J. Fixed Point Theory Appl. 18 (2016) 397–416
DOI 10.1007/s11784-015-0276-6
Published online November 28, 2015

Journal of Fixed Point Theory and Applications

Nonlinear integral equations with new admissibility types in b-metric spaces

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Abstract. In this paper, we aim to introduce new types of α-admissibility in the framework of b-metric spaces. Some examples to show the independently of each type of α-admissibility are given. Using these concepts, fixed point theorems satisfying generalized weak contractive condition in the setting of b-metric spaces are established. We furnish an illustrative example to demonstrate the validity of the hypotheses and the degree of utility of our results. As an application, we discuss the existence of a solution for the following nonlinear integral equation:

$$x(c) = \phi(c) + \int_a^b K(c, r, x(r))\, dr,$$

where $a, b \in \mathbb{R}$ such that $a < b$, $x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$), $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

Mathematics Subject Classification. 47H09, 47H10.

Keywords. Admissibility types, b-metric spaces, Hölder's inequality, nonlinear integral equations.

1. Introduction and preliminaries

In this section, we recollect some essential notations, required definitions and primary results coherent with the literature. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers and real numbers, respectively.

1.1. Altering distance functions

The classical Banach contraction principle and its applications are well known. In the recent past, many researchers extended this principle by considering relatively more general contractive mappings on various distance spaces such as metric spaces, cone metric spaces [17], complex-valued metric spaces [5], partial metric spaces [21], multiplicative metric spaces [24, 32], etc.

One of the most interested generalizations is the extension of contractive condition to the case of weak contractive condition which was first introduced by Alber and Guerre-Delabriere [4] in the setup of Hilbert spaces. Afterward, Rhoades [28] considered the class of weak contraction mappings in the setup of metric spaces and proved that the result of Alber and Guerre-Delabriere [4] is also valid in complete metric spaces. Fixed point results involving weak contraction and generalized weak contraction mappings have extensively been studied in the literature (see, e.g., [2, 10, 12, 15, 16, 20, 33] and references therein).

On the other hand, Khan, Swaleh and Sessa [19] introduced the concept of an altering distance function, which is a control function that alters distance between two points in a metric space.

Definition 1.1. A function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties hold:

(1) φ is continuous and nondecreasing;
(2) $\varphi(t) = 0$ if and only if $t = 0$.

This concept has been used by many mathematicians to prove fixed point results in a number of subsequent works. Here, we give some examples of altering distance functions.

Example 1.2. Let $\varphi_i : [0,\infty) \to [0,\infty)$, $i \in \{1, 2, \ldots, 5\}$, be defined by

(φ_1) $\varphi_1(t) = kt$, where $k > 0$,
(φ_2) $\varphi_2(t) = t^k$, where $k > 0$,
(φ_3) $\varphi_3(t) = \begin{cases} t, & t \in [0,1], \\ 1 + \sqrt{t-1}, & t \in (1,\infty), \end{cases}$
(φ_4) $\varphi_4(t) = a^t - 1$, where $a > 0$ and $a \neq 1$,
(φ_5) $\varphi_5(t) = \log(kt + 1)$, where $k > 0$.

Then φ_i is an altering distance function for all $i \in \{1, 2, \ldots, 5\}$.

In 2011, Choudhury et al. [11] generalized the concept of weak contraction mappings by using the idea of an altering distance function and proved fixed point results for such mappings.

1.2. b-metric spaces

In 1993, Czerwik [13] introduced the concept of a b-metric space which is a generalization of the ordinary metric space as follows.

Definition 1.3 (See [13]**).** Let X be a nonempty set and let $s \geq 1$ be a given real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions:

(B_1) $d(x,y) = 0$ if and only if $x = y$;
(B_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(B_3) $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x, y, z, \in X$.

Then (X,d) is called a b-metric space with coefficient $s \geq 1$.

It is obvious that the class of b-metric spaces is effectively larger than that of metric spaces since any metric space is a b-metric space with $s = 1$. The following examples show that, in general, a b-metric space need not necessarily be a metric space.

Example 1.4. Let $X = \mathbb{R}$ and let the mapping $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = |x - y|^2 \quad \text{for all } x, y \in X.$$

Then (X, d) is a b-metric space with coefficient $s = 2$.

Next, we show the generality of Example 1.4.

Example 1.5. Let (X, d) be a metric space and let the mapping $\sigma_d : X \times X \to \mathbb{R}_+$ be defined by

$$\sigma_d(x, y) = [d(x, y)]^p \quad \text{for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b-metric space with coefficient $s = 2^{p-1}$. Indeed, conditions (B_1) and (B_2) in Definition 1.3 are satisfied and thus we only need to show that condition (B_3) holds for σ_d. It should be noted that the convexity of the function $\mathbb{R}_+ \ni x \mapsto x^p$, where $1 < p < \infty$, implies

$$\left(\frac{a + c}{2}\right)^p \leq \frac{1}{2}(a^p + c^p)$$

for all $a, c \in \mathbb{R}_+$. This implies that

$$(a + c)^p \leq 2^{p-1}(a^p + c^p)$$

for all $a, c \in \mathbb{R}_+$. Therefore, for each $x, y, z \in X$, we get

$$\begin{aligned}\sigma_d(x, y) &= [d(x, y)]^p \\ &\leq \big[d(x, z) + d(z, y)\big]^p \\ &\leq 2^{p-1}\big[(d(x, z))^p + (d(z, y))^p\big] \\ &= 2^{p-1}\big[\sigma_d(x, z) + \sigma_d(z, y)\big].\end{aligned}$$

This means that condition (B_3) in Definition 1.3 holds.

