}(\omega',T\omega'),\\&\qquad\qquad\mathcal{D}(\omega,T\omega'),\mathcal{D}(\omega',T\omega)\big\}\big)\\&=\phi\big(\max\big\{\mathcal{D}(\omega,\omega'),\mathcal{D}(\omega,\omega),\mathcal{D}(\omega',\omega')\big\}\big)\\&=\phi\big(\mathcal{D}(\omega,\omega')\big).\end{aligned}$$
Similarly, as $\mathcal{D}(\omega,\omega')<\infty$, then $\mathcal{D}(\omega,\omega')=0$, so $\omega=\omega'$. □

In order to illustrate the power and usability of Theorem 5.7, we could list here several corollaries by replacing some of its hypotheses by stronger ones. For instance, the following statements are easy consequences of Theorem 5.7.

Corollary 5.8. *Let $(X,\mathcal{D})$ be a complete JS-GMS and let $T:X\to X$ be a self-mapping. Let $x_0\in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ for some $n_0\in\mathbb{N}$. Suppose that there exists $\phi\in\mathcal{F}_{\text{com}}$ such that*
$$\begin{aligned}\mathcal{D}(Tx,Ty)\leq\phi\big(\max\big\{&\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\\&\mathcal{D}(y,Tx)\big\}\big)\quad\text{for all } x,y\in\mathcal{O}_T(x_0).\end{aligned}\tag{5.7}$$
Additionally, assume that

(a) *T is continuous.*

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 $\mathcal{D}$*-converges to a fixed point* ω *of* T*. Furthermore,* $\mathcal{D}(\omega,\omega)=0$ *and*

$$\mathcal{D}(T^n x_0,\omega) \le C\,\phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D},T,x_0)\big) \quad \text{for all } n\in\mathbb{N}, n\ge n_0,$$

where $C = C_{X,\mathcal{D}}$ *is the (lowest) constant for which* $(X,\mathcal{D})$ *satisfies property* $(\mathcal{D}_3)$*.*

In addition to this, if condition (5.7) *holds for all* $x,y\in X$*, and* ω' *is another fixed point of* T *such that* $\mathcal{D}(\omega,\omega')<\infty$ *and* $\mathcal{D}(\omega',\omega')<\infty$*, then* $\omega=\omega'$*.*

Proof. It follows from Theorem 5.7 by using the trivial preorder $\mathcal{S}_X$ given by $x\mathcal{S}_X y$ for all $x,y\in X$. □

Corollary 5.9. *Let* $(X,\mathcal{D})$ *be an* $\preceq$*-nondecreasing-complete JS-GMS with respect to a partial order* $\preceq$ *on* X *and let* $T\colon X\to X$ *be an* $\preceq$*-nondecreasing self-mapping. Let* $x_0\in X$ *be a point such that* $x_0\preceq Tx_0$ *and* $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ *for some* $n_0\in\mathbb{N}$*. Suppose that there exists* $\phi\in\mathcal{F}_{\text{com}}$ *such that*

$$\begin{aligned}\mathcal{D}(Tx,Ty) \le \phi\Big(\max\big\{&\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\\ &\mathcal{D}(y,Tx)\big\}\Big) \quad \text{for all } x,y\in\mathcal{O}_T(x_0).\end{aligned} \tag{5.8}$$

Additionally, assume that

(a) T *is* $\preceq$*-nondecreasing-continuous.*

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 $\mathcal{D}$*-converges to a fixed point* ω *of* T*. Furthermore,* $\mathcal{D}(\omega,\omega)=0$ *and*

$$\mathcal{D}(T^n x_0,\omega) \le C\,\phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D},T,x_0)\big) \quad \text{for all } n\in\mathbb{N}, n\ge n_0,$$

where $C = C_{X,\mathcal{D}}$ *is the (lowest) constant for which* $(X,\mathcal{D})$ *satisfies property* $(\mathcal{D}_3)$*.*

In addition to this, if condition (5.8) *holds for all* $x,y\in X$ *such that* $x\preceq y$*, and* ω' *is another fixed point of* T *such that* $\omega\preceq\omega'$*,* $\mathcal{D}(\omega,\omega')<\infty$ *and* $\mathcal{D}(\omega',\omega')<\infty$*, then* $\omega=\omega'$*.*

Proof. It follows from Theorem 5.7 by using the a partial order $\preceq$ as the binary relation $\mathcal{S}$. □

In order not to enlarge this manuscript, we only point out that Theorem 5.7 remains true (that is, the operator T has at least a fixed point) if we do one or more of the following changes in its statement:

