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Some remarks on Lagrangian tori

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To Professor Andrzej Granas

Abstract. We illustrate the power of Floer theory for Lagrangian submanifolds through some of its applications in symplectic topology.

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1. Introduction

Floer theory for Lagrangian submanifolds has been developed in the last decades [12, 14, 15, 25, 26, 27] and it reveals many interesting features in symplectic geometry. Even though we have a general theory, it is often difficult to make explicit computations. In the case of compact toric Kähler manifolds, Lagrangian tori appear as principal orbits of the compact torus, which coincide with fibers of the moment map. We call them *Lagrangian torus fibers*. Thanks to large symmetry, Floer theory for Lagrangian torus fibers in compact Kähler toric manifolds is understood in [16, 17, 18] (see also [19]). In this note, we consider simple examples of toric manifolds, such as the product of two-dimensional spheres, and present some applications, which illustrate the effectiveness of Floer theory for Lagrangian submanifolds. The first application is an approach to distinguish symplectic embeddings of a polydisc to another polydisc and gives an alternative proof of a result due to Floer, Hofer and Wysocki [13]. The second is about homological Lagrangian monodoromy problem, which was studied by Yau [29] and Hu, Lalonde and Leclercq [23]. Lastly, we discuss a certain Lagrangian torus in the symplectic blowup of a symplectically aspherical manifold and show that it is superheavy in the sense of Entov and Polterovich [10].

2. Displacement energy of Lagrangian submanifolds

In this section, we review results in [7] (cf. [20]) on the lower bound of the displacement energy of a Lagrangian submanifold.

Let (X, ω) be a symplectic manifold. For simplicity, we assume that X is a closed manifold from now on. A diffeomorphism $\varphi : X \to X$ is called a Hamiltonian diffeomorphism if there is a smooth function $H : \mathbb{R} \times X \to \mathbb{R}$ such that φ is the time-one map ϕ_H^1 of

$$\frac{d}{dt}\phi_H^t = X_{H_t} \circ \phi_H^t,$$

where X_{H_t} is the Hamiltonian vector field of $H_t = H(t, \bullet)$. Denote by $\mathrm{Ham}(X, \omega)$ the group of Hamiltonian diffeomorphisms of (X, ω).

We recall the Hofer distance on $\mathrm{Ham}(X, \omega)$.

Definition 2.1. For $\varphi, \psi \in \mathrm{Ham}(X, \omega)$, we define

$$d_H(\varphi, \psi) = \inf\left\{\|H\| \mid \psi^{-1} \circ \varphi = \phi_H^1\right\},$$

where

$$\|H\| = \int_0^1 \left(\sup_X H_t - \inf_X H_t\right) dt.$$

It is easy to see that d_H is a bi-invariant pseudodistance on $\mathrm{Ham}(X, \omega)$. The following important theorem is due to Hofer [22], Polterovich [28] and Lalonde and McDuff [24].

Theorem 2.2. *The bi-invariant pseudodistance d_H is a bi-invariant distance on* $\mathrm{Ham}(X, \omega)$.

A closed subset A is said to be *displaceable*, if there is a Hamiltonian diffeomorphism φ such that $A \cap \varphi(A) = \emptyset$. We define the displacement energy of $A \subset X$ by

$$e^X(A) = \begin{cases} \inf\left\{\|H\| \mid A \cap \phi_H^1(A) = \emptyset\right\} & \text{if } A \text{ is displaceable,} \\ +\infty & \text{otherwise.} \end{cases}$$

For $\phi \neq \mathrm{id}$, we can find an open subset $U \subset X$ such that

$$\phi(\overline{U}) \cap \overline{U} = \emptyset.$$

Hence Theorem 2.2 is obtained once we have $e^X(\overline{U}) > 0$ for any open subset U. Such an inequality is a consequence of the so-called energy-capacity inequality.

By Darboux's theorem, we can take U as the image of a symplectic embedding of a round ball in the standard symplectic space. Thus we find that U contains Lagrangian tori coming from the product of small circles

$$S^1(\epsilon) \times \cdots \times S^1(\epsilon) \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2,$$

where $\mathbb{R}^2$ is equipped with the standard symplectic structure. Hence Theorem 2.2 also follows from the fact that $e^X(L) > 0$ for any closed Lagrangian submanifold $L \subset X$. Polterovich [28] gave a lower bound for rational Lagrangian submanifolds. Here L is called a rational Lagrangian submanifolds if the set of the integration of ω on discs in X with boundary on L is discrete in $\mathbb{R}$.

Next we quote Chekanov's theorem [7] for the lower bound of the displacement energy of a closed Lagrangian submanifold L in (X, ω). Let $\mathcal{J}(X, \omega)$ be the space of almost complex structures on X tamed by ω. For $J \in \mathcal{J}(X, \omega)$, we define $\sigma_S(X, J)$ (resp., $\sigma_D(X, L, J)$) as the minimal symplectic area of nonconstant J-holomorphic spheres in X (resp., nonconstant J-holomorphic discs in X with boundary on L). We set

$$\sigma(X, L) = \sup_{J \in \mathcal{J}(X,\omega)} \min \big\{ \sigma_S(X, J), \sigma_D(X, L, J) \big\}.$$

The lower bound for the displacement energy $e^X(L)$ of L in X is given by the following theorem due to Chekanov. In particular, this theorem implies the nondegeneracy of the Hofer distance d_H, hence Theorem 2.2.

Theorem 2.3.

$$e^X(L) \geq \sigma(X, L).$$

Chekanov used a variant of Floer theory to prove this theorem, although Floer cohomology is not necessarily defined for L. In Theorem 2.3, the lower bound is given by the minimal area of nonconstant holomorphic spheres or discs. However, the effects from those holomorphic curves to Floer theory may cancel one another. In such a case, it should be possible to get a better lower bound (see [14, Theorem J], see also [20]).