Example 1.6. The set $l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) := \left\{ \{x_n\} \subseteq \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the mapping $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p} \quad \text{for each } x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R}),$$

is a b-metric space with coefficient $s = 2^{1/p} > 1$. The above result also holds for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 1.7. Let p be a given real number in the interval $(0,1)$. The space $L_p[0,1]$ of all functions $x:[0,1]\to\mathbb{R}$ such that $\int_0^1 |x(t)|^p dt < 1$, together with the mapping $d: L_p[0,1]\times L_p[0,1]\to\mathbb{R}_+$ defined by

$$d(x,y) := \left(\int_0^1 |x(t)-y(t)|^p\, dt\right)^{1/p} \quad \text{for each } x,y\in L_p[0,1],$$

is a b-metric space with coefficient $s=2^{1/p}$.

Next, we give the concepts of b-convergence, b-Cauchy sequence, b-continuity and b-completeness in b-metric spaces.

Definition 1.8 (See [7]**).** Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(a) b-convergent if there exists $x\in X$ such that $d(x_n,x)\to 0$ as $n\to\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$;
(b) b-Cauchy if $d(x_n,x_m)\to 0$ as $n,m\to\infty$.

Each b-convergent sequence in a b-metric space has a unique limit and it is also a b-Cauchy sequence. Moreover, in general, a b-metric is not continuous. We need the following simple lemma about b-convergent sequences in the proof of our main results.

Lemma 1.9 (See [1]**).** *Let (X,d) be a b-metric space with coefficient $s\geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x,y\in X$, respectively. Then we have*

$$\frac{1}{s^2}\, d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x=y$, then we have $\lim_{n\to\infty} d(x_n,y_n)=0$. Moreover, for each $z\in X$, we have

$$\frac{1}{s}\, d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x,z).$$

Definition 1.10 (See [7]**).** Let (X,d_X) and (Y,d_Y) be two b-metric spaces.

(1) The space (X,d_X) is b-complete if every b-Cauchy sequence in X b-converges.
(2) A function $f:X\to Y$ is b-continuous at a point $x\in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.

There are several papers dealing with fixed point results in b-metric spaces (see, e.g., [8, 14, 23, 25, 30] and references therein).

1.3. The objectives of this paper

Inspired by the famous concept of α-admissibility in the setup of metric spaces due to Samet et al. [29], we introduce new types of α-admissibility in the framework of b-metric spaces. Also, we give some examples to show the independently of this concept and the α-admissibility of Samet et al. [29].

Using the new concepts, we prove some fixed point theorems satisfying generalized weak contractive condition by using altering distance functions in the setting of b-metric spaces. We furnish an illustrative example to demonstrate the validity of the hypotheses of our results. Our results generalize and improve several fixed point results in metric spaces and b-metric spaces. We also point out in some remark that many fixed point results in b-metric spaces endowed with partially ordered (or arbitrary binary relation or graph) and fixed point results for cyclic mappings can be concluded from our results. As an application, we apply our results to prove the existence of a solution for the following nonlinear integral equation:

$$x(c) = \phi(c) + \int_a^b K(c, r, x(r))\, dr, \tag{1.1}$$

where $a, b \in \mathbb{R}$ such that $a < b$, $x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$), $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

2. Main results

2.1. α-admissibility types

In this subsection, we introduce new types of α-admissibility and give some examples to show the validity of these concepts.

Definition 2.1. Let X be a nonempty set and let $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be an α-admissible mapping if the following condition holds:

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1.$$

In fact, the concept of α-admissibility was first introduced by Samet et al. [29] in the framework of metric spaces. Now we introduce a new type of α-admissibility, so called α-admissibility type S as follows.

Definition 2.2. Let X be a nonempty set, let s be a given real number such that $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be an α-admissible mapping type S if the following condition holds:

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \alpha(fx, fy) \geq s.$$

Unless otherwise specified, for a nonempty set X, a real number $s \geq 1$ and a mapping $\alpha : X \times X \to [0, \infty)$, we use $\mathcal{A}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha)$ to denote the collection of all α-admissible mappings on X and the collection of all α-admissible mappings type S on X, that is,

$$\mathcal{A}(X, \alpha) := \{f : X \to X \mid f \text{ is an } \alpha\text{-admissible mapping}\}$$

and

$$\mathcal{A}_s(X, \alpha) := \{f : X \to X \mid f \text{ is an } \alpha\text{-admissible mapping type } S\}.$$

Here we give some examples to show that the class of α-admissible mappings and the class of α-admissible mappings type S are independent; that is, $\mathcal{A}(X,\alpha) \neq \mathcal{A}_s(X,\alpha)$ in general case.

