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Decay mild solutions for two-term time fractional differential equations in Banach spaces

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Abstract. In this paper, we deal with the nonlocal Cauchy problem for a class of two-term time fractional differential equations in Banach spaces. By constructing a suitable measure of noncompactness on the space of solutions, we prove the existence of a compact set containing decay mild solutions to the mentioned problem.

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1. Introduction

We consider fractional differential equations with nonlocal conditions in a Banach space X of the form

$$D_0^{\alpha+1}u(t) + \mu D_0^{\beta}u(t) - Au(t) = F(t, u(t)), \quad t > 0,\ \mu \geq 0, \tag{1.1}$$

$$u(0) + g(u) = x_0, \quad u_t(0) + h(u) = y_0, \tag{1.2}$$

where $0 < \alpha \leq \beta \leq 1$, D_0^{α} is the Caputo fractional derivative of order α with the lower limit 0, $A : D(A) \subset X \to X$ is a closed linear operator, and A generates a strongly continuous family $\{S_{\alpha,\beta}(t)\}$ of bounded and linear operators on X, $x_0 \in X$, $y_0 \in X$.

Fractional differential equations also have been proved to be useful tools in modeling of phenomena in various fields of science and engineering. There has been significant development in fractional differential equations in recent years; see the monographs [16, 17, 27], the papers [4, 9, 18, 19, 20, 21, 25] and the references therein. Initial value problems for nonlinear fractional differential and integrodifferential equations are discussed in [2, 10, 15, 22, 26, 28], and Dirichlet or Neumann-type of problems for nonlinear fractional differential equations are studied in [12, 13]. In these papers, techniques of functional analysis such as fixed point theory, the Banach contraction principle, Leray–Schauder theory, etc. are applied for solving such kind of problems.

Equation (1.1) with classical initial conditions comes from recent investigations where a related class appears in connection with partial differential equations and Cauchy-time processes, a type of iterated stochastic processes (see [5]). In the case $0 < \alpha < 1$, $\beta = 1$, $\mu = 0$, equation (1.1) is model of a fractional diffusion-wave equation (see [25, 26]). Recently, in [15] Keyantuo, Lizama and Warma have studied equation (1.1) with classical initial conditions and the right-hand side $F = D_0^\alpha f$, in which f is a Lipschitzian function for the second variable.

Motivated by [15], we deal with the two-term time fractional differential equations with nonlocal conditions in Banach spaces. The concept of nonlocal conditions was first used by Byszewski [7]. This notion is more appropriate than the classical one to describe natural phenomena because it allows us to consider additional information, see Deng [11], Byszewski and Lakshmikantham [8]. The purpose of this paper is to use a fixed point principle for condensing maps for measures of noncompactness [14] and the theory of (α, β)-regularized families [15] to prove the existence of decay mild solutions.

The rest of the paper is organized as follows. Section 2 introduces some useful preliminaries. In addition, we construct a regular measure of noncompactness (MNC) on $BC(\mathbb{R}^+; X)$ and give a fixed point principle. In Section 3, we prove the existence of mild solutions on $[0, T]$, $T > 0$, for problem (1.1)–(1.2). Section 4 is devoted to show the decay mild solutions. In the last section, we give an example to illustrate the abstract results obtained in the paper.

2. Preliminaries

In this section, we introduce preliminary facts which are used throughout the paper.

Definition 2.1. For a function $f \in C^N(\mathbb{R}_+; X)$, the Caputo derivative of order α with the lower limit 0 is defined by

$$D_0^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(N-\alpha)} \displaystyle\int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)\, ds & \text{if } \alpha \in (N-1, N), \\ f^{(N)}(t) & \text{if } \alpha = N. \end{cases}$$

Definition 2.2. Let $\mu \geq 0$ and $0 \leq \alpha, \beta \leq 1$ be given. Let A be a closed and linear operator with domain $D(A)$ on a Banach space X. We say that A is the generator of an $(\alpha, \beta)_\mu$-regularized family if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathscr{L}(X)$ such that

$$\left\{\lambda^{\alpha+1} + \mu\lambda^\beta : \operatorname{Re}(\lambda) > \omega\right\} \subset \rho(A)$$

and

$$\lambda^\alpha \left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t) x\, dt, \quad \operatorname{Re}(\lambda) > \omega, x \in X.$$

It is known that in the case $\mu = 0$, $\alpha = 0$, this is a C_0-semigroup; while if $\mu = 0$, $\alpha = 1$, we have a cosine family. The existence and characterization of generators of $(\alpha, \beta)_\mu$-regularized families were discussed in [19]. Specifically, let A be a closed and densely defined operator. An operator A is said to be ω-sectorial of angle θ if there exist $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent is in the sector

$$\omega + S_\theta := \left\{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \theta\right\} \setminus \{\omega\}, \tag{2.1}$$

and

$$\left\|(\lambda - A)^{-1}\right\| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta. \tag{2.2}$$

The following results are established in [15].