In subsequent sections, we apply Theorem 2.3 to compute the displacement energy of certain Lagrangian tori and show some results in symplectic topology. We prepare some notation. Let $D^2(a) \subset \mathbb{C}$ be the standard disc of area a and let $S^1(a) = \partial D^2(a)$. We denote the polydisc by

$$D^{2n}(a_1, \ldots, a_n) = D^2(a_1) \times \cdots \times D^2(a_n) \subset \mathbb{C}^n$$

and the product Lagrangian torus by

$$T^n(b_1, \ldots, b_n) = S^1(b_1) \times \cdots \times S^1(b_n) \subset D^{2n}(a_1, \ldots, a_n)$$

for $b_1 < a_1, \ldots, b_n < a_n$.

In Section 3, we give an alternative proof of a theorem of Floer, Hofer and Wysocki concerning nonisotopic symplectic embeddings of a polydisc $D^4(u_1, u_2)$ to $D(1, 1)$ in the case that $u_1 + u_2 > 1$. In Section 4 (resp., Section 5), we consider the homological Lagrangian monodromy of Lagrangian tori $T^n(u_1, \ldots, u_n)$ in $D^{2n}(1, \ldots, 1)$ (resp., $S^1_{eq} \times S^1_{eq}$) in the monotone product $S^2 \times S^2$.

3. Non-Hamiltonianly isotopic Lagrangian tori

In [13], Floer, Hofer and Wysocki proved the following theorem.

Theorem 3.1. *Let $0 < u_1, u_2 < 1$ such that $u_1 + u_2 > 1$. Let*

$$\iota : D^4(u_1, u_2) \to D^4(1, 1)$$

be the standard inclusion. Then there does not exist a one-parameter family $\{\psi_t\}_{0\le t\le 1}$ *of symplectic embeddings of* $D^4(u_1,u_2)$ *to* $D^4(1,1)$ *such that*

$$\psi_0=\iota \quad \textit{and} \quad \psi_1=\sigma_{12}\circ\iota,$$

where σ_{12} *is the transposition of the two factors of* $D^4(1,1)$.

In this section, we prove the following theorem.

Theorem 3.2. *Let* u_1 *and* u_2 *be as in Theorem* 3.1. *Then there does not exist a Hamiltonian isotopy* $\{\phi_t\}_{0\le t\le 1}$ *of* $D^4(1,1)$ *such that*

$$\phi_0=\mathrm{id} \quad \textit{and} \quad \phi_1\big(T(u_1,u_2)\big)=T(u_2,u_1).$$

Firstly, we observe that Theorem 3.2 implies Theorem 3.1.

Proof of Theorem 3.2 $\Rightarrow$ *Theorem* 3.1. If necessary, we pick $0<v_1<u_1$ and $0<v_2<u_2$ such that $v_1+v_2>1$ and $v_1\neq v_2$. Without loss of generality, we may assume that $v_1>v_2$. The statement for (v_1,v_2) implies the one for (u_1,u_2). So it is enough to consider the case that $u_1>u_2$. Suppose to the contrary that there exists $\{\psi_t\}$ as in the statement of the theorem. Since $D^4(u_1,u_2)$ is simply connected, the one-parameter family $\{\psi_t\}$ of embeddings is a one-parameter family of *exact* symplectic embeddings. Hence there is a Hamiltonian isotopy φ_t such that $\psi_t=\varphi_t\circ\iota$. Pick $0<s_2<u_2<s_1<u_1$ such that $s_1+s_2>1$. Restricting φ_t to $T^2(s_1,s_2)\subset D^4(1,1)$, we get a contradiction to Theorem 3.2. □

Proof of Theorem 3.2. Without loss of generality, we assume that $u_1>u_2$. Pick a sufficiently small $\epsilon>0$ such that $u_2>1+\epsilon-u_1$. We take an open embedding of $D^4(1,1)$ to $X=S^2(1+\epsilon)\times S^2(2)$ equipped with the product symplectic structure. (Note that $D(1)\subset S^2(1+\epsilon)$, $D(1)\subset S^2(2)$, where $S^2(A)$ denotes the two-sphere with total area A.) Let $\{\phi_t\}$ be a Hamiltonian isotopy of $D^4(1,1)$ such that $\phi_0=\mathrm{id}$ and $\phi_1(T^2(u_1,u_2))=T(u_2,u_1)$. Since the trace of $T^2(u_1,u_2)$ under the isotopy is compact, we can arrange the isotopy such that ϕ_t is the identity outside a suitable compact subset (by cutting off the generating Hamiltonian). Then it extends naturally to a Hamiltonian isotopy of X. It implies that the displacement energies of $T^2(u_1,u_2)$ and $T(u_2,u_1)$ in X are the same; i.e.,

$$e^X\big(T^2(u_1,u_2)\big)=e^X\big(T^2(u_2,u_1)\big).$$

By Theorem 2.3, the displacement energy of a Lagrangian submanifold L is bounded below by the minimal symplectic area of holomorphic discs and spheres. Hence we have

$$e^X\big(T^2(u_1,u_2)\big)\ge 1+\epsilon-u_1, \qquad e^X\big(T^2(u_2,u_1)\big)\ge \min\{u_2,1+\epsilon-u_2\}.$$

For the upper bound for $e^X(T^2(u_1,u_2))$, we use the following lemma.