Example 2.3. Let $X = [0,\infty)$, let $s = 2$ and let the mappings $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be defined by

$$\alpha(x,y) = \begin{cases} \dfrac{x+y+3}{2}, & x,y \in [0,1], \\ \dfrac{|x-y|}{1+|x-y|}, & \text{otherwise} \end{cases}$$

and

$$f(x) = \begin{cases} \sin x, & x \in [0,1], \\ \cosh x, & x \in (1,\infty). \end{cases}$$

Firstly, we show that $f \in \mathcal{A}(X,\alpha)$. Assume that $x,y \in X$ such that $\alpha(x,y) \geq 1$ and so $x,y \in [0,1]$. Therefore, $fx = \sin x$, $fy = \sin y \in [0,1]$ and then $\alpha(fx,fy) \geq 1$. This shows that $f \in \mathcal{A}(X,\alpha)$. Next, we claim that $f \notin \mathcal{A}_s(X,\alpha)$. Let $x = y = 0.5$. We can see that

$$\alpha(x,y) = \alpha(0.5,0.5) = 2 \geq s,$$

but

$$\begin{aligned} \alpha(fx,fy) &= \alpha(f(0.5),f(0.5)) = \alpha(\sin 0.5, \sin 0.5) \\ &= 1.5 + \sin 0.5 < 1.5 + \sin\frac{\pi}{6} = 2 = s. \end{aligned}$$

This implies that $f \notin \mathcal{A}_s(X,\alpha)$. Therefore, we have $\mathcal{A}(X,\alpha) \nsubseteq \mathcal{A}_s(X,\alpha)$.

Example 2.4. Let $X = \mathbb{R}$, let $s = 2$ and let the mappings $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be defined by

$$\alpha(x,y) = \begin{cases} x^2+y^2, & x,y \in [3,4], \\ \min\{1,|x-y|\}, & \text{otherwise} \end{cases}$$

and

$$f(x) = \begin{cases} 3+\tanh(2x+1), & x \in [3,4], \\ \dfrac{x}{2}, & x \in [0,3) \cup (4,\infty). \end{cases}$$

Here we claim that $f \in \mathcal{A}_s(X,\alpha)$. Let $x,y \in X$ such that

$$\alpha(x,y) \geq s = 2.$$

Then $x,y \in [3,4]$ and thus

$$\begin{aligned} fx &= 3+\tanh(2x+1), \\ fy &= 3+\tanh(2y+1) \in [3,4]. \end{aligned}$$

Therefore, $\alpha(fx,fy) \geq s$. This implies that $f \in \mathcal{A}_s(X,\alpha)$. Now, we show that $f \notin \mathcal{A}(X,\alpha)$. Let $x = 1$ and $y = 2$. Then we observer that

$$\alpha(x,y) = \alpha(1,2) = 1,$$

but

$$\alpha(fx, fy) = \alpha\big(f(1), f(2)\big) = \alpha(0.5, 1) = 0.5 < 1.$$

This implies that $f \notin \mathcal{A}(X,\alpha)$. Therefore, we have $\mathcal{A}_s(X,\alpha) \nsubseteq \mathcal{A}(X,\alpha)$.

Next we give the new concepts of weak α-admissibility (see also [31]) and weak α-admissibility type S.

Definition 2.5. Let X be a nonempty set and let $\alpha : X \times X \to [0,\infty)$ be a given mapping. A mapping $f : X \to X$ is said to be a weak α-admissible mapping if the following condition holds:

$$x \in X \text{ with } \alpha(x, fx) \geq 1 \Longrightarrow \alpha(fx, ffx) \geq 1.$$

Definition 2.6. Let X be a nonempty set and let $\alpha : X \times X \to [0,\infty)$ be a given mapping. A mapping $f : X \to X$ is said to be a weak α-admissible mapping type S if the following condition holds:

$$x \in X \text{ with } \alpha(x, fx) \geq s \Longrightarrow \alpha(fx, ffx) \geq s.$$

Unless otherwise specified, for a nonempty set X, a real number $s \geq 1$ and a mapping $\alpha : X \times X \to [0,\infty)$, we use the following symbols:

$$\mathcal{WA}(X,\alpha) := \{f : X \to X \mid f \text{ is a weak } \alpha\text{-admissible mapping}\}$$

and

$$\mathcal{WA}_s(X,\alpha) := \{f : X \to X \mid f \text{ is a weak } \alpha\text{-admissible mapping type } S\}.$$

Remark 2.7. It is easy to see that the following assertions hold:

- α-admissibility implies weak α-admissibility, that is,
$$\mathcal{A}(X,\alpha) \subseteq \mathcal{WA}(X,\alpha);$$
- α-admissibility type S implies weak α-admissibility type S, that is,
$$\mathcal{A}_s(X,\alpha) \subseteq \mathcal{WA}_s(X,\alpha).$$

2.2. Fixed point results

In this subsection, we give fixed point results for mappings in classes

$$\mathcal{WA}_s(X,\alpha) \quad \text{and} \quad \mathcal{A}_s(X,\alpha).$$

Throughout this paper, unless otherwise stated, $\mathrm{Fix}(f)$ denotes the set of all fixed points of a self-mapping f on a nonempty set X, that is,

$$\mathrm{Fix}(f) := \{x \in X \mid fx = x\}.$$

Also, for each elements x and y in a b-metric space (X,d) with coefficient $s \geq 1$, let

$$M_s(x,y) := \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\right\},$$

where f is a self-mapping on X. We write $M(x,y)$ instead of $M_s(x,y)$ when $s = 1$, that is,

$$M(x,y) := \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

Definition 2.8. Let (X, d) be a b-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a given mapping and let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions. We say that a mapping $f : X \to X$ is an $(\alpha, \psi, \varphi)_s$-contraction mapping if the following condition holds:

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \psi\big(s^3 d(fx, fy)\big) \leq \psi\big(M_s(x, y)\big) - \varphi\big(M_s(x, y)\big). \tag{2.1}$$

We denote by $\Omega_s(X, \alpha, \psi, \varphi)$ the collection of all $(\alpha, \psi, \varphi)_s$-contraction mappings on a b-metric space (X, d) with coefficient $s \geq 1$.