- replace the preorder $\mathcal{S}$ by the trivial preorder $\mathcal{S}_X$ given by $x\mathcal{S}_X y$ for all $x,y\in X$ (in this case, we obtain Corollary 5.8);
- replace the preorder $\mathcal{S}$ by a partial order $\preceq$ (in this case, we obtain Corollary 5.9);
- replace the preorder $\mathcal{S}$ on X by a binary relation on X that has only to be reflexive and transitive on the orbit $\mathcal{O}_T(x_0)$;
- replace, in the contractivity condition, "for all $x,y\in\mathcal{O}_T(x_0)$" by "for all $x,y\in X$ such that $x\mathcal{S}y$";

- replace the contractivity condition (5.6) by one of the following list:
  - $\mathcal{D}(Tx,Ty) \leq \phi(\mathcal{D}(x,y))$;
  - $\mathcal{D}(Tx,Ty) \leq \phi(\max\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty)\})$;
  - $\mathcal{D}(Tx,Ty) \leq \phi(\max\{\mathcal{D}(x,y), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\})$;
  - $\mathcal{D}(Tx,Ty) \leq \phi\left(\max\left\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \frac{\mathcal{D}(x,Ty)+\mathcal{D}(y,Tx)}{2}\right\}\right)$;
  - $\mathcal{D}(Tx,Ty) \leq \phi\left(\max\left\{\mathcal{D}(x,y), \frac{\mathcal{D}(x,Tx)+\mathcal{D}(y,Ty)}{2}, \frac{\mathcal{D}(x,Ty)+\mathcal{D}(y,Tx)}{2}\right\}\right)$

  for all $x,y \in \mathcal{O}_T(x_0)$ (take into account that $(t+s)/2 \leq \max\{t,s\}$ for all $t,s \in [0,\infty]$);
- replace the contractivity condition (5.6) by one of the previous list, considered for all $x,y \in X$ such that $x\mathcal{S}y$;
- replace the function $\phi \in \mathcal{F}_{\text{com}}$ by the particular case $\phi_\lambda(t) = \lambda t$ for all $t \in [0,\infty]$, where $\lambda \in [0,1)$;
- replace the $\mathcal{S}$-nondecreasing-completeness of $(X,\mathcal{D})$ by the completeness of $\mathcal{D}$;
- replace the $\mathcal{S}$-nondecreasing-continuity of T by continuity;
- replace the condition "$\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$" by

$$\delta(\mathcal{D},T,x_0) < \infty;$$

- replace the JS-GMS by any of the abstract metric spaces that Jleli and Samet showed in [18] that could be considered as a JS-GMS: metric spaces, b-metric spaces, Hitzler–Seda metric spaces and modular spaces with the Fatou property.

5.3. Some fixed point theorems under $\mathcal{S}$-nondecreasing-regularity

The main objective of the current subsection is to introduce some fixed point theorems in which T does not necessarily satisfy any continuity condition. The regularity (or $\mathcal{S}$-regularity) of the space $(X,\mathcal{D})$ is not a sufficient property as we showed in Section 3. In order to do this, the following result will play an important role.

Proposition 5.10. *Let $\{a_n\} \subset [0,\infty)$ be a sequence of nonnegative real numbers such that $\{a_n\} \to 0$ and let $\phi \in \mathcal{F}_{\text{com}}$. If*

$$b_n = \max\left\{\phi(a_n), \phi^2(a_{n-1}), \phi^3(a_{n-2}), \ldots, \phi^n(a_1), \phi^{n+1}(a_0)\right\}$$

for all $n \in \mathbb{N}$, then $b_n < \infty$ for all $n \in \mathbb{N}$. Furthermore, $\{b_n\} \to 0$.

Proof. As $a_n < \infty$ for all $n \in \mathbb{N}$ and $\phi^m(t) \leq t < \infty$ for all $m \in \mathbb{N}$ and all $t \in [0,\infty)$, then $b_n < \infty$ for all $n \in \mathbb{N}$.