Lemma 3.3. *Let* C *be a smoothly embedded simple closed curve in* (S^2,ω). *Denote by* D_1 *and* D_2 *the discs bounded by* C. *If* $\int_{D_1}\omega\neq\int_{D_2}\omega$, *then we have*

$$e^{S^2}(C)=\min\left\{\int_{D_1}\omega,\int_{D_2}\omega\right\}.$$

Applying the above lemma, we obtain upper bounds for $e^X(T^2(u_1,u_2))$ and $e^X(T^2(u_2,u_1))$. Combining with the lower bounds obtained just before Lemma 3.3, we conclude that

$$e^X\big(T^2(u_1,u_2)\big) = 1+\epsilon-u_1, \qquad e^X\big(T^2(u_2,u_1)\big) = \min\{u_2, 1+\epsilon-u_2\}.$$

Because $u_1 > u_2$, $u_2 > 1+\epsilon-u_1$, we find that

$$e^X\big(T^2(u_1,u_2)\big) \neq e^X\big(T^2(u_2,u_1)\big),$$

which is a contradiction. □

4. Homological Lagrangian monodromy of Lagrangian tori $T^n(u,\dots,u)$ in $D^{2n}(1,\dots,1)$

Let $L \subset (X,\omega)$ be a Lagrangian submanifold. We call an automorphism f on $H_*(L;\mathbb{Z})$ a homological Lagrangian monodromy of L, if there exists a Hamiltonian diffeomorphism ϕ of (X,ω) such that $\phi(L) = L$ and f is induced by $(\phi|_L)$. The problem of homological Lagrangian monodromy has been studied by Yau [29] and Hu, Lalonde and Leclercq [23]. In this section, we discuss homological Lagrangian monodromies of $T^n(u,\dots,u)$ in $D^{2n}(1,\dots,1)$.

If $u < 1/2$, $L^n(u,\dots,u)$ is contained in the product of the 4-dimensional unit ball $D^4(1)$ and the $(2n-4)$-dimensional polydisc $D^{2n-4}(1,\dots,1)$. There are Hamiltonian diffeomorphisms ϕ on $D^{2n}(1,\dots,1)$ which induces nontrivial homological monodromy on $T^n(u,\dots,u)$. The unitary group $U(2)$ acting on $\mathbb{C}^2$ preserves the unit ball $B^4(1)$. There are elements in $U(2)$ which exchange factors of $T^2(u,u)$. For a path in $U(2)$ from the identity to such a unitary transformation, there is a Hamiltonian $H(t,\cdot)$ on $\mathbb{C}^2$. We cut off the Hamiltonian H near the boundary of the unit ball and obtain a Hamiltonian isotopy $\phi^t_{H'}$ which is the identity near the boundary of the unit ball. In this way, we can find $\phi^1_{H'}$ which does not induce the identity on the homology of $T^2(u,u)$ in $D^4(1,1)$. Then $\phi^1_{H'} \times \mathrm{id}_{D^{2n-4}(1,\dots,1)}$ preserves $T^n(1,\dots,1)$ with nontrivial homological monodromy.

We show that the homological Lagrangian monodromy of $T^n(u,\dots,u)$ in $D^{2n}(1,\dots,1)$ is trivial provided $u \geq 1/2$.

Theorem 4.1. *Suppose that $u > 1/2$. Let ϕ be a Hamiltonian diffeomorphism of $D^{2n}(1,\dots,1)$ such that*

$$\phi\big(T^n(u,\dots,u)\big) = T^n(u,\dots,u).$$

Then ϕ induces the identity on the homology $H_*(T(u,\dots,u);\mathbb{Z})$.

Proof. Let $\mathbf{s} = (s_1,\dots,s_n)$ with sufficiently small s_i. Then

$$L_{\mathbf{s}} = T(u+s_1,\dots,u+s_n)$$

is a Lagrangian torus in $D^{2n}(1,\dots,1)$. The image $L'_{\mathbf{s}} = \phi(L_{\mathbf{s}})$ is also a family of Lagrangian tori. Note that the germ of a versal family of Lagrangian submanifolds at $L = T(u,\dots,u)$ is realized by the germ of $H^1(L;\mathbb{R})$ around the origin 0. Denote by $d\theta_i \in H^1(T^n;\mathbb{R})$ the cohomology class such that

$$\left\langle d\theta_i, S^1_j \right\rangle = 2\pi\delta_{ij},$$

where S^1_j is the cycle represented by

$$\{1\} \times \cdots \times \partial D^2(1) \times \cdots \times \{1\},$$

where $\partial D^2(1)$ is put in the jth factor. We identify $\mathbf{s}$ and $(1/2\pi)\sum_i s_i d\theta_i$.

Hence we find that, if $\mathbf{s}$ is sufficiently close to zero, $L'_{\mathbf{s}}$ is Hamiltonian isotopic to $T((u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s})))$, where

$$\big(u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s})\big) = (u,\dots,u) + \big(\phi^{-1}\big)^*(s_1,\dots,s_n).$$

Since $u > 1/2$, there exists $\delta > 0$ such that

$$u'_i(\mathbf{s}) + u'_j(\mathbf{s}) > 1 \quad \text{for } |\mathbf{s}|_\infty < \delta,\ 1 \le i,j \le n.$$

Here $|\mathbf{s}|_\infty = \max\{s_1,\dots,s_n\}$. We may assume that

$$\delta < \min\{u - 1/2, 1-u\}.$$

We choose a sufficiently small positive real number ϵ such that

$$0 < \epsilon < 2u - 1 - 2\delta$$

and

$$u'_i(\mathbf{s}) + u'_j(\mathbf{s}) > 1 + \epsilon \quad \text{for } |\mathbf{s}|_\infty < \delta, 1 \le i,j \le n.$$

Let

$$X^{(i)} = S^2(2) \times \cdots \times S^2(1+\epsilon) \times \cdots \times S^2(2)$$

with the product symplectic structure. (Note that $S^2(1+\epsilon)$ is put as the ith factor.) We can compute the displacement energy of $L_{\mathbf{s}}$ in $X^{(i)}$ in a similar way as in the proof of Theorem 3.2. For $\mathbf{s}$ with $|\mathbf{s}|_\infty < \delta$, we have

$$e^{X^{(i)}}(L_{\mathbf{s}}) = 1 + \epsilon - (u + s_i).$$

On the other hand, we find that

$$e^{X^{(i)}}\big(T(u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s}))\big) = \min\big\{u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s}), 1+\epsilon - u'_i(\mathbf{s})\big\}.$$

Since ϕ is a Hamiltonian diffeomorphism and $L'_{\mathbf{s}} = \phi(L_{\mathbf{s}})$ are Hamiltonian isotopic to $T(u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s}))$, we find that

$$e^{X^{(i)}}(L_{\mathbf{s}}) = e^{X^{(i)}}\big(T(u'_1(\mathbf{s}),\dots,u'_n(\mathbf{s}))\big).$$

