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# Relation-theoretic contraction principle

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**Abstract.** In this paper, we present yet another new and novel variant of classical Banach contraction principle on a complete metric space endowed with a binary relation which, under universal relation, reduces to Banach contraction principle. In process, we observe that various kinds of binary relations, such as partial order, preorder, transitive relation, tolerance, strict order, symmetric closure, etc., utilized by earlier authors in several well-known metrical fixed point theorems can be weakened to the extent of an arbitrary binary relation.

**Mathematics Subject Classification.** 47H10, 54H25.

**Keywords.** Complete metric spaces, binary relations, contraction mappings.

## 1. Introduction

The classical Banach contraction principle [1] continues to be an indispensable and effective tool in theory as well as applications within and beyond Mathematics, which guarantees the existence and uniqueness of fixed points of contraction self-mappings defined on complete metric spaces besides offering a constructive procedure to compute the fixed point of the underlying mapping. In the recent past, many authors extended this theorem employing relatively more general contractive mappings on various types of spaces.

In this paper, we extend the classical Banach contraction principle to a complete metric space endowed with a binary relation. In this context, the contraction condition is relatively weaker than usual contraction as it is required to hold only on those elements which are related under the underlying relation rather than the whole space. Particularly, under the universal relation, our result reduces to Banach contraction principle.

## 2. Preliminaries

In this section, to make our exposition self-contained, we present the relevant background material needed to prove our result. In what follows, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ and

$\mathbb{R}$ denote the sets of positive integers, nonnegative integers, rational numbers and real numbers, respectively (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

**Definition 2.1 (See [8]).** Let $X$ be a nonempty set. A subset $\mathcal{R}$ of $X^2$ is called a binary relation on $X$.

Notice that for each pair $x, y \in X$, one of the following conditions holds:

(i) $(x, y) \in \mathcal{R}$; which amounts to saying that "$x$ is $\mathcal{R}$-related to $y$" or "$x$ relates to $y$ under $\mathcal{R}$." Sometimes, we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$;

(ii) $(x, y) \notin \mathcal{R}$; which means that "$x$ is not $\mathcal{R}$-related to $y$" or "$x$ does not relate to $y$ under $\mathcal{R}$."

Trivially, $X^2$ and $\emptyset$ being subsets of $X^2$ are binary relations on $X$, which are respectively called the universal relation (or full relation) and empty relation. Another important relation of this kind is the relation

$$\triangle_X = \{(x, x) : x \in X\},$$

called the identity relation or the diagonal relation on $X$.

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation, but for the sake of simplicity, we write only "binary relation" instead of "nonempty binary relation."

**Definition 2.2.** Let $\mathcal{R}$ be a binary relation defined on a nonempty set $X$ and $x, y \in X$. We say that $x$ and $y$ are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

**Proposition 2.3.** *If $(X, d)$ is a metric space, $\mathcal{R}$ is a binary relation on $X$, $T$ is a self-mapping on $X$ and $\alpha \in [0, 1)$, then the following contractivity conditions are equivalent:*

(I) $d(Tx, Ty) \leq \alpha d(x, y)\ \forall\, x, y \in X$ *with* $(x, y) \in \mathcal{R}$,

(II) $d(Tx, Ty) \leq \alpha d(x, y)\ \forall\, x, y \in X$ *with* $[x, y] \in \mathcal{R}$.

*Proof.* The implication (II)$\Rightarrow$(I) is trivial. Conversely, suppose that (I) holds. Take $x, y \in X$ with $[x, y] \in \mathcal{R}$. If $(x, y) \in \mathcal{R}$, then (II) directly follows from (I). Otherwise, if $(y, x) \in \mathcal{R}$, then using the symmetry of $d$ and (I), we obtain

$$d(Tx, Ty) = d(Ty, Tx) \leq \alpha d(y, x) = \alpha d(x, y).$$

This shows that (I)$\Rightarrow$(II). □

**Definition 2.4 (See [8, 9]).** A binary relation $\mathcal{R}$ defined on a nonempty set $X$ is called

- *reflexive* if $(x, x) \in \mathcal{R}\ \forall\, x \in X$,
- *irreflexive* if $(x, x) \notin \mathcal{R}\ \forall\, x \in X$,
- *symmetric* if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,
- *antisymmetric* if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$,
- *transitive* if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$,
- *complete*, *connected* or *dichotomous* if $[x, y] \in \mathcal{R}\ \forall\, x, y \in X$,
- *weakly complete*, *weakly connected* or *trichotomous* if $[x, y] \in \mathcal{R}$ or $x = y$ $\forall\, x, y \in X$.