Fix $\varepsilon > 0$ arbitrarily, and we are going to find $k_0 \in \mathbb{N}$ such that $b_n \leq \varepsilon$ for all $n \geq k_0$. Indeed, as $\{a_n\} \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$a_n \leq \varepsilon \quad \text{for all } n \geq n_0.$$

Notice that, for $n \geq n_0$,

$$\begin{aligned} b_n &= \max\left\{\phi\left(a_n\right), \phi^2(a_{n-1}), \phi^3(a_{n-2}), \ldots, \phi^n(a_1), \phi^{n+1}(a_0)\right\} \\ &= \max_{0\leq i\leq n} \phi^{n+1-i}(a_i) = \max\left\{\max_{0\leq i\leq n_0-1} \phi^{n+1-i}(a_i), \max_{n_0\leq i\leq n} \phi^{n+1-i}(a_i)\right\}. \end{aligned}$$

As $\phi^m(t) \leq t$ for all $t \in [0, \infty)$ and all $m \in \mathbb{N}$, then

$$\max_{n_0\leq i\leq n} \phi^{n+1-i}(a_i) \leq \max_{n_0\leq i\leq n} a_i \leq \varepsilon.$$

Therefore, for all $n \geq n_0$,

$$\begin{aligned} b_n &= \max\left\{\max_{0\leq i\leq n_0-1} \phi^{n+1-i}(a_i), \max_{n_0\leq i\leq n} \phi^{n+1-i}(a_i)\right\} \\ &\leq \max\left\{\max_{0\leq i\leq n_0-1} \phi^{n+1-i}(a_i), \varepsilon\right\}. \end{aligned} \tag{5.9}$$

Next, let $t_0 = \max\{a_0, a_1, a_2, \ldots, a_{n_0-1}\}$. If $t_0 = 0$, then $a_i = 0$ for all $i \in \{0, 1, \ldots, n_0 - 1\}$. In such a case, $\phi^{n+1-i}(a_i) = \phi^{n+1-i}(0) = 0$ for all $i \in \{0, 1, \ldots, n_0 - 1\}$, and (5.9) guarantees that $b_n \leq \varepsilon$ for all $n \geq n_0$. On the contrary case, assume that $t_0 > 0$. Since $\phi \in \mathcal{F}_{\text{com}}$, there exists $m_0 \in \mathbb{N}$ (we can assume that $m_0 > n_0$) such that

$$\phi^n(t_0) \leq \varepsilon \quad \text{for all } n \geq m_0.$$

Let $k_0 = m_0 + n_0 \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n \geq k_0$, then

$$n \geq k_0 = m_0 + n_0 > m_0 + n_0 - 2 \Longrightarrow n - n_0 + 2 > m_0.$$

If $i \in \{0, 1, \ldots, n_0 - 1\}$, then

$$\begin{aligned} i \leq n_0 - 1 &\Longrightarrow -n_0 + 1 \leq -i \\ &\Longrightarrow n - n_0 + 1 \leq n - i \\ &\Longrightarrow n - n_0 + 2 \leq n + 1 - i. \end{aligned}$$

Therefore,

$$m_0 < n - n_0 + 2 \leq n + 1 - i \quad \text{for all } i \in \{0, 1, \ldots, n_0 - 1\}.$$

As each ϕ^m is a nondecreasing function, then

$$\phi^{n+1-i}(a_i) \leq \phi^{n+1-i}(t_0) \leq \varepsilon \quad \text{for all } i \in \{0, 1, \ldots, n_0 - 1\}.$$

Hence, by (5.9), for all $n \geq k_0 \geq n_0$,

$$b_n \leq \max\left\{\max_{0\leq i\leq n_0-1} \phi^{n+1-i}(a_i), \varepsilon\right\} \leq \max\{\varepsilon, \varepsilon\} = \varepsilon,$$

which implies that $\{b_n\} \to 0$. □

Given a self-mapping $T : X \to X$ of a JS-GMS $(X, \mathcal{D})$ and a point $x_0 \in X$, we will use the notation

$$\mathcal{O}'_T(x_0) = \mathcal{O}_T(x_0) \cup \left\{ \omega \in X : \lim_{n\to\infty} \mathcal{D}(T^n x_0, \omega) = 0 \right\}.$$

By Proposition 2.5, the second part of $\mathcal{O}'_T(x_0)$ contains, at most, a single point.

In the next result, we shall assume that $\delta(\mathcal{D}, T, x_0) < \infty$ rather than "$\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$". Obviously, the second condition is more general but, when it is satisfied, we can take $y_0 = T^{n_0}x_0$ for which $\delta(\mathcal{D}, T, y_0) < \infty$. Nevertheless, in order to make the proof easier, we directly suppose $\delta(\mathcal{D}, T, x_0) < \infty$.