Lemma 2.3. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$ and let A be an ω-sectorial operator of angle $\beta\pi/2$. Then A generates an exponentially bounded $(\alpha, \beta)_\mu$-regularized family $S_{\alpha\beta}(t)$.*

Lemma 2.4. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$ and $\omega < 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$. Then A generates an $(\alpha, \beta)_\mu$-regularized family $S_{\alpha\beta}(t)$ satisfying the estimate*

$$\|S_{\alpha\beta}(t)\| \le \frac{C}{1 + |\omega|(t^{\alpha+1} + \mu t^\beta)}, \quad t \ge 0, \tag{2.3}$$

for some constant $C > 0$ depending only on α, β.

We now look for suitable concept of mild solutions to problem (1.1)–(1.2). Denoting by $\mathscr{L}$ the Laplace transform for X-valued functions acting on $\mathbb{R}_+$, putting $v(t) = F(t, u(t))$ and applying the Laplace transform to (1.1)–(1.2), we have

$$\begin{aligned}&\left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)\mathscr{L}[u](\lambda)\\ &\quad = \lambda^\alpha u(0) + \lambda^{\alpha-1}u_t(0) + \mu\lambda^{\beta-1}u(0) + \mathscr{L}[v](\lambda), \quad \mathrm{Re}(\lambda) > \omega.\end{aligned}$$

So

$$\begin{aligned}\mathscr{L}[u](\lambda) &= \lambda^\alpha\left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)^{-1}u(0) + \lambda^{\alpha-1}\left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)^{-1}u_t(0)\\ &\quad + \mu\lambda^{\beta-1}\left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)^{-1}u(0) + \left(\lambda^{\alpha+1} + \mu\lambda^\beta - A\right)^{-1}\mathscr{L}[v](\lambda)\end{aligned}$$

for all λ such that $\mathrm{Re}(\lambda) > \omega$, $\lambda^{\alpha+1} + \mu\lambda^\beta \in \rho(A)$. Let $S_{\alpha,\beta}(t)$ be an $(\alpha, \beta)_\mu$-regularized family generated by A, then

$$\begin{aligned}\mathscr{L}[u](\lambda) &= \mathscr{L}[S_{\alpha,\beta}](\lambda)[x_0 - g(u)] + \mathscr{L}[\varphi_1]\mathscr{L}[S_{\alpha,\beta}](\lambda)[y_0 - h(u)]\\ &\quad + \mu\mathscr{L}[\varphi_{1+\alpha-\beta}]\mathscr{L}[S_{\alpha,\beta}](\lambda)[x_0 - g(u)]\\ &\quad + \mathscr{L}[S_{\alpha,\beta}](\lambda)\mathscr{L}[\varphi_\alpha](\lambda)\mathscr{L}[v](\lambda), \quad \mathrm{Re}(\lambda) > \omega,\end{aligned}$$

where

$$\varphi_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,\ \beta > 0.$$

Inversion of the Laplace transform shows that

$$\begin{aligned} u(t) = {} & S_{\alpha,\beta}(t)[x_0 - g(u)] + (\varphi_1 * S_{\alpha,\beta})(t)[y_0 - h(u)] \\ & + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)[x_0 - g(u)] + (S_{\alpha,\beta} * \varphi_\alpha * v)(t). \end{aligned} \tag{2.4}$$

In this paper, we assume that $f(t, u(t)) = \varphi_\alpha * F(t, u(t))$. Motivated by (2.4), we give the following definition of mild solutions.

Definition 2.5. Let $0 < \alpha \le \beta \le 1$ and $\mu \ge 0$. A function $u \in C(\mathbb{R}_+, X)$ is said to be a mild solution of problem (1.1)–(1.2) if it satisfies

$$\begin{aligned} u(t) = {} & S_{\alpha,\beta}(t)[x_0 - g(u)] + (\varphi_1 * S_{\alpha,\beta})(t)[y_0 - h(u)] \\ & + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)[x_0 - g(u)] \\ & + \int_0^t S_{\alpha,\beta}(t-\tau) f\big(\tau, u(\tau)\big)\, d\tau \end{aligned} \tag{2.5}$$

for each $t \in \mathbb{R}_+$ and $(x_0, y_0) \in X \times X$.

In the following part, we recall the knowledge of the measures of noncompactness in Banach spaces. Among them, Hausdorff measure of noncompactness is important. Next, we mention the condensing maps and fixed point principles for condensing maps. We denote the collection of all nonempty bounded subsets in X by B_X, and the norm of space $C([0,T]; X)$ by $\|\cdot\|_C$, with $\|u\|_C = \sup_{t\in[0,T]} \|u(t)\|_X$.