Theorem 2.9. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:*

(S_1) $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;
(S_2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq s$;
(S_3) α *has a transitive property type* S, *that is, for* $x, y, z \in X$,

$$\alpha(x, y) \geq s \text{ and } \alpha(y, z) \geq s \Longrightarrow \alpha(x, z) \geq s;$$

(S_4) f *is* b-*continuous.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By the given condition (S_2), there exists $x_0 \in X$ such that

$$\alpha(x_0, fx_0) \geq s.$$

Now we define the Picard iteration sequence $\{x_n\}$ by

$$x_{n+1} := fx_n$$

for all $n \in \mathbb{N} \cup \{0\}$. If there is $\tilde{n} \in \mathbb{N} \cup \{0\}$ so that $x_{\tilde{n}} = x_{\tilde{n}+1}$, then we have $x_{\tilde{n}} \in \mathrm{Fix}(f)$ and hence the conclusion holds. So we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. It follows that

$$d(x_n, x_{n+1}) > 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Here we show that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

It follows from $f \in \mathcal{WA}_s(X, \alpha)$ and $\alpha(x_0, fx_0) \geq s$ that

$$\alpha(x_1, x_2) = \alpha(fx_0, ffx_0) \geq s. \tag{2.3}$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq s \tag{2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows from $f \in \Omega_s(X, \alpha, \psi, \varphi)$ that inequality (2.4) implies that

$$\begin{aligned} \psi\big(d(fx_n, fx_{n+1})\big) &\leq \psi\big(s^3 d(fx_n, fx_{n+1})\big) \\ &\leq \psi\big(M_s(x_n, x_{n+1})\big) - \varphi\big(M_s(x_n, x_{n+1})\big) \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} M_s(x_n, x_{n+1}) &= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \\ &\qquad \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s} \Bigg\} \\ &= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s} \Bigg\} \\ &= \max\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \}. \end{aligned}$$

If $M_s(x_{n^*}, x_{n^*+1}) = d(x_{n^*+1}, x_{n^*+2})$ for some $n^* \in \mathbb{N} \cup \{0\}$, then inequality (2.5) implies that

$$\begin{aligned} \psi\big(d(fx_{n^*}, fx_{n^*+1})\big) &\le \psi\big(d(x_{n^*+1}, x_{n^*+2})\big) - \varphi\big(d(x_{n^*+1}, x_{n^*+2})\big) \\ &< \psi\big(d(x_{n^*+1}, x_{n^*+2})\big), \end{aligned}$$

which is a contradiction. Therefore,

$$M_s(x_n, x_{n+1}) = d(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.5), we have

$$\begin{aligned} \psi\big(d(x_{n+1}, x_{n+2})\big) &= \psi\big(d(fx_n, fx_{n+1})\big) \\ &\le \psi\big(d(x_n, x_{n+1})\big) - \varphi\big(d(x_n, x_{n+1})\big) \\ &< \psi\big(d(x_n, x_{n+1})\big) \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since ψ is a nondecreasing mapping, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$. Since $\{d(x_n, x_{n+1})\}$ is bounded below, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (2.6), we get

$$\psi(r) \le \psi(r) - \varphi(r) \le \psi(r).$$

This implies that $\varphi(r) = 0$ and thus $r = 0$. Hence, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Thus (2.2) holds.

Next, we prove that $\{x_n\}$ is a b-Cauchy sequence in X. Assume to the contrary that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \ge k$ and

$$d(x_{m(k)}, x_{n(k)}) \ge \epsilon \tag{2.8}$$

and $n(k)$ is the smallest number such that (2.8) holds. From (2.8), we have

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.9}$$

By (B_3), (2.8) and (2.9), we get

$$\begin{aligned}\epsilon &\leq d\big(x_{m(k)}, x_{n(k)}\big)\\ &\leq s\big[d\big(x_{m(k)}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big)\big]\\ &< s\epsilon + d\big(x_{n(k)-1}, x_{n(k)}\big).\end{aligned} \tag{2.10}$$

Taking the limit supremum as $k \to \infty$ in (2.10), using (2.7) we get

$$\epsilon \leq \limsup_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) \leq s\epsilon. \tag{2.11}$$

Again, using (B_3), we obtain

$$d\big(x_{m(k)}, x_{n(k)}\big) \leq s\big[d\big(x_{m(k)}, x_{n(k)+1}\big) + d\big(x_{n(k)+1}, x_{n(k)}\big)\big] \tag{2.12}$$

and

$$d\big(x_{m(k)}, x_{n(k)+1}\big) \leq s\big[d\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{n(k)}, x_{n(k)+1}\big)\big]. \tag{2.13}$$

Taking the limit supremum as $k \to \infty$ in (2.12) and (2.13), from (2.7) and (2.11), we get

$$\epsilon \leq s\left(\limsup_{k\to\infty} d\big(x_{m(k)}, x_{n(k)+1}\big)\right) \tag{2.14}$$

and

$$\limsup_{k\to\infty} d\big(x_{m(k)}, x_{n(k)+1}\big) \leq s^2\epsilon. \tag{2.15}$$

