J. Fixed Point Theory Appl. 17 (2015) 541–555
DOI 10.1007/s11784-015-0233-4
Published online May 15, 2015

Journal of Fixed Point Theory and Applications

Fixed points and Cauchy sequences in semimetric spaces

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Abstract. There have been numerous attempts recently to extend many of the metric standard fixed point theorems to a more general semimetric context. In many instances a weakened form of the triangle inequality is involved and the space is assumed to be complete. Thus Cauchy sequences play a central role. One of the standard tests to determine when a sequence is Cauchy in a metric space (X, d) is the summation criterion: If $\{p_n\} \subset X$ and $\sum_{i=1}^{\infty} d(p_i, p_{i+1}) < \infty$, then $\{p_n\}$ is Cauchy. In this note we examine instances in which this criterion plays a critical role.

Mathematics Subject Classification. 54H25, 47H10.

Keywords. Fixed points, Cauchy sequences, contraction mappings, metric spaces, semimetric spaces, generalized metric spaces, Caristi's theorem.

1. Introduction

It is well known that the question of assigning a "distance" between two points of an abstract set is fundamental in geometry. According to Blumenthal [3, p. 31], this concept has its origins in the late nineteenth century in axiomatic studies of de Tilly [26]. In his 1928 treatise [22], Menger used the term *halbmetrischer Raume, or semimetric space*, to describe the same concept. We begin by summarizing the results of Wilson's seminal paper [29] on semimetric spaces.

Definition 1.1. Let X be a set and let $d : X \times X \to \mathbb{R}^+$ be a mapping satisfying, for each $a, b \in X$, the following axioms:

(I) $d(a, b) = 0$ if and only if $a = b$;
(II) $d(a, b) = d(b, a)$.

Then, d is a *semimetric* on X and the pair (X, d) is called a *semimetric space.*

In such a space, convergence of sequences is defined in the usual way: A sequence $\{x_n\} \subseteq X$ is said to *converge* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$.

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Also a sequence is said to be *Cauchy* (or d-Cauchy) if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$. The space (X, d) is said to be *complete* if every Cauchy sequence has a limit.

With such a broad definition of distance, three problems are immediately obvious:

(i) there is nothing to assure that limits are unique (thus the space need not be Hausdorff);
(ii) a convergent sequence need not be a Cauchy sequence;
(iii) the mapping $d(a, \cdot) : X \to \mathbb{R}$ need not even be continuous.

Therefore, it is unlikely there could be an effective topological theory in such a setting. These three problems are immediately resolved with the introduction of the triangle inequality:

(III) Triangle inequality: *With X and d as in Definition* 1.1 *assume also that for each $a, b, c \in X$,*

$$d(a, b) \leq d(a, c) + d(c, b).$$

Definition 1.2. A pair (X, d) satisfying axioms (I), (II), and (III) is called a *metric space.*[1]

The focus of this note is on classes of spaces which lie strictly between the semimetric and metric spaces.

2. Relaxing the triangle inequality

The object of this note is to discuss fixed point properties in spaces which lie strictly between the semimetric spaces and metric spaces. These are spaces that satisfy relaxed forms of the triangle inequality. The following concept is one example of such a class of spaces. This concept has long history; see Heinonen [14] and in particular the Notes at the end of Chapter 14 of that book. In this connection Heinonen refers to [5, Chapters 5–10].

Definition 2.1. A semimetric space (X, d) is said to be a *quasi-metric space* (or *b-metric space*) if there exists $s \geq 1$ such that for each $x, y, z \in X$,

$$d\,(x, y) \leq s\,[d\,(x, z) + d\,(z, y)]\,. \tag{2.1}$$

The term b-metric space for this concept is due to Czerwik [9, 10]. This relaxation of the triangle inequality is also discussed by Fagin and Stockmeyer [12] who called this new distance measure *nonlinear elastic matching* (NEM). The authors of that paper remarked that this measure has been used, for example, in [8] for trademark shapes and in [21] to measure ice floes. These distance spaces are called relaxed$_t$ metric spaces in [11]. Xia [30] has also used this semimetric distance to study the optimal transport path between probability measures. Peppo [23] used the term *quasi-distance* for this concept.

[1] The term "metric space" for spaces satisfying axioms (I), (II), and (III), is apparently due to Hausdorff [13].

Remark 2.2. Another definition of a quasi-metric is given recently by Alsulami et al. [1]. In the terminology of [1], a quasi-metric is a metric without the assumption of symmetry of the distance. Thus a *quasi-metric* on a set X in the sense of [1] is a function $q : X \times X \to \mathbb{R}^+$ which satisfies, for all $x, y, z \in X$, the following conditions:

(1) $q(x, y) = 0$ if and only if $x = y$;
(2) $q(x, y) \leq q(x, z) + q(z, y)$.