Definition 2.6. A function $\Phi : B_X \to [0, +\infty)$ is called a measure of noncompactness (MNC) in X if

$$\Phi(\overline{\mathrm{co}}\,\Omega) = \Phi(\Omega) \quad \forall\, \Omega \in B_X,$$

where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC Φ in X is called

(i) monotone if for all $\Omega_1, \Omega_2 \in B_X$, $\Omega_1 \subset \Omega_2$ implies $\Phi(\Omega_1) \le \Phi(\Omega_2)$;
(ii) nonsingular if $\Phi(\{x\} \cup \Omega) = \Phi(\Omega)$ for all $x \in X$ and all $\Omega \in B_X$;
(iii) invariant with respect to union with a compact set if $\Phi(K \cup \Omega) = \Phi(\Omega)$ for every relatively compact $K \subset X$ and $\Omega \in B_X$;
(iv) algebraically semi-additive if $\Phi(\Omega_1 + \Omega_2) \le \Phi(\Omega_1) + \Phi(\Omega_2)$ for any $\Omega_1, \Omega_2 \in B_X$;
(v) regular if $\Phi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of measures of noncompactness is the Hausdorff MNC $\chi(\cdot)$ which is defined as follows:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\} \tag{2.6}$$

for all $\Omega \in B_X$.

For $T > 0$, it is known that the Hausdorff MNC on $C([0,T], \mathbb{R}^n)$ is given by (see [1])

$$\chi_T(\Omega) = \frac{1}{2} \lim_{\delta\to 0} \sup_{u\in\Omega} \max_{t,s\in[0,T], |t-s|<\delta} \|u(t) - u(s)\|_{\mathbb{R}^n}. \tag{2.7}$$

The last measure can be seen as the modulus of equicontinuity of a subset in $C([0,T];\mathbb{R}^n)$. In $C([0,T];X)$ with X being of infinite dimension, there is no such formulation as (2.7). However, if $\Omega\subset C([0,T];X)$ is an equicontinuous set, then

$$\chi_T(\Omega)=\sup_{t\in[0,T]}\chi\big(\Omega(t)\big), \tag{2.8}$$

here χ is the Hausdorff MNC in X.

Consider the space $BC(\mathbb{R}^+;X)$ of bounded continuous functions on $\mathbb{R}^+$ taking values on X. Denote by π_T the restriction operator on this space; i.e., $\pi_T(u)$ is the restriction of u on $[0,T]$. Then

$$\chi_\infty(D)=\sup_{T>0}\chi_T\big(\pi_T(D)\big),\quad D\subset BC(\mathbb{R}^+;X), \tag{2.9}$$

is an MNC. We give some measures of noncompactness as follows:

$$d_T(D)=\sup_{u\in D}\sup_{t\geq T}\|u(t)\|_X, \tag{2.10}$$

$$d_\infty(D)=\lim_{T\to\infty}d_T(D), \tag{2.11}$$

$$\chi^*(D)=\chi_\infty(D)+d_\infty(D). \tag{2.12}$$

The regularity of MNC χ^* is proved in [2, Lemma 2.6]. Then the following property is evident.

Proposition 2.7. *Let χ be the Hausdorff MNC on a Banach space X, $\Omega\in B_X$. Then there exists a sequence $\{x_n\}_{n=1}^\infty\subset\Omega$ such that*

$$\chi(\Omega)\leq 2\chi\big(\{x_n\}_{n=1}^\infty\big)+\varepsilon\quad\forall\,\varepsilon>0. \tag{2.13}$$

We have the following estimate whose proof can be found in [14].

Proposition 2.8 (See [14]**).** *Let χ be the Hausdorff MNC on a Banach space X and let $\{u_n\}_{n=1}^\infty\subset L_1(0,T;X)$ such that $\|u_n(t)\|_X\leq v(t)$, for every $n\in\mathbb{N}^*$ and a.e. $t\in[0,T]$, for some $v\in L_1(0,T)$. Then we have*

$$\chi\left(\left\{\int_0^t u_n(s)\,dx\right\}\right)\leq 2\int_0^t\chi\big(\{u_n(t)\}\big)\,ds \tag{2.14}$$

for $t\in[0,T]$.

To end this section, we recall a fixed point principle for condensing maps that will be used in the next sections.

Definition 2.9 (See [3]**).** Let X be a Banach space, χ an MNC on X and $\emptyset\neq D\subset X$. A continuous map $\Phi:D\to X$ is said to be condensing with respect to χ (χ-condensing) if for all $\Omega\in B_D$, the relation

$$\chi(\Omega)\leq\chi\big(\Phi(\Omega)\big)$$

implies the relative compactness of Ω.

Theorem 2.10 (See [14]). *Let X be a Banach space, χ an MNC on X, D a bounded convex closed subset of X, and let $\Phi : D \to D$ be a χ-condensing map. Then the fixed set of Φ,*

$$\mathrm{Fix}(\Phi) = \{x \in D : x = \Phi(x)\}$$

is a nonempty compact set.

3. Existence result

In formulation of problem (1.1)–(1.2), we assume the following.