Note that $u > 1/2$, $u'_j(\mathbf{s}) > 1+\epsilon - u'_i(\mathbf{s})$ for $\mathbf{s}$ with $|\mathbf{s}|_\infty < \delta$ and $j = 1,\dots,n$. We find that $1+\epsilon-(u+s_i) = 1+\epsilon-u'_i(\mathbf{s})$, namely, $u'_i(\mathbf{s}) = u + s_i$. Using the identities for $i = 1,\dots,n$, we find that ϕ^* is the identity on $H^1(L;\mathbb{Z})$. Since $H^*(L;\mathbb{Z})$ is generated by $H^1(L;\mathbb{Z})$, ϕ^* is the identity on $H^*(L;\mathbb{Z})$. Therefore, we find that

$$\phi_* : H_*\big(T(u,\dots,u);\mathbb{Z}\big) \to H_*\big(T(u,\dots,u);\mathbb{Z}\big)$$

is the identity. □

The case that $u = 1/2$ can be also done by looking at the homological monodromy of $T(u,\dots,u)$, $u > 1/2$, which is sufficiently close to $1/2$.

Remark 4.2. Hu, Lalonde and Leclercq [23] showed that the homological monodromy of a Hamiltonian loop of Lagrangian submanifolds is trivial if $\pi_2(M, L) = 0$.

5. Homological Lagrangian monodromy of the product of equators in the monotone product of 2-spheres

Let S^2 be the unit sphere in $\mathbb{R}^3$, $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ and ω the standard area form. The rotation around the z-axis is a Hamiltonian S^1-action with the moment map $h(x, y, z) = z$. Denote by S^1_{eq} the equator of the 2-sphere S^2; i.e.,

$$S^1_{eq} = S^2 \cap \{z = 0\}.$$

The total area of S^2 is 4π and S^1_{eq} cut S^2 into two discs of area 2π. In this section, we show the following theorem.

Theorem 5.1. *Let φ of $S^2 \times S^2$ be a Hamiltonian diffeomorphism such that $\varphi(S^1_{eq} \times S^1_{eq}) = S^1_{eq} \times S^1_{eq}$. Then φ induces*

$$\varphi_* = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix},$$

where $\epsilon_1, \epsilon_2 = \pm 1$ on $H_1(S^1_{eq} \times S^1_{eq}; \mathbb{Z}) \cong \mathbb{Z}[S^1_{eq} \times \{pt\}] \oplus \mathbb{Z}[\{pt\} \times S^1_{eq}]$.

Remark 5.2. For each $\varepsilon_i \in \{\pm 1\}$, $i = 1, 2$, by suitable rotations of the first and second factor, it is easy to find a Hamiltonian diffeomorphism φ such that

$$\varphi_*\big([S^1_{eq} \times \{pt\}]\big) = \varepsilon_1 \big[S^1_{eq} \times \{pt\}\big]$$

and

$$\varphi_*\big([\{pt\} \times S^1_{eq}]\big) = \varepsilon_2 \big[\{pt\} \times S^1_{eq}\big].$$

Firstly, we use Floer cohomology with coefficients in local systems to show a constraint for homological Lagrangian monodromy of $S^1_{eq} \times S^1_{eq}$. We work with Floer theory over the universal Novikov field, which is the field of fractions of the universal Novikov ring

$$\Lambda_0 = \left\{ \sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\}.$$

Remark 5.3. The universal Novikov ring was introduced in [14, 15] with $\mathbb{Z}$-grading. In this note, we suppress the variable e and work with $\mathbb{Z}/2\mathbb{Z}$-grading as in [16, 17, 18].

Proposition 5.4. *Let φ be a Hamiltonian diffeomorphism such that*

$$\varphi(S^1_{eq} \times S^1_{eq}) = S^1_{eq} \times S^1_{eq}.$$

Then we have

$$\varphi_* \neq \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}.$$

Proof. Let $\rho : \pi_1(S^1_{eq} \times S^1_{eq}) \to \{\pm 1\}$ be a representation. Denote by $\mathcal{L}_\rho$ the corresponding flat $\mathbb{R}$-bundle on $S^1_{eq} \times S^1_{eq}$. Then we have

$$HF\big((S^1_{eq} \times S^1_{eq}, \mathcal{L}_\rho), (S^1_{eq} \times S^1_{eq}, \mathcal{L}_\rho); \Lambda\big) \cong (\Lambda)^{\oplus 4},$$

in particular, not zero. Denote by $\rho_{(\varepsilon_1,\varepsilon_2)}$ the representation given by

$$\rho_{(\varepsilon_1,\varepsilon_2)}\big([S^1_{eq} \times \{pt\}]\big) = \varepsilon_1 \quad \text{and} \quad \rho_{(\varepsilon_1,\varepsilon_2)}\big([\{pt\} \times S^1_{eq}]\big) = \varepsilon_2.$$

Then we have

$$HF\big((S^1_{eq} \times S^1_{eq}, \mathcal{L}_{\rho_{(+1,-1)}}), (S^1_{eq} \times S^1_{eq}, \mathcal{L}_{\rho_{(-1,+1)}}); \Lambda\big) = 0. \tag{5.1}$$

These computations are also obtained as special cases in [16, 18]. In the context of [16], these local systems $\mathcal{L}_\rho$ correspond to critical points of the potential function of the Lagrangian submanifold $S^1_{eq} \times S^1_{eq} \subset S^2 \times S^2$. Different local systems correspond to different critical points. The left-hand side of (5.1) is well defined, since the critical values of the potential function at critical points corresponding to ρ_{+-} and ρ_{-+} coincide.