Theorem 5.11. *Let $(X, \mathcal{D})$ be an $\mathcal{S}$-nondecreasing-complete JS-GMS with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta(\mathcal{D}, T, x_0) < \infty$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\begin{aligned} \mathcal{D}(Tx, Ty) \le \phi\big(\max\big\{&\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty), \\ &\mathcal{D}(y, Tx)\big\}\big) \quad \text{for all } x, y \in \mathcal{O}'_T(x_0). \end{aligned} \tag{5.10}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega \in X$ that verifies $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(T^n x_0, \omega) \le C_{X,\mathcal{D}}\, \phi^n\big(\delta(\mathcal{D}, T, x_0)\big) \quad \text{for all } n \in \mathbb{N}.$$

Additionally, assume that

(b) *$\mathcal{D}(\omega, T\omega) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ and, if $\phi(\mathcal{D}(\omega, T\omega)) > 0$, then*

$$C_{X,\mathcal{D}}\, \phi\big(\mathcal{D}(\omega, T\omega)\big) < \mathcal{D}(\omega, T\omega).$$

Then ω is a fixed point of T.

Furthermore, if condition (5.10) holds for all $x, y \in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega \mathcal{S} \omega'$, $\mathcal{D}(\omega, \omega') < \infty$ and $\mathcal{D}(\omega', \omega') < \infty$, then $\omega = \omega'$.

Proof. As $\mathcal{O}_T(x_0) \subseteq \mathcal{O}'_T(x_0)$, it follows from Theorem 5.5 that the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy, and it $\mathcal{D}$-converges to a point $\omega \in X$ verifying (5.4) and (5.5). Suppose that $\mathcal{D}(\omega, T\omega) < \infty$ and $\mathcal{D}(x_0, T\omega) < \infty$. Since

$$\{x_n\} \xrightarrow{\mathcal{D}} \omega,$$

then $\omega \in \mathcal{O}'_T(x_0)$. Let us define

$$a_n = \max\big\{\mathcal{D}(x_n, \omega), \mathcal{D}(x_n, x_{n+1}), \mathcal{D}(\omega, x_{n+1})\big\} \quad \text{for all } n \in \mathbb{N}.$$

As $\{x_n\} \xrightarrow{\mathcal{D}} \omega$ and $\{x_n\}$ is $\mathcal{D}$-Cauchy, then $\{a_n\} \to 0$. Also

$$\mathcal{D}(x_n, x_{n+1}) \le \delta(\mathcal{D}, T, x_0) < \infty,$$
$$\mathcal{D}(x_n, \omega) = \mathcal{D}(T^n x_0, \omega) \le C_{X,\mathcal{D}}\, \phi^n\big(\delta(\mathcal{D}, T, x_0)\big) < \infty$$

for all $n \in \mathbb{N}$, then $a_n < \infty$ for all $n \in \mathbb{N}$. By Proposition 5.10,

$$\left\{ b_n = \max\left\{ \phi\left(a_n\right),\, \phi^2(a_{n-1}),\, \phi^3(a_{n-2}), \ldots,\, \phi^n(a_1),\, \phi^{n+1}(a_0) \right\} \right\} \to 0$$

and $b_n < \infty$ for all $n \in \mathbb{N}$. We claim that $\mathcal{D}\left(x_n, T\omega\right) < \infty$ for all $n \in \mathbb{N}$. Indeed, by hypothesis, $\mathcal{D}\left(x_0, T\omega\right) < \infty$. Assume that, for some $n \in \mathbb{N}$, we have that $\mathcal{D}\left(x_n, T\omega\right) < \infty$. Therefore, as $\omega \in \mathcal{O}'_T(x_0)$,

$$\begin{aligned}
\mathcal{D}\left(x_{n+1}, T\omega\right) &= \mathcal{D}\left(Tx_n, T\omega\right) \\
&\leq \phi\Big(\max\big\{ \mathcal{D}\left(x_n, \omega\right), \mathcal{D}\left(x_n, Tx_n\right), \mathcal{D}\left(\omega, T\omega\right), \\
&\qquad \mathcal{D}\left(x_n, T\omega\right), \mathcal{D}\left(\omega, Tx_n\right) \big\}\Big) \\
&\leq \phi\Big(\max\big\{ \mathcal{D}\left(x_n, \omega\right), \mathcal{D}\left(x_n, x_{n+1}\right), \mathcal{D}\left(\omega, T\omega\right), \\
&\qquad \mathcal{D}\left(x_n, T\omega\right), \mathcal{D}\left(\omega, x_{n+1}\right) \big\}\Big) \\
&= \max\Big\{ \phi\Big(\max\big\{ \mathcal{D}\left(x_n, \omega\right), \mathcal{D}\left(x_n, x_{n+1}\right), \mathcal{D}\left(\omega, x_{n+1}\right) \big\}\Big), \\
&\qquad \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi\big(\mathcal{D}\left(x_n, T\omega\right)\big) \Big\} \\
&= \max\Big\{ \phi(a_n), \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi\big(\mathcal{D}\left(x_n, T\omega\right)\big) \Big\}.
\end{aligned} \tag{5.11}$$