(G) The function $g : C([0,T];X) \to X$ obeys the following conditions:

  (i) g is continuous on X, and

$$\|g(u)\|_X \leq \theta_g(\|u\|_C) \tag{3.1}$$

  for all $u \in C([0,T];X)$, where $\theta_g : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing;

  (ii) there exists a nonnegative constant η_g such that

$$\chi\big(g(\Omega)\big) \leq \eta_g \chi_T(\Omega) \tag{3.2}$$

  for all bounded sets $\Omega \subset C([0,T];X)$.

(H) The function $h : C([0,T];X) \to X$ satisfies the following conditions:

  (i) for all $u \in C([0,T];X)$,

$$\|h(u)\|_X \leq \theta_h(\|u\|_C), \tag{3.3}$$

  where $\theta_h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function;

  (ii) there exists a nonnegative constant η_h such that

$$\chi\big(h(\Omega)\big) \leq \eta_h \chi_T(\Omega) \tag{3.4}$$

  for all bounded sets $\Omega \subset C([0,T];X)$.

(F) The nonlinear function $f : \mathbb{R}^+ \times X \to X$ satisfies the following conditions:

  (i) $f(\cdot, u(\cdot))$ is measurable for each $u(\cdot) \in X$, $f(t,\cdot)$ is continuous for a.e. $t \in [0,T]$, and

$$\|f\big(t,v\big)\|_X \leq m(t)\theta_f(\|v\|_X) \tag{3.5}$$

  for all $v \in X$, where $m \in L_1(0,T)$, $\theta_f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function;

  (ii) there exists $\eta_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\eta_f \in L_1(0,T)$ and

$$\chi\big(f(t,\Omega)\big) \leq \eta_f(t)\chi(\Omega) \tag{3.6}$$

  for all bounded sets $\Omega \subset X$.

Remark 3.1. Let us give some comments on assumptions (G)(ii), (H)(ii) and (F)(ii).

(1) If g, h are Lipschitzian, then (3.2) and (3.4) are satisfied. These conditions is also satisfied with $\theta_g = \theta_h = 0$ if g, h are completely continuous.

(2) If $f(t,\cdot)$ satisfies the Lipschitzian condition for the second variable, i.e,

$$\big\|f\big(t,u_1(t)\big)-f\big(t,u_2(t)\big)\big\|_X \le k_f(t)\|u_1-u_2\|_C$$

for some $k_f \in L_1(0,T)$, then (3.6) is satisfied. Furthermore, if $f(t,\cdot)$ is completely continuous (for each fixed t), then (3.6) is obviously fulfilled with $\eta_f = 0$.

We denote

$$\mathcal{M} := \{u \in C([0,T];X) : \|u\|_C \le R\},$$

where $R > 0$ is given. We conclude that $\mathcal{M}$ is a bounded convex closed subset of $C([0,T];X)$. For each $u \in \mathcal{M}$, we define the solution operator $\Phi : \mathcal{M} \to C([0,T];X)$ as follows:

$$\begin{aligned}\Phi(u)(t) = {} & S_{\alpha,\beta}(t)[x_0 - g(u)] + (\varphi_1 * S_{\alpha,\beta})(t)[y_0 - h(u)] \\ & + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)[x_0 - g(u)] \\ & + \int_0^t S_{\alpha,\beta}(t-\tau) f\big(\tau, u(\tau)\big)\, d\tau.\end{aligned} \tag{3.7}$$

Then u is a mild solution of problem (1.1)–(1.2) if it is a fixed point of the solution operator Φ.

Thanks to the assumptions imposed on g, h, f, then Φ is continuous on $\mathcal{M}$. We denote

$$\begin{aligned} M &:= \sup_{t\in[0,T]} \|S_{\alpha,\beta}(t)\|_{\mathscr{L}(X)}, \\ \Lambda_T &:= \sup_{t\in[0,T]} \|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathscr{L}(X)}, \\ \Theta_T &:= \sup_{t\in[0,T]} \|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathscr{L}(X)}. \end{aligned}$$

Lemma 3.2. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$, and let A be an ω-sectorial operator of angle $\beta\pi/2$. If hypotheses* (G), (H), (F) *are satisfied and*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\bigg[& (M + \mu\Theta_T)\theta_g(n) + \Lambda_T \theta_h(n) \\ & + \theta_f(n) \sup_{t\in[0,T]} \int_0^t \|S_{\alpha,\beta}(t-\tau)\| m(\tau)\, d\tau \bigg] < 1,\end{aligned} \tag{3.8}$$

then there exists $R > 0$ such that $F(\mathcal{M}) \subset \mathcal{M}$.