From (2.14) and (2.15), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\big(x_{m(k)}, x_{n(k)+1}\big) \leq s^2\epsilon. \tag{2.16}$$

Similarly, we can show that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\big(x_{m(k)+1}, x_{n(k)}\big) \leq s^2\epsilon. \tag{2.17}$$

Finally, we obtain

$$\begin{aligned}&d\big(x_{m(k)+1}, x_{n(k)+1}\big)\\ &\quad\leq s\big[d\big(x_{m(k)+1}, x_{m(k)}\big) + d\big(x_{m(k)}, x_{n(k)+1}\big)\big]\\ &\quad\leq sd\big(x_{m(k)+1}, x_{m(k)}\big) + s^2\big[d\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{n(k)}, x_{n(k)+1}\big)\big].\end{aligned} \tag{2.18}$$

Taking the limit supremum as $k \to \infty$ in (2.18), we have

$$\limsup_{k\to\infty} d\big(x_{m(k)+1}, x_{n(k)+1}\big) \leq s^3\epsilon. \tag{2.19}$$

Using (B_3) again, we have

$$\begin{aligned}&d\big(x_{m(k)}, x_{n(k)}\big)\\ &\quad\leq s\big[d\big(x_{m(k)}, x_{m(k)+1}\big) + d\big(x_{m(k)+1}, x_{n(k)}\big)\big]\\ &\quad\leq sd\big(x_{m(k)}, x_{m(k)+1}\big) + s^2\big[d\big(x_{m(k)+1}, x_{n(k)+1}\big) + d\big(x_{n(k)+1}, x_{n(k)}\big)\big].\end{aligned} \tag{2.20}$$

Taking the limit supremum as $k \to \infty$ in (2.20) and using (2.7) and (2.11), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d\big(x_{m(k)+1}, x_{n(k)+1}\big). \tag{2.21}$$

From (2.19) and (2.21), we get

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d\big(x_{m(k)+1}, x_{n(k)+1}\big) \leq s^3\epsilon. \tag{2.22}$$

Using the transitivity property type S of α, we get

$$\alpha\big(x_{m(k)}, x_{n(k)}\big) \geq s.$$

Since $f \in \Omega_s(X, \alpha, \psi, \varphi)$, we have

$$\begin{aligned}
&\psi\big(s^3 d(x_{m(k)+1}, x_{n(k)+1})\big)\\
&\qquad = \psi\big(s^3 d(fx_{m(k)}, fx_{n(k)})\big)\\
&\qquad \leq \psi\big(M_s(x_{m(k)}, x_{n(k)})\big) - \varphi\big(M_s(x_{m(k)}, x_{n(k)})\big),
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
&M_s\big(x_{m(k)}, x_{n(k)}\big)\\
&\quad = \max\Big\{d\big(x_{m(k)}, x_{n(k)}\big), d\big(x_{m(k)}, fx_{m(k)}\big), d\big(x_{n(k)}, fx_{n(k)}\big),\\
&\qquad\qquad \frac{d\big(x_{m(k)}, fx_{n(k)}\big) + d\big(x_{n(k)}, fx_{m(k)}\big)}{2s}\Big\}\\
&\quad = \max\Big\{d\big(x_{m(k)}, x_{n(k)}\big), d\big(x_{m(k)}, x_{m(k)+1}\big), d\big(x_{n(k)}, x_{n(k)+1}\big),\\
&\qquad\qquad \frac{d\big(x_{m(k)}, x_{n(k)+1}\big) + d\big(x_{n(k)}, x_{m(k)+1}\big)}{2s}\Big\}.
\end{aligned}$$

Taking the limit supremum as $k \to \infty$ in the above equation and using (2.7), (2.11), (2.16) and (2.17), we have

$$\begin{aligned}
\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} &\leq \limsup_{k\to\infty} M_s\big(x_{m(k)}, x_{n(k)}\big)\\
&\leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} &\leq \liminf_{k\to\infty} M_s\big(x_{m(k)}, x_{n(k)}\big)\\
&\leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.
\end{aligned}$$

Taking the limit supremum as $k \to \infty$ in (2.23), we have

$$\begin{aligned}\psi(s\epsilon) &= \psi\left(s^3\left(\frac{\epsilon}{s^2}\right)\right)\\ &\leq \psi\Big(s^3 \limsup_{k\to\infty} d\big(x_{m(k)+1}, x_{n(k)+1}\big)\Big)\\ &\leq \psi\left(\limsup_{k\to\infty} M_s\left(x_{m(k)}, x_{n(k)}\right)\right) - \varphi\left(\liminf_{k\to\infty} M_s\left(x_{m(k)}, x_{n(k)}\right)\right)\\ &\leq \psi(s\epsilon) - \varphi(\epsilon).\end{aligned} \tag{2.24}$$

This implies that $\varphi(\epsilon) = 0$ and then $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence. By b-completeness of the b-metric space X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

By b-continuity of f, we get

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

From the triangle inequality, we have

$$d(x, fx) \leq s\big[d(x, fx_n) + d(fx_n, fx)\big] \tag{2.25}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$d(x, fx) = 0$$

and then $fx = x$. This shows that $\mathrm{Fix}(f) \neq \emptyset$. □

Example 2.10. Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Then (X, d) is a b-complete b-metric space with coefficient $s = 2$. Define mappings $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$fx = \begin{cases} \sinh^{-1}\dfrac{x}{4}, & x \in [0, 16/3],\\ \ln(6x - 29), & x \in (16/3, \infty)\end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1 + x + \cosh(1 + 2y), & x, y \in [0, 16/3],\\ 1 + \tanh(x - y), & \text{otherwise.}\end{cases}$$

Also, we define two altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = rt \quad \text{and} \quad \varphi(t) = (r - 1)t$$

for all $t \in [0, \infty)$, where $r \in (1, 2)$.