On the other hand, since φ is a Hamiltonian diffeomorphism, we have

$$\begin{aligned} &HF\big((S^1_{eq} \times S^1_{eq}, \mathcal{L}_\rho), (S^1_{eq} \times S^1_{eq}, \varphi^*\mathcal{L}_\rho); \Lambda\big) \\ &\qquad \cong HF\big((S^1_{eq} \times S^1_{eq}, \mathcal{L}_\rho), (S^1_{eq} \times S^1_{eq}, \mathcal{L}_\rho); \Lambda\big). \end{aligned}$$

Hence $\rho \circ \varphi_* = \rho$, which implies the conclusion. □

Remark 5.5. During the proof, we used that Floer cohomology with some local system is nonzero. Since Floer cohomology for Lagrangian intersection is invariant under Hamiltonian deformation, we note that $S^1_{eq} \times S^1_{eq}$ is Hamiltonianly nondisplaceable. This fact is already proved in [25, 26, 27] using Floer cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Next, we compute the displacement energy of Lagrangian tori parametrized by a neighborhood of 0 in $H^1(S^1_{eq} \times S^1_{eq}; \mathbb{R})$.

Consider the $S^1 \times S^1$-action on $S^2 \times S^2$ given by the product of the S^1-actions on two factors of $S^2 \times S^2$. Then a moment map is given by

$$\mu : (\mathbf{p}, \mathbf{q}) \in S^2 \times S^2 \mapsto \big(h(\mathbf{p}), h(\mathbf{q})\big) \in \mathbb{R}^2.$$

The image of μ is $[-1, 1] \times [-1, 1]$. Then we identify $S^1_{eq} \times S^1_{eq}$ with $\mu^{-1}(0, 0)$. We set $L_{\mathbf{t}} = \mu^{-1}(t_1, t_2)$.

Proposition 5.6. *The displacement energy of $L_{\mathbf{t}}$ is given by*

$$e^{S^2 \times S^2}(L_{\mathbf{t}}) = \begin{cases} \min\{2\pi(1 - |t_1|), 2\pi(1 - |t_2|)\}, & \mathbf{t} = (t_1, t_2) \neq (0, 0) \\ +\infty, & \mathbf{t} = (t_1, t_2) = (0, 0). \end{cases}$$

Proof. As we mentioned in Remark 5.5, $S^1_{eq} \times S^1_{eq}$ is Hamiltonianly nondisplaceable, i.e., its displacement energy is $+\infty$.

Applying Lemma 3.3 to the first or second factor of $S^2 \times S^2$, we find that the displacement energy $e^{S^2 \times S^2}(L_{\mathbf{t}})$ is at most

$$\min\big\{2\pi(1 - |t_1|), 2\pi(1 - |t_2|)\big\}.$$

The other inequality follows from Theorem 2.3. Let J_0 be a complex structure on S^2, i.e., the Riemann sphere. Consider

$$pr_1^* J_0 \oplus pr_2^* J_0 \quad \text{on } S^2 \times S^2.$$

Here $pr_i : S^2 \times S^2 \to S^2$, $i = 1, 2$, is the projection to the ith factor. The symplectic area of any nonconstant holomorphic sphere is at least 4π. For a nonconstant holomorphic disc

$$w : \left(D^2, \partial D^2\right) \to \left(S^2 \times S^2, S^1_{eq} \times S^1_{eq}\right),$$

either $pr_1 \circ w$ or $pr_2 \circ w$ is a nonconstant holomorphic disc and its symplectic area is at least

$$\min\left\{2\pi(1 - |t_1|), 2\pi(1 - |t_2|)\right\}.$$

Hence we obtain the proposition. □

Proof of Theorem 5.1. We can prove that $\varphi(L_{\mathbf{t}})$ is Hamiltonianly isotopic to $L_{(\varphi^{-1})^*\mathbf{t}}$. Since φ is a symplectomorphism of $S^2 \times S^2$, we have

$$e^{S^2\times S^2}(L_{\mathbf{t}}) = e^{S^2\times S^2}\left(\varphi(L_{\mathbf{t}})\right) = e^{S^2\times S^2}\left(L_{(\varphi^{-1})^*\mathbf{t}}\right). \tag{5.2}$$

Write

$$\mathbf{t}' = (t_1', t_2') = (\varphi^{-1})^*\mathbf{t}.$$

Then $t_1' = at_1 + bt_2$, $t_2' = ct_1 + dt_2$ for some integers a, b, c, d. By Proposition 5.6 and (5.2), we find that either $a = \pm 1$, $d = \pm 1$, $b = c = 0$ or $a = d = 0$, $b = \pm 1$, $c = \pm 1$. (For example, look at the level sets of the function $\mathbf{t} \mapsto e^{S^2\times S^2}(L_{\mathbf{t}})$.)

Proposition 5.4 excludes the second possibility and we obtain the conclusion. □

Remark 5.7. Using [14, Theorem J], see its proof in [20], we can show a generalization of Proposition 5.6.

We consider Hamiltonian circle actions on (S^2, ω) and $(S^2, c\omega)$, $c > 0$, by rotations. Denote by

$$\mu : S^2 \times S^2 \to [-1, 1] \times [-c, c]$$

its moment map. Let

$$L(u_1, u_2) = \mu^{-1}(u_1, u_2)$$

in $X = (S^2, \omega) \times (S^2, c\omega)$, $c > 0$. Then the displacement energy is given by the following proposition.

Proposition 5.8. *The displacement energy of $L(u_1, u_2)$ is given by*

$$e^X\left(L(u_1, u_2)\right) = \begin{cases} \min\{2\pi(1 - |u_1|), 2\pi(c - |u_2|)\}, & u_1 \neq 0, u_2 \neq 0, \\ 2\pi(1 - |u_1|), & u_1 \neq 0, u_2 = 0, \\ 2\pi(c - |u_2|), & u_1 = 0, u_2 \neq 0, \\ +\infty, & (u_1, u_2) = (0, 0). \end{cases} \tag{5.3}$$

When $c > 1$ and $u_1 = 0$, the minimal area of holomorphic discs with boundary on $L(u_1, u_2)$ is 2π. There are two relative homotopy classes of $(D^2, \partial D^2) \to (S^2 \times S^2, L(u_1, u_2))$ represented by such holomorphic discs and their contributions to Floer complex cancel each other. In such a case, [14, Theorem J] gives a better lower bound than Chekanov's theorem.