**Definition 2.5 (See** [8, 9, 4, 14, 15]**).** A binary relation $\mathcal{R}$ defined on a nonempty set $X$ is called

- *strict order* or *sharp order* if $\mathcal{R}$ is irreflexive and transitive,
- *near-order* if $\mathcal{R}$ is antisymmetric and transitive,
- *pseudo-order* if $\mathcal{R}$ is reflexive and antisymmetric,
- *quasi-order* or *preorder* if $\mathcal{R}$ is reflexive and transitive,
- *partial order* if $\mathcal{R}$ is reflexive, antisymmetric and transitive,
- *simple order* if $\mathcal{R}$ is weakly complete strict order,
- *weak order* if $\mathcal{R}$ is complete preorder,
- *total order*, *linear order* or *chain* if $\mathcal{R}$ is complete partial order,
- *tolerance* if $\mathcal{R}$ is reflexive and symmetric,
- *equivalence* if $\mathcal{R}$ is reflexive, symmetric and transitive.

**Remark 2.6.** Clearly, universal relation $X^2$ defined on a nonempty set $X$ remains a complete equivalence relation.

**Definition 2.7 (See**[8]**).** Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$.

(1) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
(2) The reflexive closure of $\mathcal{R}$, denoted by $\mathcal{R}^{\#}$, is defined to be the set $\mathcal{R} \cup \triangle_X$ (i.e., $\mathcal{R}^{\#} := \mathcal{R} \cup \triangle_X$). Indeed, $\mathcal{R}^{\#}$ is the smallest reflexive relation on $X$ containing $\mathcal{R}$.
(3) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on $X$ containing $\mathcal{R}$.

**Remark 2.8.** If $\preceq$ is a partial order on $X$, then

(a) the associated strict order (often denoted by $\prec$) is defined as
$$x \prec y \Longleftrightarrow x \preceq y \text{ and } x \neq y,$$
(b) the associated dual relation (often denoted by $\succeq$) is defined as
$$x \succeq y \Longleftrightarrow y \preceq x \quad (\text{i.e., } \succeq := \preceq^{-1}),$$
(c) the associated tolerance relation (often denoted by $\prec\succ$) is defined as
$$x \prec\succ y \Longleftrightarrow x \preceq y \text{ or } x \succeq y \quad (\text{i.e., } \prec\succ := \preceq^{s}).$$

**Proposition 2.9.** *For a binary relation $\mathcal{R}$ defined on a nonempty set $X$,*
$$(x, y) \in \mathcal{R}^s \Longleftrightarrow [x, y] \in \mathcal{R}.$$

*Proof.* The observation is straightforward as
$$\begin{aligned}(x, y) \in \mathcal{R}^s &\Longleftrightarrow (x, y) \in \mathcal{R} \cup \mathcal{R}^{-1}\\ &\Longleftrightarrow (x, y) \in \mathcal{R} \text{ or } (x, y) \in \mathcal{R}^{-1}\\ &\Longleftrightarrow (x, y) \in \mathcal{R} \text{ or } (y, x) \in \mathcal{R}\\ &\Longleftrightarrow [x, y] \in \mathcal{R}.\end{aligned}$$
□

**Definition 2.10.** Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0.$$

In the following lines, we extend a weaker version of the notion of $d$-self-closeness of a partial order $\preceq$ (defined by Turinici [16]) to an arbitrary binary relation.

**Definition 2.11.** Let $(X, d)$ be a metric space. A binary relation $\mathcal{R}$ defined on $X$ is called $d$-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and

$$x_n \xrightarrow{d} x,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

The following definition is a variant of the notion of $F$-invariant subset of $X^6$ (for the mapping $F : X^3 \to X$) introduced by Charoensawan [3] and the notion of $F$-closed subset of $X^4$ (for the mapping $F : X^2 \to X$) introduced by Kutbi et al. [7] to the mapping $T : X \to X$.