Since all terms in the maximum are finite, then $\mathcal{D}\left(x_{n+1}, T\omega\right) < \infty$, which completes the induction. We have just proved that, for all $n \in \mathbb{N}$,

$$\begin{gathered}
\mathcal{D}\left(x_n, T\omega\right) < \infty, \\
\mathcal{D}\left(x_{n+1}, T\omega\right) \leq \max\Big\{ \phi(a_n), \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi\big(\mathcal{D}\left(x_n, T\omega\right)\big) \Big\}.
\end{gathered} \tag{5.12}$$

As ϕ is nondecreasing, applying (5.12), we deduce that

$$\begin{aligned}
\phi\big(\mathcal{D}\left(x_n, T\omega\right)\big) &\leq \phi\Big(\max\big\{ \phi(a_{n-1}), \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi\big(\mathcal{D}\left(x_{n-1}, T\omega\right)\big) \big\}\Big) \\
&= \max\Big\{ \phi^2(a_{n-1}), \phi^2\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi^2\big(\mathcal{D}\left(x_{n-1}, T\omega\right)\big) \Big\}.
\end{aligned} \tag{5.13}$$

By joining (5.12) and (5.13), and taking into account that

$$\phi^2\big(\mathcal{D}\left(\omega, T\omega\right)\big) \leq \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),$$

we obtain that

$$\begin{aligned}
\mathcal{D}\left(x_{n+1}, T\omega\right) &\leq \max\Big\{ \phi(a_n),\, \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi\big(\mathcal{D}\left(x_n, T\omega\right)\big) \Big\} \\
&\leq \max\Big\{ \phi(a_n),\, \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi^2(a_{n-1}), \phi^2\big(\mathcal{D}\left(\omega, T\omega\right)\big), \\
&\qquad \phi^2\big(\mathcal{D}\left(x_{n-1}, T\omega\right)\big) \Big\} \\
&= \max\Big\{ \phi(a_n),\, \phi^2(a_{n-1}),\, \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi^2\big(\mathcal{D}\left(x_{n-1}, T\omega\right)\big) \Big\}.
\end{aligned}$$

Repeating this process n times, we derive that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{D}\left(x_{n+1}, T\omega\right) &\leq \max\Big\{ \phi(a_n),\, \phi^2(a_{n-1}), \ldots,\, \phi^n(a_1),\, \phi^{n+1}(a_0), \\
&\qquad \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi^{n+1}\big(\mathcal{D}\left(x_0, T\omega\right)\big) \Big\} \\
&= \max\Big\{ b_n,\, \phi\big(\mathcal{D}\left(\omega, T\omega\right)\big),\, \phi^{n+1}\big(\mathcal{D}\left(x_0, T\omega\right)\big) \Big\}.
\end{aligned} \tag{5.14}$$

Next, we consider two cases.