6. Superheavy Lagrangian torus in the symplectic blowup of symplectically aspherical manifolds

Entov and Polterovich developed the theory of Calabi quasi-morphisms and (partial) symplectic quasi-states in [8, 9, 10]. When the quantum cohomology ring $QH^*(X)$ of (X, ω) is isomorphic as rings to a direct sum of fields, they constructed a quasi-morphism

$$\mu_{\mathbf{e}} : \widetilde{\mathrm{Ham}}(X, \omega) \to \mathbb{R}$$

for each unit $\mathbf{e}$ of a field factor of $QH^*(X)$. Here $\widetilde{\mathrm{Ham}}(X, \omega)$ denotes the universal covering group of the Hamiltonian diffeomorphism group of (X, ω). The Calabi property states that the quasi-morphism $\mu_{\mathbf{e}}$ coincides with the Calabi homomorphism [6] on the subgroup of Hamiltonian diffeomorphisms supported on U, which is displaceable by a Hamiltonian diffeomorphism of (X, ω). Namely, if the support H_t is contained in such a U for $0 \leq t \leq 1$, then the following equality holds:

$$\mu_{\mathbf{e}}\big(\{\phi_H^t\}_{0 \leq t \leq 1}\big) = \int_0^1 \int_X H_t \omega^n \, dt.$$

This is the property which relates Calabi quasi-morphisms and displaceability of subsets in X. As mentioned in [11], it was D. McDuff who manifested that it is enough for Entov–Polterovich's construction of Calabi quasi-morphisms that the quantum cohomology ring is isomorphic as rings to the direct sum of a field F and some ring R, and $\mathbf{e}$ is the unit of a field F.

From a Calabi quasi-morphism $\mu_{\mathbf{e}}$, a symplectic quasi-state $\zeta_{\mathbf{e}}$ is defined by

$$\zeta_{\mathbf{e}}(F) = \frac{\int_X F\omega^n}{\mathrm{Vol}(X, \omega)} - \frac{\mu_{\mathbf{e}}\big(\{\phi_F^t\}_{0 \leq t \leq 1}\big)}{\mathrm{Vol}(X, \omega)},$$

where

$$\mathrm{Vol}(X, \omega) = \int_X \omega^n$$

and $\{\phi_F^t\}$ is the Hamiltonian flow generated by $F \in C^\infty(X)$. It is shown that $\zeta_{\mathbf{e}}$ extends to $C^0(X) \to \mathbb{R}$, which is called a symplectic quasi-state.

Definition 6.1. A subset $S \subset X$ is said to be *superheavy* with respect to (the Calabi quasi-morphism associated with) $\mathbf{e}$ if $\zeta_{\mathbf{e}}(F) = 0$ for F vanishing on S.

If S is superheavy with respect to $\mathbf{e}$, S cannot be displaced by any symplectomorphism ψ, which is symplectically isotopic to the identity [10].

In particular, for such S,

$$\phi_H^1(S) \cap S \neq \emptyset$$

for any Hamiltonian diffeomorphism ϕ_H^1.

Remark 6.2. In this note, $\mathbf{e}$ is the unit of a field factor of $QH^*(X)$. Entov and Polterovich constructed a partial symplectic quasi-state associated with any idempotent of $QH^*(X)$. There are notions of heavyness and superheavyness with respect to (the partial symplectic quasi-state associated with) $\mathbf{e}$. In general, superheavyness implies heavyness, but the latter is weaker than the former. In the case of symplectic quasi-states, these conditions are equivalent.

For a Lagrangian submanifold L in (X, ω), we have the following two sufficient conditions for Hamiltonianly nondisplaceability of L:

(1) the Floer cohomology of L is defined and nonzero;
(2) there exists $\mathbf{e}$, which is the unit of a field factor of the quantum cohomology of (X, ω), such that L is (super)heavy with respect to $\mathbf{e}$.

Theorem 6.6 below gives a relation between these two conditions.

Now we prepare our setup. Let (X, ω) be a closed $2n$-dimensional symplectic manifold such that $\omega(A) = 0$ for any $A \in \pi_2(X)$. We call such a symplectic manifold a symplectically aspherical manifold. Denote by

$$B^{2n}(c) \subset \mathbb{R}^{2n}$$

a round ball of radius $\sqrt{c/\pi}$, i.e., a ball of symplectic capacity c. Suppose that there exists a symplectic embedding $\Psi : B^{2n}(c) \to X$. Then for $c' \in (0, c)$, we have the symplectic blowup $\widehat{X}_{c'}$, which is obtained by the symplectic cutting construction applied to $X \setminus \Phi(B^{2n}(c'))$ along its boundary.

We consider the quantum cohomology ring with coefficients in the universal Novikov field Λ.

Lemma 6.3. *Let (X, ω) be as above. Then the quantum cohomology ring decomposes into the direct sum*

$$QH^*\big(\widehat{X}_{c'}; \Lambda\big) \cong \bigoplus_{k=1}^{n-1} \Lambda \mathbf{e}_k \oplus R,$$

where $\mathbf{e}_i$, $i = 1, \dots, n-1$, are idempotents and R is some ring.

Proof. Let E be the class of the exceptional divisor in $\widehat{X}_{c'}$. Pick any compatible almost complex structure J, which is standard in a tubular neighborhood of E. We denote by

$$\pi : \widehat{X}_{c'} \to X$$

the contraction of E and we set

$$H = \pi^*\big(H^{>0}(X; \Lambda)\big) \subset H^*\big(\widehat{X}_{c'}; \Lambda\big).$$

Note that the quantum cohomology is isomorphic to the ordinary cohomology as Λ-modules. We regard H as a Λ-submodule of $QH^*(\widehat{X}_{c'}; \Lambda)$.