Proof. Assume to the contrary that for each $n \in \mathbb{N}$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $\|u_n\|_C \le n$ but $\|\Phi(u_n)\|_C > n$. From the formulation of Φ, we have

$$\begin{aligned}\|\Phi(u_n)(t)\|_X \le\ & \|S_{\alpha,\beta}(t)\|\big(\|x_0\| + \|g(u_n)\|\big)\\ & + \|\varphi_1 * S_{\alpha,\beta}(t)\|\big(\|y_0\| + \|h(u_n)\|\big)\\ & + \mu\|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|\big(\|x_0\| + \|g(u_n)\|\big)\\ & + \int_0^t \|S_{\alpha,\beta}(t-\tau)\|\,\|f(\tau,u_n(\tau))\|\,d\tau\\ \le\ & M\big(\|x_0\| + \theta_g(n)\big) + \Lambda_T\big(\|y_0\| + \theta_h(n)\big) + \mu\Theta_T\big(\|x_0\| + \theta_g(n)\big)\\ & + \theta_f(n)\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,m(\tau)\,d\tau.\end{aligned}$$

From the inequality above, it follows that

$$\begin{aligned}\|\Phi(u_n)\|_C \le\ & \big(M + \mu\Theta_T\big)\big(\|x_0\| + \theta_g(n)\big) + \Lambda_T\big(\|y_0\| + \theta_h(n)\big)\\ & + \theta_f(n)\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,m(\tau)\,d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}1 < \frac{1}{n}\Big[&\big(M + \mu\Theta_T\big)\big(\|x_0\| + \theta_g(n)\big) + \Lambda_T\big(\|y_0\| + \theta_h(n)\big)\\ & + \theta_f(n)\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,m(\tau)\,d\tau\Big].\end{aligned}$$

Passing to the limit in the last inequality, one gets a contradiction. The proof is just completed. □

In order to deploy the fixed point theory for condensing maps, we will establish the so-called MNC estimate for the solution operator Φ.

Lemma 3.3. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$, and let A be an ω-sectorial operator of angle $\beta\pi/2$. If hypotheses* (G), (H), (F) *are satisfied, then*

$$\begin{aligned}\chi_T\big(\Phi(D)\big) \le \Big[&(M + \mu\Theta_T)\eta_g + \Lambda_T\eta_h\\ & + 2\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\Big]\chi_T(D)\end{aligned} \tag{3.9}$$

for all bounded sets $D \subset \mathcal{M}$.

Proof. Setting

$$\Phi_1(u)(t) = S_{\alpha,\beta}(t)\big[x_0 - g(u)\big] + \mu\big(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}\big)(t)\big[x_0 - g(u)\big],$$

$$\Phi_2(u)(t) = \big(\varphi_1 * S_{\alpha,\beta}\big)(t)\big[y_0 - h(u)\big],$$

$$\Phi_3(u)(t) = \int_0^t S_{\alpha,\beta}(t-\tau)f\big(\tau,u(\tau)\big)\,d\tau,$$

we have

$$\chi_T\big(\Phi(D)\big) \le \chi_T\big(\Phi_1(D)\big) + \chi_T\big(\Phi_2(D)\big) + \chi_T\big(\Phi_3(D)\big). \tag{3.10}$$

(1) For every $z_1, z_2 \in \Phi_1(D)$, there exist $u_1, u_2 \in D$ such that for $t \in [0,T]$,

$$z_i(t) = \Phi_1(u_i)(t), \quad i = 1, 2.$$

We have

$$\begin{aligned}\|z_1(t) - z_2(t)\| = {} & \|S_{\alpha,\beta}(t)\|\,\|g(u_1) - g(u_2)\| \\ & + \mu\big\|\big(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}\big)(t)\big\|\,\big\|g(u_1) - g(u_2)\big\|.\end{aligned}$$

It implies that

$$\|z_1 - z_2\|_C \le (M + \mu\Theta_T)\big\|g(u_2) - g(u_1)\big\|.$$

Hence,

$$\chi_T\big(\Phi_1(D)\big) \le (M + \mu\Theta_T)\chi\big(g(D)\big) \le (M + \mu\Theta_T)\eta_g\chi_T(D). \tag{3.11}$$

(2) By similar arguments as above, we get

$$\chi_T\big(\Phi_2(D)\big) \le \Lambda_T\eta_h\,\chi_T(D). \tag{3.12}$$

(3) Apply Proposition 2.7 again, there exists $\{u_n\}_{n=1}^\infty \subset D$ such that for every $\varepsilon > 0$, we obtain

$$\chi_T\big(\Phi_3(D)\big) \le 2\chi_T\big(\{\Phi_3(u_n)\}_{n=1}^\infty\big) + \varepsilon. \tag{3.13}$$

We also have $\{\Phi_3(u_n)(t)\}$ which is an equicontinuous set of functions. We invoke Proposition 2.8 to deduce that

$$\begin{aligned}\chi\big(\{\Phi_3(u_n)(t)\}\big) & \le 2\int_0^t \chi\big(S_{\alpha,\beta}(t-\tau)f\big(\tau, u_n(\tau)\big)\big)\,d\tau \\ & \le 2\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\,\chi_T(u_n).\end{aligned}$$

It is inferred that

$$\chi_T\big(\{\Phi_3(u_n)\}\big) \le 2\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\,\chi_T(u_n). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\chi_T\big(\Phi_3(D)\big) \le 2\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\,\chi_T(D). \tag{3.15}$$