Now we show that $f \in \Omega_s(X,\alpha,\psi,\varphi)$. Suppose that $x, y \in X$ so that $\alpha(x,y) \geq s = 2$ and hence $x, y \in [0, 16/3]$. Using the mean value theorem simultaneously for the inverse hyperbolic sine function, we obtain

$$\begin{aligned}\psi\big(2^3 d(fx,fy)\big) &= 8r|fx - fy|^2 \\ &= 8r\left|\sinh^{-1}\frac{x}{4} - \sinh^{-1}\frac{y}{4}\right|^2 \\ &\leq 8r\left|\frac{x}{4} - \frac{y}{4}\right|^2 \\ &= \frac{r}{2}|x-y|^2 \\ &\leq M_s(x,y) \\ &\leq \psi\big(M_s(x,y)\big) - \varphi\big(M_s(x,y)\big).\end{aligned}$$

This implies that (2.1) holds and thus $f \in \Omega_s(X,\psi,\varphi)$.

It is easy to see that $f \in \mathcal{WA}_s(X,\alpha)$. Indeed, if $x \in X$ such that

$$\alpha(x,fx) \geq s = 2,$$

then $x, fx \in [0,16/3]$. This implies that $ffx \in [0,16/3]$ and hence

$$\alpha(fx,ffx) \geq s.$$

Also, we can see that f is continuous and there is $x_0 = 1$ such that

$$\begin{aligned}\alpha(x_0,fx_0) = \alpha(1,f(1)) &= \alpha\left(1, \sinh^{-1}\frac{1}{4}\right) \\ &= 1 + 1 + \cosh\left(1 + 2\sinh^{-1}\frac{1}{4}\right) \\ &\geq 2 = s.\end{aligned}$$

Therefore, all the conditions of Theorem 2.9 are satisfied. Then we can conclude that $\mathrm{Fix}(f) \neq \emptyset$. In this example, it is easy to see that $0 \in \mathrm{Fix}(f)$.

Theorem 2.11. *Let (X,d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be two altering distance functions and let $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

- (S_1) *$f \in \Omega_s(X,\alpha,\psi,\varphi) \cap \mathcal{WA}_s(X,\alpha)$;*
- (S_2) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq s$;*
- (S_3) *α has a transitive property type S;*
- $(\widetilde{S_4})$ *X is α_s-regular, that is, if $\{x_n\}$ is a sequence in X such that*

$$\alpha(x_n,x_{n+1}) \geq s$$

for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n,x) \geq s$ for all $n \in \mathbb{N}$.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Following the proof of Theorem 2.9, we obtain that $\{x_n\}$ is a b-Cauchy sequence in the b-complete b-metric space (X,d). By b-completeness of X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0, \tag{2.26}$$

that is, $x_n \to x$ as $n \to \infty$. By α_s-regularity of X, we have

$$\alpha(x_n, x) \geq s$$

for all $n \in \mathbb{N}$. It follows from $f \in \Omega_s(X,\alpha,\psi,\varphi)$ that

$$\psi\big(s^3 d(fx_n, fx)\big) \leq \psi\big(M_s(x_n,x)\big) - \varphi\big(M_s(x_n,x)\big), \tag{2.27}$$

where

$$M_s(x_n,x) = \max\left\{d(x_n,x), d(x_n,fx_n), d(x,fx), \frac{d(x_n,fx)+d(x,fx_n)}{2s}\right\}.$$

Taking the limit supremum as $n \to \infty$ in (2.27) and using Lemma 1.9, we get

$$\begin{aligned}
\psi\big(d(x,fx)\big) &\leq \psi\big(s^2 d(x,fx)\big) \\
&= \psi\left(s^3 \frac{1}{s}\, d(x,fx)\right) \\
&\leq \psi\left(s^3 \limsup_{n\to\infty} d(x_{n+1}, fx)\right) \\
&\leq \psi\left(\limsup_{n\to\infty} M_s(x_n,x)\right) - \varphi\left(\liminf_{n\to\infty} M_s(x_n,x)\right) \\
&\leq \psi\big(d(x,fx)\big) - \varphi\big(d(x,fx)\big),
\end{aligned}$$

which implies that $\varphi(d(x,fx)) = 0$. It follows that $d(x,fx) = 0$, equivalently, $x = fx$ and thus $\mathrm{Fix}(f) \neq \emptyset$. This completes the proof. □

Next, we use Remark 2.7 to establish the following results for the class $\mathcal{A}_s(X,\alpha)$.