- If $\phi(\mathcal{D}(\omega, T\omega)) = 0$, it follows from (5.14) that
$$0 \leq \mathcal{D}(x_{n+1}, T\omega) \leq \max\left\{b_n,\, \phi^{n+1}\big(\mathcal{D}(x_0, T\omega)\big)\right\} \quad \text{for all } n \in \mathbb{N}.$$
Letting $n \to \infty$, we deduce that $\lim_{n\to\infty} \mathcal{D}(x_n, T\omega) = 0$. Hence, $(\mathcal{D}_3)$ leads to
$$\mathcal{D}(\omega, T\omega) \leq C \limsup_{n\to\infty} \mathcal{D}(x_{n+1}, T\omega) = C \lim_{n\to\infty} \mathcal{D}(x_n, T\omega) = 0.$$
Therefore, $\mathcal{D}(\omega, T\omega) = 0$ and $T\omega = \omega$ by $(\mathcal{D}_1)$.
- Assume that $\phi(\mathcal{D}(\omega, T\omega)) \in (0, \infty)$. In this case, by hypothesis,
$$\phi\big(\mathcal{D}(\omega, T\omega)\big) < \frac{\mathcal{D}(\omega, T\omega)}{C_{X,\mathcal{D}}}. \tag{5.15}$$
In order to prove that $T\omega = \omega$, we are going to show that $\mathcal{D}(\omega, T\omega) = 0$ reasoning by contradiction. On the contrary case, if $\mathcal{D}(\omega, T\omega) > 0$, taking into account that $\mathcal{D}(\omega, T\omega) < \infty$, we have that
$$\mathcal{D}(\omega, T\omega) \in (0, \infty).$$
Taking $\varepsilon = \phi(\mathcal{D}(\omega, T\omega)) > 0$, we can find $n_0 \in \mathbb{N}$ such that
$$b_n \leq \phi\big(\mathcal{D}(\omega, T\omega)\big) \quad \text{and} \quad \phi^{n+1}\big(\mathcal{D}(x_0, T\omega)\big) \leq \phi\big(\mathcal{D}(\omega, T\omega)\big)$$
for all $n \geq n_0$. Then, from (5.14),
$$\begin{aligned}\mathcal{D}(x_{n+1}, T\omega) &\leq \max\left\{b_n,\, \phi\big(\mathcal{D}(\omega, T\omega)\big),\, \phi^{n+1}\big(\mathcal{D}(x_0, T\omega)\big)\right\}\\ &= \phi\big(\mathcal{D}(\omega, T\omega)\big)\end{aligned}$$
for all $n \geq n_0$. Using $(\mathcal{D}_3)$ and (5.15), we conclude that
$$\begin{aligned}\mathcal{D}(\omega, T\omega) &\leq C \limsup_{n\to\infty} \mathcal{D}(x_{n+1}, T\omega) \leq C\, \phi\big(\mathcal{D}(\omega, T\omega)\big)\\ &< C\, \frac{\mathcal{D}(\omega, T\omega)}{C} = \mathcal{D}(\omega, T\omega),\end{aligned}$$
which is a contradiction. Then $\mathcal{D}(\omega, T\omega) = 0$ and $T\omega = \omega$, so ω is a fixed point of T.

The rest of the proof follows, point by point, as in the proof of Theorem 5.7. □

A natural way to guarantee the condition "if $\phi(\mathcal{D}(\omega, T\omega)) > 0$, then $C_{X,\mathcal{D}}\, \phi(\mathcal{D}(\omega, T\omega)) < \mathcal{D}(\omega, T\omega)$" consists in assuming that $\phi(t) < t/C_{X,\mathcal{D}}$ for all $t \in (0, \infty)$. In this case, we obtain the following consequence.

Corollary 5.12. *Let $(X, \mathcal{D})$ be an $\mathcal{S}$-nondecreasing-complete JS-GMS with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta(\mathcal{D}, T, x_0) < \infty$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*
$$\begin{aligned}\mathcal{D}(Tx, Ty) \leq \phi\big(\max\{&\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty),\\ &\mathcal{D}(y, Tx)\}\big) \quad \textit{for all } x, y \in \mathcal{O}'_T(x_0).\end{aligned}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 *converges to a point* $\omega \in X$ *that verifies* $\mathcal{D}(\omega,\omega) = 0$ *and*

$$\mathcal{D}(T^n x_0, \omega) \le C_{X,\mathcal{D}}\, \phi^n\big(\delta(\mathcal{D}, T, x_0)\big) \quad \textit{for all } n \in \mathbb{N}.$$

Additionally, assume that

(b′) $\mathcal{D}(\omega, T\omega) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ *and* $\phi(t) < t/C_{X,\mathcal{D}}$ *for all* $t \in (0,\infty)$.

Then ω *is a fixed point of* T.

Furthermore, if condition (5.10) *holds for all* $x, y \in X$ *such that* $x\mathcal{S}y$, *and* ω' *is another fixed point of* T *such that* $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega') < \infty$ *and* $\mathcal{D}(\omega',\omega') < \infty$, *then* $\omega = \omega'$.

In the next statement, we slightly change the points for which the contractivity condition must hold, and we involve a kind of regularity. Hence, the following result is not a direct consequence of Theorem 5.11. However, their proofs are very similar.