**Definition 2.12.** Let $X$ be a nonempty set and $T$ a self-mapping on $X$. A binary relation $\mathcal{R}$ defined on $X$ is called $T$-closed if for any $x, y \in X$,

$$(x, y) \in \mathcal{R} \Longrightarrow (Tx, Ty) \in \mathcal{R}.$$

**Proposition 2.13.** *Let $X$, $T$ and $\mathcal{R}$ be the same as in Definition* 2.12. *If $\mathcal{R}$ is $T$-closed, then $\mathcal{R}^s$ is also $T$-closed.*

**Definition 2.14 (See [13]).** Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$. A subset $E$ of $X$ is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

**Definition 2.15 (See [6]).** Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$. For $x, y \in X$, a path of length $k$ (where $k$ is a natural number) in $\mathcal{R}$ from $x$ to $y$ is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_k = y$,
(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i$ $(0 \leq i \leq k-1)$.

Notice that a path of length $k$ involves $k+1$ elements of $X$, although they are not necessarily distinct.

In this paper, we use the following notations:

(i) $F(T) =$ the set of all fixed points of $T$,
(ii) $X(T; \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$,
(iii) $\Upsilon(x, y, \mathcal{R}) :=$ the class of all paths in $\mathcal{R}$ from $x$ to $y$.

## 3. Fixed point theorem

Now, we state and prove our main result, which runs as follows.

**Theorem 3.1.** *Let $(X, d)$ be a complete metric space, $\mathcal{R}$ a binary relation on $X$ and $T$ a self-mapping on $X$. Suppose that the following conditions hold:*

(a) *$X(T; \mathcal{R})$ is nonempty,*
(b) *$\mathcal{R}$ is $T$-closed,*
(c) *either $T$ is continuous or $\mathcal{R}$ is $d$-self-closed,*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \forall\, x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

*Then $T$ has a fixed point. Moreover, if*

(e) *$\Upsilon(x, y, \mathcal{R}^s)$ is nonempty, for each $x, y \in X$,*

*then $T$ has a unique fixed point.*

*Proof.* Let $x_0$ be an arbitrary element of $X(T; \mathcal{R})$. Define the sequence $\{x_n\}$ of Picard iterates, i.e., $x_n = T^n(x_0)$ for all $n \in \mathbb{N}_0$. As $(x_0, Tx_0) \in \mathcal{R}$, using assumption (b), we obtain

$$\big(Tx_0, T^2x_0\big), \big(T^2x_0, T^3x_0\big), \ldots, \big(T^nx_0, T^{n+1}x_0\big), \ldots \in \mathcal{R}$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

Thus the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Applying the contractivity condition (d) to (3.1), we deduce, for all $n \in \mathbb{N}_0$, that

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}),$$

which by induction yields that

$$d(x_{n+1}, x_{n+2}) \leq \alpha^{n+1}\, d(x_0, Tx_0) \quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

Using (3.2) and triangular inequality, for all $n \in \mathbb{N}_0$, $p \in \mathbb{N}$, $p \geq 2$, we have

$$\begin{aligned} d(x_{n+1}, x_{n+p}) &\leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \big(\alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{n+p-1}\big)\, d(x_0, Tx_0) \\ &= \alpha^n\, d\big(x_0, Tx_0\big) \sum_{j=1}^{p-1} \alpha^j \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which implies that the sequence $\{x_n\}$ is Cauchy in $X$. As $(X, d)$ is complete, there exists $x \in X$ such that

$$x_n \xrightarrow{d} x.$$

Now, in lieu of (c), assume that $T$ is continuous, we have

$$x_{n+1} = T(x_n) \xrightarrow{d} T(x).$$

Owing to the uniqueness of limit, we obtain $T(x) = x$, i.e., $x$ is a fixed point of $T$.

Alternately, let us assume that $\mathcal{R}$ is $d$-self-closed. As $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and

$$x_n \xrightarrow{d} x,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with

$$[x_{n_k}, x] \in \mathcal{R} \quad \forall\, k \in \mathbb{N}_0.$$

Using (d), Proposition 2.3, $[x_{n_k}, x] \in \mathcal{R}$ and $x_{n_k} \xrightarrow{d} x$, we obtain

$$d(x_{n_k+1}, Tx) = d(Tx_{n_k}, Tx) \leq \alpha\, d(x_{n_k}, x) \to 0 \quad \text{as } k \to \infty$$

so that $x_{n_k+1} \xrightarrow{d} T(x)$. Again, owing to the uniqueness of limit, we obtain $T(x) = x$ so that $x$ is a fixed point of $T$.