Corollary 2.12. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, let $\psi,\varphi : [0,\infty) \to [0,\infty)$ be altering distance functions and let $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

- $(\widetilde{S_1})$ *$f \in \Omega_s(X,\alpha,\psi,\varphi) \cap \mathcal{A}_s(X,\alpha)$;*
- (S_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;*
- (S_3) *α has a transitive property type S;*
- (S_4) *f is b-continuous.*

Then $\mathrm{Fix}(f) \neq \emptyset$*.*

Corollary 2.13. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, let $\psi,\varphi : [0,\infty) \to [0,\infty)$ be altering distance functions and let $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

- $(\widetilde{S_1})$ *$f \in \Omega_s(X,\alpha,\psi,\varphi) \cap \mathcal{A}_s(X,\alpha)$;*

(S_2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq s$;
(S_3) α *has a transitive property type* S;
($\widetilde{S_4}$) X *is* α_s*-regular.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Remark 2.14. Theorems 2.9 and 2.11 and Corollaries 2.12 and 2.13 extend and improve various fixed point results in b-metric spaces. Also, our results generalize and complement the following well-known fixed point results in metric spaces.

- The very famous Banach contraction mapping principle [6], Kannan's fixed point result [18] (see also Reich's work [26, 27]), Chatterjea's fixed point result [9] in the ordinary metric spaces.
- Alber et al.'s fixed point result [4] in the framework of Hilbert spaces.
- Rhoades's fixed point result [28].
- Dutta and Choudhury's fixed point result [15].

Moreover, it has been pointed out in some studies that the following fixed point results can be concluded from our result under some suitable (weak) α-admissible and (weak) α-admissible mappings type S.

- Fixed point results in b-metric spaces endowed with a binary relation such as strict order (or sharp order), near-order, pseudo-order, quasi-order (or preorder), partial order, simple order, weak order, total order (or linear order or chain), tolerance, equivalence, etc. (see [3]).
- Fixed point results in b-metric spaces endowed with graph.
- Fixed point results for cyclic mappings.

3. Applications: Existence of a solution for a nonlinear integral equation

The theory of differential and integral equations nowadays is a large subject of mathematics which found in the last three decades numerous applications in physics, mechanics, engineering, bioengineering, control theory and other fields connected with real-world problems.

In this section, we prove an existence theorem for a solution of the following nonlinear integral equation by using our main results in the previous section:

$$x(c) = \phi(c) + \int_a^b K(c, r, x(r))\, dr, \tag{3.1}$$

where $a, b \in \mathbb{R}$ such that $a < b$, $x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$), $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

Theorem 3.1. *Consider the nonlinear integral equation* (3.1)*. Suppose that the following conditions hold:*

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing in the third order;*

(ii) *there exists $p > 1$ satisfying the following condition: for each $r, c \in [a, b]$ and $x, y \in X$ with $x(w) \leq y(w)$ for all $w \in [a, b]$, we have*

$$\left|K(c, r, x(r)) - K(c, r, y(r))\right| \leq \zeta(c, r)\big(\Upsilon(|x(r) - y(r)|^p)\big),$$

where $\zeta : [a, b] \times [a, b] \to [0, \infty)$ is a continuous function satisfying

$$\sup_{c\in[a,b]} \left(\int_a^b \zeta(c, r)^p \, dr\right) < \frac{1}{2^{3p^2-3p}(b - a)^{p-1}}$$

and $\Upsilon : [0, \infty) \to [0, \infty)$ is an altering distance function satisfying the following conditions:

(Υ_1) $\frac{d}{dt}[\Upsilon(t)] < 1$ *for all $t > 0$,*

(Υ_2) $\Upsilon(t) < t$ *for all $t > 0$;*

(iii) *there exists $x_0 \in X$ such that $x_0(c) \leq \phi(c) + \int_a^b K(c, r, x_0(r))\, dr$ for all $c \in [a, b]$.*

Then the nonlinear integral equation (3.1) *has a solution.*

Proof. Let $X = C[a, b]$ and let $f : X \to X$ be defined by

$$(fx)(c) = \phi(c) + \int_a^b K(c, r, x(r))\, dr$$

for all $x \in X$ and $c \in [a, b]$. Clearly, X with the b-metric $d : X \times X \to \mathbb{R}_+$ given by

$$d(x, y) = \sup_{c\in[a,b]} \left|x(c) - y(c)\right|^p$$

for all $x, y \in X$, is a b-complete b-metric space with coefficient $s = 2^{p-1}$.

Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2^{p-1}, & x(c) \leq y(c) \text{ for all } c \in [a, b], \\ \tau, & \text{otherwise}, \end{cases}$$

where $0 < \tau < 2^{p-1}$. It is easy to see that α has a transitive property type S. Since K is nondecreasing in the third order, we get

$$f \in \mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha).$$

From (iii), we get $\alpha(x_0, fx_0) \geq 2^{p-1} = s$. Also, we get that condition $(\widetilde{S_4})$ in Theorem 2.11 holds (see [22]).

Next, we define functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = t^p \quad \text{and} \quad \varphi(t) = t^p - (\Upsilon(t))^p$$

for all $t \in [0, \infty)$. It should be noted that ψ is an altering distance function. Since Υ is an altering distance function and it satisfies conditions (Υ_1) and (Υ_2), then φ is also an altering distance function.