Theorem 5.13. *Let* $(X,\mathcal{D})$ *be an* $\mathcal{S}$*-nondecreasing-complete JS-GMS with respect to a preorder* $\mathcal{S}$ *and let* $T : X \to X$ *be an* $\mathcal{S}$*-nondecreasing self-mapping. Let* $x_0 \in X$ *be a point such that* $x_0\mathcal{S}Tx_0$ *and* $\delta(\mathcal{D}, T, x_0) < \infty$. *Suppose that there exists* $\phi \in \mathcal{F}_{\mathrm{com}}$ *such that*

$$\begin{aligned}\mathcal{D}(Tx, Ty) \le \phi\big(\max\{&\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty),\\ &\mathcal{D}(y,Tx)\}\big) \quad \textit{for all } x,y \in X \textit{ such that } x\mathcal{S}y.\end{aligned} \tag{5.16}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 *converges to a point* $\omega \in X$ *that verifies* $\mathcal{D}(\omega,\omega) = 0$ *and*

$$\mathcal{D}(T^n x_0, \omega) \le C_{X,\mathcal{D}}\, \phi^n\big(\delta(\mathcal{D}, T, x_0)\big) \quad \textit{for all } n \in \mathbb{N}.$$

Additionally, assume that

(b″) $(X,\mathcal{D})$ *is* $\mathcal{S}$*-nondecreasing-regular,* $\mathcal{D}(\omega, T\omega) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ *and, if* $\phi(\mathcal{D}(\omega, T\omega)) > 0$, *then*

$$C_{X,\mathcal{D}}\, \phi\big(\mathcal{D}(\omega, T\omega)\big) < \mathcal{D}(\omega, T\omega)$$

(*this last condition can be replaced by the fact that* $\phi(t) < t/C_{X,\mathcal{D}}$ *for all* $t \in (0,\infty)$).

Then ω *is a fixed point of* T.

Furthermore, if ω' *is another fixed point of* T *such that* $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega') < \infty$ *and* $\mathcal{D}(\omega',\omega') < \infty$, *then* $\omega = \omega'$.

Notice that, in the previous result, the space $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, but it does not have to be regular (see Example 4.3).

Proof. As the contractivity condition (5.16) holds for all $x, y \in X$ such that $x\mathcal{S}y$, Lemma 5.2 ensures that it also holds for all $x, y \in \mathcal{O}_T(x_0)$, so Theorem 5.5 guarantees that the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy, and it $\mathcal{D}$-converges to a point $\omega \in X$ verifying (5.4) and (5.5). As $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, we deduce that $x_n\mathcal{S}\omega$ for all $n \in \mathbb{N}$. Thus, as the contractivity condition (5.16) is applicable to x_n and ω, we can repeat, point by point, the arguments of the proof of Theorem 5.11 in order to conclude that ω is a fixed point of T. □

We can also deduce a large list of corollaries from Theorems 5.11 and 5.13 as we commented in the last lines of Subsection 5.2. However, we leave this task to the reader.

5.4. Fixed point theorems under an easier contractive condition

Finally, in this subsection, we wish to highlight some fixed point theorems in the context of JS-GMSs under the stronger contractivity condition

$$\mathcal{D}(Tx, Ty) \leq \phi\big(\mathcal{D}(x, y)\big)$$

considered over an appropriate subset of X. For example, the following statement is an immediate consequence of Corollary 5.8.

Corollary 5.14. *Let $(X, \mathcal{D})$ be a complete JS-GMS and let $T : X \to X$ be a continuous self-mapping. Let $x_0 \in X$ be a point such that $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\mathrm{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\big(\mathcal{D}(x, y)\big) \quad \text{for all } x, y \in \mathcal{O}_T(x_0). \tag{5.17}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(T^n x_0, \omega) \leq C\, \phi^{n-n_0}\big(\delta_{n_0}(\mathcal{D}, T, x_0)\big) \quad \text{for all } n \in \mathbb{N}, n \geq n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X, \mathcal{D})$ satisfies property $(\mathcal{D}_3)$.

In addition to this, if condition (5.17) holds for all $x, y \in X$, and ω' and ω'' are two fixed points of T such that $\mathcal{D}(\omega', \omega'') < \infty$, then $\omega' = \omega''$.

In the next result, we observe that if the contractivity condition holds for all $x, y \in \mathcal{O}'_T(x_0)$, then we can avoid the continuity of T. In fact, when the contractivity condition holds for all pairs of points, we can deduce it.

Theorem 5.15. *Let $(X, \mathcal{D})$ be a complete JS-GMS and let $T : X \to X$ be a self-mapping. Let $x_0 \in X$ be a point such that $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\mathrm{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\big(\mathcal{D}(x, y)\big) \quad \text{for all } x, y \in \mathcal{O}'_T(x_0). \tag{5.18}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega, \omega) = 0$.

In addition to this, if (5.18) holds for all $x, y \in X$, then T is continuous. Moreover, if ω' and ω'' are two fixed points of T such that $\mathcal{D}(\omega', \omega'') < \infty$, then $\omega' = \omega''$.