To prove uniqueness, take $x, y \in F(T)$, i.e.,

$$T(x) = x \quad \text{and} \quad T(y) = y. \tag{3.3}$$

By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \ldots, z_k\}$) of some finite length $k$ in $\mathcal{R}^s$ from $x$ to $y$ so that

$$z_0 = x, \quad z_k = y, \quad [z_i, z_{i+1}] \in \mathcal{R} \quad \text{for each } i \ (0 \leq i \leq k-1). \tag{3.4}$$

As $\mathcal{R}$ is $T$-closed, by using Proposition 2.13, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \quad \text{for each } i \ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \tag{3.5}$$

Making use of (3.3), (3.4), (3.5), triangular inequality, assumption (d) and Proposition 2.3, we obtain

$$\begin{aligned} d(x, y) = d(T^n z_0, T^n z_k) &\leq \sum_{i=0}^{k-1} d(T^n z_i, T^n z_{i+1}) \\ &\leq \alpha \sum_{i=0}^{k-1} d(T^{n-1} z_i, T^{n-1} z_{i+1}) \\ &\leq \alpha^2 \sum_{i=0}^{k-1} d(T^{n-2} z_i, T^{n-2} z_{i+1}) \\ &\leq \cdots \leq \alpha^n \sum_{i=0}^{k-1} d(z_i, z_{i+1}) \\ &\to 0 \quad \text{as } n \to \infty \end{aligned}$$

so that $x = y$. Hence $T$ has a unique fixed point. □

If $\mathcal{R}$ is complete or $X$ is $\mathcal{R}^s$-directed, then the following consequence is worth recording.

**Corollary 3.2.** *Theorem* 3.1 *remains true if we replace condition* (e) *by one of the following conditions (besides retaining the rest of the hypotheses):*

($e'$) *$\mathcal{R}$ is complete,*
($e''$) *$X$ is $\mathcal{R}^s$-directed.*

*Proof.* If $(e')$ holds, then for each $x, y \in X$, $[x, y] \in \mathcal{R}$, which amounts to saying that $\{x, y\}$ is a path of length 1 in $\mathcal{R}^s$ from $x$ to $y$ so that $\Upsilon(x, y, \mathcal{R}^s)$ is nonempty. Hence Theorem 3.1 gives rise to the conclusion.

Otherwise, if $(e'')$ holds, then for each $x, y \in X$, there exists $z \in X$ such that $[x, z] \in \mathcal{R}$ and $[y, z] \in \mathcal{R}$ so that $\{x, z, y\}$ is a path of length 2 in $\mathcal{R}^s$ from $x$ to $y$. Hence $\Upsilon(x, y, \mathcal{R}^s)$ is nonempty, for each $x, y \in X$ and again by Theorem 3.1 the conclusion is immediate. □

Now, we consider some special cases, wherein our result deduces several well-known fixed point theorems of the existing literature.

(1) Under the universal relation (i.e., $\mathcal{R} = X^2$), Theorem 3.1 reduces to the classical Banach contraction principle. Clearly, under the universal relation, (a), (b), (c) and (e) trivially hold.
(2) On setting $\mathcal{R} = \preceq$, the partial order in Theorem 3.1, we obtain Theorems 2.1, 2.2 and 2.3 of Nieto and Rodríguez-López [10]. Clearly, assumption (b) (i.e., $\preceq$ is $T$-closed) is equivalent to the increasing property of $T$.
(3) By setting $\mathcal{R} = \succeq$, the dual relation associated with a partial order $\preceq$ in Theorem 3.1, we obtain Theorems 2.4 and 2.5 of Nieto and Rodríguez-López [10]. Clearly, assumption (b) (i.e., $\succeq$ is $T$-closed) is equivalent to the increasing property of $T$.
(4) Particularizing $\mathcal{R}$ by the preorder $\preccurlyeq$ in Theorem 3.1, we obtain Theorem 1 of Turinici [19].
(5) Particularizing $\mathcal{R}$ by the transitive relation $\preccurlyeq$ in Theorem 3.1, we obtain the natural versions of Theorems 2.2 and 2.4 of Ben-El-Mechaiekh [2], which is also indicated in [2, Remark 2.3].
(6) By choosing $\mathcal{R} = \prec\succ$, the tolerance relation associated with a partial order $\preceq$ in Theorem 3.1, we obtain Theorem 2.1 of Turinici [17] and Theorem 2.1 of Turinici [18], which are in fact sharpened versions of the main result of Ran and Reurings [12] and Nieto and Rodríguez-López [11], respectively. For further details, one can consult Turinici [17, 18].
(7) Putting $\mathcal{R} = \prec$, the strict order associated with a partial order $\preceq$ in Theorem 3.1, we obtain a fixed point theorem for a strict increasing mapping, which is a unidimensional variant of coupled fixed point theorem of Ghods et al. [5].
(8) Taking the symmetric closure $\mathcal{R}^s$ of an arbitrary relation $\mathcal{R}$ in Theorem 3.1, we obtain Corollary 2.12 of Samet and Turinici [13]. Notice that assumption (b) (i.e., $\mathcal{R}^s$ is $T$-closed) is equivalent to the comparative property of $T$ and assumption (c) (i.e., $\mathcal{R}^s$ is $d$-self-closed) is equivalent to the regular property of $(X, d, \mathcal{R}^s)$.