Finally, we show that $f \in \Omega_S(X, \alpha, \psi, \varphi)$. To prove this fact, we first choose

$$q := \frac{p}{p - 1} \in \mathbb{R},$$

that is, $1/p + 1/q = 1$. Assume that $x, y \in X$ such that

$$\alpha(x,y) \geq s = 2^{p-1},$$

that is, $x(c) \leq y(c)$ for all $c \in [a,b]$. From (i), (ii) and the Hölder inequality, for each $c \in [a,b]$ we get

$$\begin{aligned}
&\big(2^{3p-3}\big|(fx)(c) - (fy)(c)\big|\big)^p \\
&\leq 2^{3p^2-3p}\left(\int_a^b \big|K(c,r,x(r)) - K(c,r,y(r))\big|\,dr\right)^p \\
&\leq 2^{3p^2-3p}\left[\left(\int_a^b 1^q\,dr\right)^{1/q}\left(\int_a^b \big|K(c,r,x(r)) - K(c,r,y(r))\big|^p\,dr\right)^{1/p}\right]^p \\
&\leq 2^{3p^2-3p}(b-a)^{p/q}\left(\int_a^b \zeta(c,r)^p\big(\Upsilon\big(|x(r)-y(r)|^p\big)\big)^p\,dr\right) \\
&\leq 2^{3p^2-3p}(b-a)^{p/q}\left(\int_a^b \zeta(c,r)^p\big(\Upsilon(d(x,y))\big)^p\,dr\right) \\
&\leq 2^{3p^2-3p}(b-a)^{p/q}\left(\int_a^b \zeta(c,r)^p\big(\Upsilon(M_s(x,y))\big)^p\,dr\right) \\
&= 2^{3p^2-3p}(b-a)^{p-1}\left(\int_a^b \zeta(c,r)^p\,dr\right)\big(\Upsilon(M_s(x,y))\big)^p \\
&< \big(\Upsilon(M_s(x,y))\big)^p \\
&= M_s(x,y)^p - \big[M_s(x,y)^p - (\Upsilon(M_s(x,y)))^p\big].
\end{aligned}$$

This implies that

$$\begin{aligned}
\psi\big(s^3 d(fx,fy)\big) &= \big(s^3 d(fx,fy)\big)^p \\
&= \left(2^{3p-3}\sup_{t\in[a,b]}\big|(fx)(t) - (fy)(t)\big|\right)^p \\
&\leq M_s(x,y)^p - \big[M_s(x,y)^p - (\Upsilon(M_s(x,y)))^p\big] \\
&= \psi\big(M_s(x,y)\big) - \varphi\big(M_s(x,y)\big)
\end{aligned}$$

for all $x, y \in X$. It follows that $f \in \Omega_S(X, \alpha, \psi, \varphi)$. Thus all the conditions of Theorem 2.11 are satisfied and hence f has a fixed point in X (namely, $\widehat{x}$). It follows that $\widehat{x}$ is a solution of the nonlinear integral equation (3.1). □

Now we consider some special cases of the function Υ, wherein Theorem 3.1 deduces the following results.

Corollary 3.2. *Consider the nonlinear integral equation* (3.1)*. Suppose that the following conditions hold:*

(i) $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing at the third order;*

(ii′) *there exists* $p > 1$ *satisfying the following condition: for each* $r, c \in [a, b]$ *and* $x, y \in X$ *with* $x(w) \leq y(w)$ *for all* $w \in [a, b]$, *we have*

$$\left|K(c, r, x(r)) - K(c, r, y(r))\right| \leq \zeta(c, r) \left(\sinh^{-1} \left|x(r) - y(r)\right|^p\right),$$

where $\zeta : [a, b] \times [a, b] \to [0, \infty)$ *is a continuous function satisfying*

$$\sup_{c\in[a,b]} \left(\int_a^b \zeta(c, r)^p \, dr\right) < \frac{1}{2^{3p^2-3p}(b - a)^{p-1}};$$

(iii) *there exists* $x_0 \in X$ *such that* $x_0(c) \leq \phi(c) + \int_a^b K(c, r, x_0(r))dr$ *for each* $c \in [a, b]$.

Then the nonlinear integral equation (3.1) *has a solution.*

Proof. From Theorem 3.1 by taking $\Upsilon(t) = \sinh^{-1} t$, we get the result. □

Corollary 3.3. *Consider the nonlinear integral equation* (3.1)*. Suppose that the following conditions hold:*

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing at the third order;*

(ii″) *there exists* $p > 1$ *satisfying the following condition: for each* $r, c \in [a, b]$ *and* $x, y \in X$ *with* $x(w) \leq y(w)$ *for all* $w \in [a, b]$, *we have*

$$\left|K(c, r, x(r)) - K(c, r, y(r))\right| \leq \zeta(c, r)\big(\ln\left(1 + |x(r) - y(r)|^p\right)\big),$$

where $\zeta : [a, b] \times [a, b] \to [0, \infty)$ *is a continuous function satisfying*

$$\sup_{c\in[a,b]} \left(\int_a^b \zeta(c, r)^p \, dr\right) < \frac{1}{2^{3p^2-3p}(b - a)^{p-1}};$$

(iii) *there exists* $x_0 \in X$ *such that* $x_0(c) \leq \phi(c) + \int_a^b K(c, r, x_0(r))dr$ *for each* $c \in [a, b]$.

Then the nonlinear integral equation (3.1) *has a solution.*

Proof. From Theorem 3.1 by taking $\Upsilon(t) = \ln(1 + t)$, we get the result. □

Acknowledgements

The author would like to thank the Thailand Research Fund and Thammasat University for financial support, under Grant No. TRG5780013, during the preparation of this manuscript.

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