Since X is symplectically aspherical, all nonconstant pseudoholomorphic rational curves are contained in E. Denote by α (resp., p) the Poincaré dual of $[E]$ (resp., the point class $[pt]$). Then we find that the relation of the quantum multiplication between α and p is

$$\alpha^n = (-1)^n\big(-p + \alpha T^{c'}\big), \quad \alpha \cdot p = p \cdot p = 0.$$

We set

$$z = -\exp\left(-\frac{\pi\sqrt{-1}}{n-1}\right)\alpha T^{-\frac{c'}{n-1}}, \qquad w = \exp\left(-\frac{\pi\sqrt{-1}}{n-1}\right)T^{-\frac{nc'}{n-1}}p.$$

Then we have

$$z^n = w + z, \quad z \cdot w = w \cdot w = 0$$

and we find that z, w and the unit 1 of $QH^*(\widehat{X}_{c'};\Lambda)$ generate a subring

$$\Lambda[z,w]/\big(z^n - w - z, wz, w^2\big)$$

of $QH^*(\widehat{X}_{c'};\Lambda)$.

Define

$$\mathbf{e}_k = \frac{1}{n-1}\left(\chi_k([1])w + \sum_{j=1}^{n-1}\chi_k([j])z^j\right) \quad \text{for } k = 1,\dots,n-1,$$

where $\chi_k : \mathbb{Z}/(n-1)\mathbb{Z} \to \mathbb{C}^*$ is the character given by

$$\chi_k([j]) = \exp\big(2jk\pi\sqrt{-1}/(n-1)\big)$$

and

$$\mathbf{e}_n = 1 - \sum_{k=1}^{n-1}\mathbf{e}_k.$$

Using the orthogonality of characters and that 1 is the unit, we find that

$$\mathbf{e}_k \cdot \mathbf{e}_\ell = \delta_{k\ell}\mathbf{e}_k \quad \text{for } k,\ell = 1,\dots,n$$

and $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}, w, \mathbf{e}_n\}$ is linearly independent over Λ. Clearly, we have $w \cdot \mathbf{e}_k = 0$, $k = 1,\dots,n-1$ and $w \cdot \mathbf{e}_n = w$. Set

$$R' = \Lambda w \oplus \Lambda\mathbf{e}_n,$$

which is multiplicatively closed. Note that the dimension of

$$\Lambda[z,w]/\big(z^n - w - z, wz, w^2\big)$$

as a Λ-vector space is $n+1$. We have the following decomposition as a ring:

$$\Lambda[z,w]/\big(z^n - w - z, wz, w^2\big) \cong \bigoplus_{k=1}^{n-1}\Lambda\mathbf{e}_k \oplus R'. \tag{6.1}$$

In particular, $\bigoplus_{k=1}^{n-1}\Lambda\mathbf{e}_k$ is the direct sum of $(n-1)$ copies of the ground field Λ as a ring. Recall that

$$w = \exp\big(-\pi\sqrt{-1}/(n-1)\big)\,T^{-nc'/(n-1)}p$$

and p is the Poincaré dual of the point class. Thus we have $w \in H$.

Define

$$R = \Lambda\mathbf{e}_n \oplus H,$$

which contains R'. Since $\dim_\Lambda QH^*(\widehat{X}_{c'};\Lambda) = \dim_\Lambda H^*(X) + (n-1)$ and $R \cap \bigoplus_{k=1}^{n-1} \Lambda \mathbf{e}_k = 0$, we obtain the direct sum decomposition

$$QH^*\big(\widehat{X}_{c'};\Lambda\big) \cong \bigoplus_{k=1}^{n-1} \Lambda \mathbf{e}_k \oplus R$$

as Λ-vector spaces. Since X is symplectically aspherical, we have $\mathbf{e}_k \cdot x = 0$ for any $x \in H = \pi^*(H^{>0}(X))$ for $k = 1, \dots, n-1$ and R is closed under the quantum multiplication. By the definition of $\mathbf{e}_n$, we have that $\mathbf{e}_n$ is an idempotent and that $\mathbf{e}_k \cdot \mathbf{e}_n = 0$ for $k = 1, \dots, n-1$. Combining these observations above, we find that

$$QH^*\big(\widehat{X}_{c'};\Lambda\big) \cong \bigoplus_{k=1}^{n-1} \Lambda \mathbf{e}_k \oplus R$$

as rings. □

Theorem 6.4. *Let (X,ω) be a closed $2n$-dimensional symplectically aspherical manifold. Suppose that there exists a symplectic embedding*

$$\Psi : B^{2n}(c) \to X.$$

For c' with $0 < c' < (n-1)c/n$, the symplectic blowup $\widehat{X}_{c'}$ contains a Lagrangian torus L, which is superheavy with respect to $\mathbf{e}_i$, $i = 1, \dots, n-1$.

For the proof, we recall some results.

Theorem 6.5. *Let $L \subset (M,\omega)$ be a Lagrangian submanifold such that the Floer cohomology $HF((L,\mathcal{L}),(L,\mathcal{L});\Lambda)$ with coefficients in a local system $\mathcal{L}$ is defined. Then there is a ring homomorphism*

$$i_{qm}^* : QH^*(M;\Lambda) \to HF\big((L,\mathcal{L}),(L,\mathcal{L});\Lambda\big).$$

The ring structure on $HF((L,\mathcal{L}),(L,\mathcal{L});\Lambda)$ is induced from the operation $\mathfrak{m}_2$ with the grading shift taken into account [14, 15] (see also [3]). We can use not only local systems but also *weak bounding cochains* as well as *bulk deformations* to deform the Floer complex; see [14, 15]. The construction of i_{qm}^* is due to [14, Theorem 3.8.62]. The assertion that i_{qm}^* is a ring homomorphism is due to [18, Section 2.6] for the toric case and due to [1] in the general case. See [4] for related results.

The following result is a special case of [21, Theorem 18.8]. See [2, 4] for related results.