Finally, we furnish two illustrative examples in support of Theorem 3.1, which do not satisfy the hypotheses of the previous results [10, 19, 2, 17, 18, 12, 11, 5, 13] but have fixed points.

**Example 3.3.** Let $X = \mathbb{R}$ and $d = |x - y|$, then $(X, d)$ is a complete metric space. Define a binary relation $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : x - y \geq 0,\ x \in \mathbb{Q}\}$ on $X$. Consider a mapping $T : X \to X$ defined by

$$T(x) = 4 + \frac{1}{3}\, x.$$

Clearly, $\mathcal{R}$ is $T$-closed and $T$ is continuous. Now, for $x, y \in X$ with $(x, y) \in \mathcal{R}$, we have

$$d(Tx, Ty) = \left| \left(4 + \frac{1}{3}\, x\right) - \left(4 + \frac{1}{3}\, y\right) \right| = \frac{1}{3}\, |x - y| = \frac{1}{3}\, d(x, y) < \frac{2}{5}\, d(x, y),$$

i.e., $T$ satisfies assumption (d) of Theorem 3.1 for $\alpha = 2/5$. Thus all the conditions (a)–(d) of Theorem 3.1 are satisfied and $T$ has a fixed point in $X$. Moreover, here assumption (e) of Theorem 3.1 also holds and therefore, $T$ has a unique fixed point (namely, $x = 6$).

Notice that the underlying binary relation $\mathcal{R}$ is a near-order. Indeed, $\mathcal{R}$ is nonreflexive, nonirreflexive as well as nonsymmetric and hence it is not a preorder, partial order, strict order or tolerance and also never turns out to be a symmetric closure of any binary relation.

**Example 3.4.** Consider $X = [0, 2]$ equipped with usual metric $d = |x - y|$ so that $(X, d)$ is a complete metric space. Define a binary relation

$$\mathcal{R} = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2)\}$$

on $X$ and the mapping $T : X \to X$ defined by

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } 1 < x \leq 2. \end{cases}$$

Clearly, $\mathcal{R}$ is $T$-closed but $T$ is not continuous. Take an $\mathcal{R}$-preserving sequence $\{x_n\}$ such that

$$x_n \xrightarrow{d} x$$

so that $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. Here one may notice that

$$(x_n, x_{n+1}) \notin \{(0, 2)\}$$

so that

$$(x_n, x_{n+1}) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \quad \forall\, n \in \mathbb{N}_0,$$

which gives rise to $\{x_n\} \subset \{0, 1\}$. As $\{0, 1\}$ is closed, we have $[x_n, x] \in \mathcal{R}$. Therefore, $\mathcal{R}$ is $d$-self-closed. By a routine calculation, one can verify assumption (d) of Theorem 3.1 with $\alpha = 1/2$. Thus all the conditions (a)–(d) of Theorem 3.1 are satisfied and $T$ has a fixed point in $X$ (namely, $x = 0$).

Notice that in Example 3.4, the binary relation $\mathcal{R}$ is not one of the earlier known standard binary relations such as reflexive, irreflexive, symmetric, antisymmetric, transitive, complete and weakly complete.

Here, it is fascinating to point out that corresponding theorems contained in [10, 19, 2, 17, 18, 12, 11, 5, 13] cannot be used in the context of the foregoing examples (i.e., Examples 3.3 and 3.4), which substantiate the utility of Theorem 3.1 over corresponding several noted results. Thus, in all, we have extended all the classical results to an arbitrary binary relation.

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