Proof. Let us consider on X the trivial preorder $\mathcal{S}_X$ given by $x\mathcal{S}_X y$ for all $x, y \in X$. Then T is $\mathcal{S}_X$-nondecreasing and $(X, \mathcal{D})$ is $\mathcal{S}_X$-nondecreasing-complete. Theorem 5.5 ensures us that $\{x_n = T^n x_0\}_{n\in\mathbb{N}}$ is an $\mathcal{S}_X$-nondecreasing, $\mathcal{D}$-Cauchy sequence. Since $(X, \mathcal{D})$ is complete, there is an $\omega \in X$ such that

$$\{T^n x_0\} \xrightarrow{\mathcal{D}} \omega.$$

As $\omega \in \mathcal{O}'_T(x_0)$, we observe that

$$\mathcal{D}\left(T^{n+1}x_0, T\omega\right) = \mathcal{D}\left(TT^n x_0, T\omega\right) \leq \phi\big(\mathcal{D}\left(T^n x_0, \omega\right)\big) \quad \text{for all } n \in \mathbb{N}.$$

Taking into account that $\{\mathcal{D}\left(T^n x_0, \omega\right)\} \to 0$ and ϕ is continuous at $t = 0$, with $\phi(0) = 0$, we deduce that

$$\{T^{n+1}x_0\} \xrightarrow{\mathcal{D}} \omega.$$

Hence $T\omega = \omega$ by Proposition 2.5, and ω is a fixed point of T. By using $(\mathcal{D}_3)$,

$$\mathcal{D}\left(\omega, \omega\right) \leq C \limsup_{m\to\infty} \mathcal{D}\left(T^m x_0, \omega\right) = 0,$$

so $\mathcal{D}\left(\omega, \omega\right) = 0$.

Next, assume that (5.18) holds for all $x, y \in X$, and let $z \in X$ be an arbitrary point. We claim that T is continuous at z. Indeed, let $\{y_n\} \subseteq X$ be a sequence such that

$$\{y_n\} \xrightarrow{\mathcal{D}} z.$$

Then $\{\mathcal{D}\left(y_n, z\right)\} \to 0$. By using the contractivity condition (5.18), we derive that

$$\mathcal{D}\left(Ty_n, Tz\right) \leq \phi\big(\mathcal{D}\left(y_n, z\right)\big) \quad \text{for all } n \in \mathbb{N}.$$

Since ϕ is continuous at $t = 0$, with $\phi(0) = 0$, we deduce that

$$\{Ty_n\} \xrightarrow{\mathcal{D}} Tz.$$

Therefore, T is continuous at z.

Finally, if ω' and ω'' are two fixed points of T such that $\mathcal{D}(\omega', \omega'') < \infty$, then

$$\mathcal{D}\left(\omega', \omega''\right) = \mathcal{D}\left(T\omega', T\omega''\right) \leq \phi\big(\mathcal{D}\left(\omega', \omega''\right)\big),$$

which means that $\omega' = \omega''$. □

Corollary 5.16. *Theorem* 2.6 *immediately follows from Theorem* 5.15.

Proof. It is only necessary to apply Theorem 5.15 using $n_0 = 0$ and $\phi(t) = k\,t$ for all $t \in [0, \infty]$, where $k \in [0, 1)$. □

Notice that Theorem 5.15 improves Theorem 2.6 in three aspects:

(1) the function $\phi \in \mathcal{F}_{\text{com}}$ is arbitrary;
(2) the number $n_0 \in \mathbb{N}$ is arbitrary;
(3) we deduce that T is continuous.

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Erdal Karapınar
Department of Mathematics
Atilim University
06836 İncek, Ankara,
Turkey
and
Nonlinear Analysis and Applied Mathematics Research Group (NAAM)
King Abdulaziz University
Jeddah
Saudi Arabia
e-mail: erdalkarapinar@yahoo.com

Donal O'Regan
School of Mathematics
Statistics and Applied Mathematics
National University of Ireland
Galway
Ireland
e-mail: donal.oregan@nuigalway.ie

Antonio Francisco Roldán López de Hierro
Department of Quantitative Methods for Economics and Business
University of Granada
Granada
Spain
and
PAIDI Research Group FQM-268
University of Jaén
Jaén
Spain
e-mail: aroldan@ugr.es, afroldan@ujaen.es

Naseer Shahzad
Operator Theory and Applications Research Group
Department of Mathematics
King Abdulaziz University
P.O. Box 80203
Jeddah, 21589
Saudi Arabia
e-mail: nshahzad@kau.edu.sa