Combining (3.10), (3.11), (3.12) and (3.15) yields

$$\begin{aligned}\chi_T\big(\Phi(D)\big) \le \Bigg[& (M + \mu\Theta_T)\eta_g + \Lambda_T\eta_h \\ & + 2\sup_{t\in[0,T]}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\Bigg]\chi_T(D).\end{aligned} \tag{3.16}$$

The proof is completed. □

Theorem 3.4. *Let* $0 < \alpha \le \beta \le 1$, $\mu > 0$, *and let* A *be an* ω*-sectorial operator of angle* $\beta\pi/2$. *If hypotheses* (G), (H), (F) *are satisfied. Then problem* (1.1)–(1.2) *has at least one mild solution on* $[0, T]$ *provided that*

$$\lim_{n\to\infty} \frac{1}{n}\Big[(M + \mu\Theta_T)\theta_g(n) + \Lambda_T\theta_h(n) + \theta_f(n) \sup_{t\in[0,T]} \int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\| m(\tau)\, d\tau\Big] < 1, \tag{3.17}$$

$$l := (M + \mu\Theta_T)\eta_g + \Lambda_T\eta_h + 2 \sup_{t\in[0,T]} \int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\| \eta_f(\tau)\, d\tau < 1. \tag{3.18}$$

Proof. By inequality (3.18), the solution operator Φ is a χ_T-condensing. Indeed, if $D \subset \mathcal{M}$ is a bounded set such that $\chi_T(D) \le \chi_T(\Phi(D))$, applying Lemma 3.3, we obtain

$$\chi_T(D) \le \chi_T\big(\Phi(D)\big) \le l\chi_T(D).$$

Therefore $\chi_T(D) = 0$ and D is a relative compactness.

By assumption (3.17), applying Lemma 3.2, we have $F(\mathcal{M}) \subset \mathcal{M}$. Applying Theorem 2.10, the χ_T-condensing map Φ defined by (3.7) has a fixed set $\mathrm{Fix}(\Phi) \subset \mathcal{M}$ which is compact, and is not an empty set. It implies that problem (1.1)–(1.2) has a mild solution $u(t)$, $t \in [0, T]$, described by (2.5). □

4. Existence of decay mild solutions

In this section, we consider the solution operator Φ on the following space:

$$\mathcal{M}_\infty = \{u \in BC(\mathbb{R}^+; X) : u(t) \to 0 \text{ as } t \to \infty\}.$$

We are going to prove that $\Phi(\mathcal{M}_\infty) \subset \mathcal{M}_\infty$ and, using the MNC χ^* defined by (2.12), to prove that Φ is a χ^*-condensing map on $\mathcal{M}_\infty$. In hypotheses (G), (H), (F), we consider the conditions of g, h, f for any $T > 0$. The norm $\|\cdot\|_C$ is replaced by the norm $\|\cdot\|_\infty$, $\|u\|_\infty = \sup_{t\in\mathbb{R}^+} \|u(t)\|_X$ for every $u \in BC(\mathbb{R}^+; X)$. The condition $m, \eta_f \in L_1(0, T)$ is replaced by the condition $m, \eta_f \in L^\infty(\mathbb{R}^+)$. Furthermore, we assume that

(A) A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$, $0 < \alpha \le \beta \le 1$, $\mu > 0$.

Lemma 4.1. *If hypothesis* (A) *is satisfied, then*

$$S_{\alpha,\beta}(t), (\varphi_1 * S_{\alpha,\beta})(t), (\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t) \to 0 \quad \textit{as } t \to \infty.$$

The lemma is proved by using Lemma 2.4 (see the proof of [15, Theorem 4.3] for more details).

Lemma 4.2. *Let hypotheses* (G), (H), (F), (A) *hold. Then we always have* $\Phi(\mathcal{M}_\infty) \subset \mathcal{M}_\infty$.

Proof. Let $u \in \mathcal{M}_\infty$ with $\|u\|_\infty = R > 0$. For every $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$, we get

$$\|S_{\alpha,\beta}(t)\| < \varepsilon, \quad \|(\varphi_1 * S_{\alpha,\beta})(t)\| < \varepsilon, \quad \|(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\| < \varepsilon.$$

We find that for every $t \in \mathbb{R}^+$,

$$\begin{aligned} \|\Phi(u)(t)\|_X \leq & \left\|S_{\alpha,\beta}(t)[x_0 - g(u)] + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)[x_0 - g(u)]\right\|_X \\ & + \left\|(\varphi_1 * S_{\alpha,\beta})(t)[y_0 - h(u)]\right\|_X \\ & + \left\|\int_0^t S_{\alpha,\beta}(t-\tau) f\big(\tau, u(\tau)\big)\, d\tau\right\|_X \\ =: & \, P + Q + K. \end{aligned} \tag{4.1}$$