Theorem 6.6. *Let $\mathbf{e}$ be an idempotent in $QH^*(M;\Lambda)$. If the Floer cohomology $HF((L,\mathcal{L}),(L,\mathcal{L});\Lambda)$ of L with coefficients in a local system $\mathcal{L}$ is defined and $i_{qm}^*(\mathbf{e}) \neq 0$, then L is heavy with respect to $\mathbf{e}$. Moreover, if $\mathbf{e}$ is the unit of a field factor of $QH^*(M;\Lambda)$, then L is superheavy with respect to $\mathbf{e}$.*

Proof of Theorem 6.4. Since $c' < (n-1)c/n$, we find that the Lagrangian torus $T = S^1(c'/n-1) \times \cdots \times S^1(c'/(n-1)) \subset B^{2n}(c)$. Here $S^1(c'/(n-1))$ is the round circle of area $c'/(n-1)$. We find that

$$\Psi(T) \subset X \setminus \Psi\big(B^{2n}(c')\big)$$

and denote by L the Lagrangian torus in $\widehat{X}_{c'}$ corresponding to $\Psi(T)$. Denote by $U_c(E)$ the union of the image of $B^{2n}(c) \setminus B^{2n}(c')$ in $\widehat{X}_{c'}$ and E.

To study the Floer theory of L, we need to know the pseudoholomorphic discs with boundary on L. Since X contains no pseudoholomorphic rational curves, we can show that all pseudoholomorphic discs with boundary on L are contained in $U_c(E)$ using the compactness argument as in [5].

Therefore, the computation reduces to the case of $L \subset U_c(E)$. Note that $U_c(E)$ is considered as a J-convex domain in the one-point blowup $(\widehat{\mathbb{C}^n})_{c'}$ of $\mathbb{C}^n$ such that the symplectic area of a line $\ell \in E \cong \mathbb{C}P^{n-1}$ is c'.

Therefore, we can use the machinery for computing Lagrangian Floer theory of torus fibers in toric manifolds; see, e.g., [16]. In particular, we can define Floer cohomology for such Lagrangian submanifolds. For

$$L = S^1\big(c'/(n-1)\big) \times \cdots \times S^1\big(c'/(n-1)\big) \subset \big(\widehat{\mathbb{C}^n}\big)_{c'},$$

the potential function of L is given by

$$\mathfrak{PO}^L(y_1, \dots, u_n) = \big(y_1 + \cdots + y_n + y_1 \cdots y_n\big)\, T^{c'/(n-1)}.$$

Its critical points are

$$y_1 = \cdots = y_n = \exp\left(\pi\sqrt{-1}/(n-1)\right) \cdot \exp\left(2\pi k\sqrt{-1}/(n-1)\right),$$

$k = 1, \dots, n-1$. Namely, there are $(n-1)$ local systems $\mathcal{L}_k$, $k = 1, \dots, n-1$, such that the Floer cohomology $HF((L, \mathcal{L}_k), (L, \mathcal{L}_k); \Lambda)$ is nonzero.

In order to apply Theorem 6.6 to L and $\mathbf{e}_k$, $k = 1, \dots, n-1$, we need the following proposition.

Proposition 6.7. *Let $L \subset \widehat{X}_{c'}$ be as in Theorem 6.4. For the ring homomorphism $i^*_{qm} : QH^*(\widehat{X}_{c'}; \Lambda) \to HF((L, \mathcal{L}_k), (L, \mathcal{L}_k); \Lambda)$, $k = 1, \dots, n-1$, we obtain*

$$i^*_{qm}(\alpha) = -\exp\left(\pi\sqrt{-1}/(n-1)\right) \exp\left(2\pi k\sqrt{-1}/(n-1)\right) T^{c'/(n-1)} \cdot 1_{(L,\mathcal{L}_k)},$$

where $1_{(L,\mathcal{L}_k)} \in HF((L, \mathcal{L}_k), (L, \mathcal{L}_k); \Lambda)$ is the unit.

Proof. Recall that α is the Poincaré dual of $[E]$. Pick a point q in L, which is a Lagrangian torus fiber in $(\widehat{\mathbb{C}}^n)_{c'}$. There is a unique pseudoholomorphic disc u of Maslov index 2 passing through q such that the algebraic intersection number of u and E is 1. We note the following facts. The symplectic area of u is $c'/(n-1)$. The holonomy of $\mathcal{L}_k$ along the boundary of u is $y_1 \cdots y_n$, which is $-\exp(\pi\sqrt{-1}/(n-1)) \exp(2\pi k\sqrt{-1}/(n-1))$. By definition, $i^*_{qm}(\alpha)$ is determined by these data and we obtain the conclusion. □

Recall that

$$z = -\exp\left(-\pi\sqrt{-1}/(n-1)\right) \alpha T^{-c'/(n-1)}.$$

Hence Proposition 6.7 implies that

$$i^*_{qm}(z) = \exp\left(2\pi k\sqrt{-1}/(n-1)\right) 1_{(L,\mathcal{L}_k)}$$

in $HF((L, \mathcal{L}_k), (L, \mathcal{L}_k); \Lambda)$. By Theorem 6.5,

$$i^*_{qm}(z^j) = \exp\left(2\pi kj\sqrt{-1}/(n-1)\right) 1_{(L,\mathcal{L}_k)} = \chi_k([j]) 1_{(L,\mathcal{L}_k)}.$$

Pick a point pt outside of $U_c(E)$. Since any holomorphic disc with boundary on L is contained in $U_c(E)$, no holomorphic disc with boundary on L passes through pt. Thus we find that $i^*_{qm}(w) = 0$. Therefore, for the local system $\mathcal{L}_k$, $k = 1, \ldots, n-1$, we have

$$i^*_{qm}(\mathbf{e}_\ell) = \begin{cases} 1_{(L,\mathcal{L}_k)} & \text{if } k + \ell \equiv 0 \mod n-1, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 6.6, we find that L is superheavy with respect to $\mathbf{e}_k$, where $k = 1, \ldots, n-1$. □

Remark 6.8. We expect that a similar result holds for symplectically non-uniruled manifold X; i.e., all genus 0 Gromov–Witten invariants involving the point class vanish.

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