Then for any $t > T$, we have

$$P \leq \varepsilon(1+\mu)\big(\|x_0\| + \theta_g(R)\big), \quad Q \leq \varepsilon\left(\|y_0\| + \theta_h(R)\right) \tag{4.2}$$

and

$$K \leq \|(\varphi_1 * S_{\alpha,\beta})(t)\| \, \|m\|_{L^\infty(\mathbb{R}_+)} \, \theta_f(R) \leq \varepsilon \|m\|_{L^\infty(\mathbb{R}_+)} \, \theta_f(R). \tag{4.3}$$

From (4.1), (4.2) and (4.3), we obtain $\|\Phi(u)(t)\|_X \to 0$ as $t \to \infty$ for all $u \in \mathcal{M}_\infty$. The proof is completed. □

Lemma 4.3. *Let hypotheses* (G), (H), (F), (A) *hold. Then we have*

$$\chi^*\big(\Phi(D)\big) \leq \left[(C + \mu\Theta_\infty)\eta_g + \Lambda_\infty \eta_h + \sup_{t\geq 0} \int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\| \eta_f(\tau)\, d\tau\right] \chi^*(D) \tag{4.4}$$

for all bounded sets $D \subset M_\infty$.

Proof. Let $D \subset M_\infty$ be a bounded set. We have

$$\chi^*\big(\Phi(D)\big) = \chi_\infty\big(\Phi(D)\big) + d_\infty\big(\Phi(D)\big). \tag{4.5}$$

Thanks to Lemma 3.3 and hypothesis (A), we obtain the following estimates:

$$\chi_\infty\big(\Phi(D)\big) \leq \chi_\infty\big(\Phi_1(D)\big) + \chi_\infty\big(\Phi_2(D)\big) + \chi_\infty\big(\Phi_3(D)\big), \tag{4.6}$$

$$\chi_\infty\big(\Phi_1(D)\big) \leq (C + \mu\Theta_\infty)\eta_g \, \chi_\infty(D), \tag{4.7}$$

$$\chi_\infty\big(\Phi_2(D)\big) \leq \Lambda_\infty \eta_h \, \chi_\infty(D), \tag{4.8}$$

$$\chi_\infty\big(\Phi_3(D)\big) \leq \sup_{t\geq 0} \int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\| \eta_f(\tau)\, d\tau \, \chi_\infty(D). \tag{4.9}$$

From (4.6)–(4.9), we have

$$\chi_\infty\big(\Phi(D)\big) \le \left[(C+\mu\Theta_\infty)\,\eta_g + \Lambda_\infty\eta_h + \sup_{t\ge 0}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\right]\chi_\infty(D). \tag{4.10}$$

Next, we find that

$$d_\infty\big(\Phi(D)\big) = \lim_{T\to\infty} d_T\big(\Phi(D)\big), \quad d_T\big(\Phi(D)\big) = \sup_{u\in D}\sup_{t\ge T}\|\Phi(u)(t)\|_X.$$

Applying Lemma 4.2, we obtain

$$d_\infty\big(\Phi(D)\big) = 0. \tag{4.11}$$

From (4.5), (4.10) and (4.11), we obtain (4.4). The proof is completed. □

Theorem 4.4. *Let hypotheses* (G), (H), (F), (A) *hold. Then problem* (1.1)–(1.2) *has at least one mild solution* $u \in \mathcal{M}_\infty$ *provided that*

$$l_\infty := \left[(C+\mu\Theta_\infty)\,\eta_g + \Lambda_\infty\eta_h + \sup_{t\ge 0}\int_0^t \big\|S_{\alpha,\beta}(t-\tau)\big\|\,\eta_f(\tau)\,d\tau\right] < 1, \tag{4.12}$$

and (3.8) *holds for all* $T > 0$.

Proof. By inequality (4.12), the solution operator Φ is χ^*-condensing. Indeed, if $D \subset \mathcal{M}_\infty$ is bounded such that $\chi^*(D) \le \chi^*(\Phi(D))$. Applying Lemma 4.3, we obtain

$$\chi^*(D) \le \chi^*\big(\Phi(D)\big) \le l_\infty \chi^*(D).$$

Therefore $\chi^*(D) = 0$, and so D is relative compactness.

Thanks to Lemma 4.2, we have

$$\Phi(\mathcal{M}_\infty) \subset \mathcal{M}_\infty.$$

Applying Theorem 2.10, the χ^*-condensing operator Φ defined by (3.7) has a fixed set $\mathrm{Fix}(\Phi) \subset \mathcal{M}_\infty$ which is compact, and is not an empty set. This confirms that problem (1.1)–(1.2) has a mild solution $u(t)$, $t \in \mathbb{R}^+$, described by (2.5) which satisfies $\lim_{t\to\infty}\|u(t)\| = 0$. □

5. An example

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ smooth enough. Let the operator

$$L := \sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial^2}{\partial x_i \partial x_j}$$

have the property

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ x\in\Omega$$

with $c > 0$. With $0 < \alpha \le \beta \le 1$, $\mu > 0$, $a_0 > 0$, we consider the following problem:

$$D_0^{\alpha+1} u(t,x) + \mu D_0^{\beta} u(t,x) - Lu(t,x) + a_0 u(t,x) = F\big(t,x,u(t,x)\big), \tag{5.1}$$

$$\begin{aligned} u(0,x) + \int_\Omega k(x,y)\, u(0,y)\, dy &= u_0(x), \\ u_t(0,x) + \sum_{i=1}^{n} C_i u(t_i,x) &= u_1(x), \end{aligned} \tag{5.2}$$

$$u|_{\partial\Omega} = 0, \tag{5.3}$$

where $0 \le t_1 < t_2 < \cdots < t_n < +\infty$, $C_1, \ldots, C_n$ are positive constants and the function $k : \Omega \times \Omega \to L_2(\Omega)$ satisfies

$$\int_\Omega \int_\Omega |k(x,y)|^2 \, dx\, dy = C < +\infty. \tag{5.4}$$

Let $X = L^2(\Omega)$, $A = L - a_0$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then problem (5.1)–(5.3) is in the form of the abstract model (1.1)–(1.2) with

$$\begin{aligned} F\big(t,u(t)\big)(x) &= F\big(t,x,u(t,x)\big), \\ g(u)(x) &= \int_\Omega k(x,y)\, u(0,y)\, dy, \\ h(u)(x) &= \sum_{i=1}^{n} C_i u(t_i,x). \end{aligned}$$

(A) It is known that (see [23, Theorem 3.6]) $L = A + a_0$ is a sectorial operator of angle $\pi/2$ (and hence of angle $\beta\pi/2$). Therefore, we have that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega = -a_0 < 0$.

Now we give a description for the functions g, h and f.

(G) $g : BC(\mathbb{R}_+, X) \to X$ is continuous.
  (i) $\|g(u)\|_X \le C\|u(0)\|_X \le C\|u\|_\infty$.
  (ii) By [24, Theorem 8.83], g is a compact operator, so $\chi(g(D)) = 0$ for all bounded sets $D \subset BC(\mathbb{R}_+; X)$. Therefore, we can choose $\eta_g = 0$.

(H) $h : BC(\mathbb{R}_+, X) \to X$ is continuous.
  (i) For all $u \in C(\mathbb{R}^+; X)$, we have

$$\|h(u)\|_X = \left\| \sum_{i=1}^{n} C_i u(t_i,x) \right\|_X \le \sum_{i=1}^{n} C_i \|u(t_i,x)\|_X \le \sum_{i=1}^{n} C_i \|u\|_\infty. \tag{5.5}$$

(ii) Next, for every $u_1, u_2 \in BC(\mathbb{R}_+, X)$, we get

$$\begin{aligned}\|h(u_1) - h(u_2)\|_X &= \left\|\sum_{i=1}^{n} C_i\big[u_1(t_i, x) - u_2(t_i, x)\big]\right\|_X \\ &\le \sum_{i=1}^{n} C_i\|u_1(t_i, x) - u_2(t_i, x)\|_X \\ &\le \sum_{i=1}^{n} C_i\|u_1 - u_2\|_\infty.\end{aligned}$$

Therefore,

$$\chi\big(h(D)\big) \le \sum_{i=1}^{n} C_i\chi_\infty(D) \tag{5.6}$$

for every bounded set $D \subset BC(\mathbb{R}^+; X)$.

(F) Suppose that $f(t, u(t)) = \varphi_\alpha * F(t, u(t))$. The nonlinear function f : $\mathbb{R}^+ \times X \to X$ satisfies the following conditions:

(i) $f(\cdot, u(\cdot))$ is measurable for each $u(\cdot) \in X$, $f(t, \cdot)$ is continuous for a.e. $t \in \mathbb{R}_+$, and

$$\|f\big(t, v\big)\|_X \le m(t)\,\theta_f(\|v\|_X) \tag{5.7}$$

for all $v \in X$, where $m \in L^\infty(\mathbb{R}_+)$, $\theta_f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function.

(ii) There exists $\eta_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\eta_f \in L^\infty(\mathbb{R}^+)$ and

$$\chi\big(f(t, \Omega)\big) \le \eta_f(t)\,\chi(\Omega) \tag{5.8}$$

for all bounded sets $\Omega \subset X$.

Under the above settings, applying Theorem 4.4, one can state that problem (5.1)–(5.3) has at least one mild solution in $\mathcal{M}_\infty$, provided that

$$\Lambda_\infty \sum_{i=1}^{n} C_i + \sup_{t\ge 0} \int_0^t \big\|S_{\alpha,\beta}(t - \tau)\big\|\,\eta_f(\tau)\,d\tau < 1.$$

6. Conclusion

In this paper, we discussed the existence of decay mild solutions for two-term time fractional differential equations with nonlocal conditions in Banach spaces. The result of the existence has been established under general settings via measures of noncompactness, which is more extensive than that in [15]. Furthermore, we obtained the existence of decay mild solutions u with $u(t) \to 0$ as $t \to \infty$. The results are illustrated with a well-analyzed example in Section 5.

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