In 2004, Turinici [28] introduced the following notions: Let X be a set and let $e : X \times X \to \mathbb{R}^+$ be a mapping. Then, e is said to be

(i) *pseudometric*, provided it is reflexive: $e(x, x) = 0$ for all $x \in X$;
(ii) *triangular*, provided $e(x, z) \leq e(x, y) + e(y, z)$ for all $x, y, z \in X$;
(iii) *sufficient*, provided $e(x, y) = 0$ implies $x = y$.

In this case, the quasi-metric appearing in [1] is a sufficient triangular pseudometric defined earlier in [28].

A semimetric space (X, d) is said to have the *metric boundedness property* (see [11]) if there exist a metric ρ on X and positive constants c_1 and c_2 such that for each $x, y \in X$,

$$c_1 \rho(x, y) \leq d(x, y) \leq c_2 \rho(x, y).$$

It is almost immediate that the metric boundedness property implies that the semimetric is a quasi-metric space since in this case for each $x, y, z \in X$,

$$\begin{aligned} d(x, y) &\leq c_2 \rho(x, y) \\ &\leq c_2 [\rho(x, z) + \rho(z, y)] \\ &\leq c_2 \left[c_1^{-1} d(x, z) + c_1^{-1} d(z, y)\right] \\ &= c_2 c_1^{-1} [d(x, z) + d(z, y)]. \end{aligned}$$

It is also noted in [11] that while the converse is not true, rather surprisingly the converse is true if one replaces the relaxed triangle inequality (2.1) with the *relaxed polygonal inequality*, which asserts that there is a constant $s \geq 1$ such that for all $n \in \mathbb{N}$ and $x, y, x_1, \ldots, x_{n-1} \in X$,

$$d(x, y) \leq s [d(x, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, y)].$$

Following [11], we call a space which satisfies the relaxed polygonal inequality an *s-relaxed$_p$ metric space.*

Theorem 2.3 (See [11]). *A semimetric space has the metric boundedness property if and only if it is an s-relaxed$_p$ metric space.*

In any quasi-metric space limits are unique. However, it is easy to see that the distance function need not be continuous. In fact if $\{q_n\} \subset X$ and if $\lim_{n\to\infty} q_n = q$, then for any $p \in X$ all that can be said is that

$$s^{-1} d(p, q) \leq \liminf_{n\to\infty} d(p, q_n) \leq \limsup_{n\to\infty} d(p, q_n) \leq s d(p, q).$$

In general,

$$\lim_{n\to\infty} d(p, q_n) = d(p, q) \Longleftrightarrow s = 1.$$

Indeed, open balls in such spaces need not be open sets. This prompts us to suggest a strengthening of the notion of quasi-metric spaces that remedies this defect. The following concept was introduced in [19].

Definition 2.4. A semimetric space (X, d) is said to be a *strong quasi-metric space* (called an *sb-metric space* in [19]) if there exists $s \geq 1$ such that for each $x, y, z \in X$,

$$d(x, y) \leq d(y, z) + sd(x, z). \tag{2.2}$$

The next result shows that strong quasi-metric spaces are precisely those quasi-metric spaces that satisfy condition (2.6) of [30].

Proposition 2.5 (See [19]). *A semimetric space (X, d) is a strong quasi-metric space if and only if there exists $s \geq 1$ such that for each $p, q, r, t \in X$,*

$$|d(p, q) - d(r, t)| \leq s\,[d(p, r) + d(q, t)]. \tag{2.3}$$

Proof. Suppose that (X, d) is a strong quasi-metric space with constant $s \geq 1$. Then there exists $s \geq 1$ such that for all $p, q, r, t \in X$,

$$\begin{aligned} d(p, q) &\leq d(p, r) + sd(q, r) \\ &\leq d(r, t) + sd(p, t) + sd(q, r) \end{aligned}$$

from which

$$d(p, q) - d(t, r) \leq s\,[d(p, t) + d(q, r)].$$

A similar argument shows that

$$d(t, r) - d(p, q) \leq s\,[d(t, p) + d(r, q)];$$

hence

$$|d(p, q) - d(t, r)| \leq s\,[d(p, r) + d(q, t)].$$

Thus a strong quasi-metric space satisfies (2.3). The converse is trivial; merely take $q = t$. □

Proposition 2.6 (See [19]). *If a semimetric space (X, d) is a strong quasi-metric space, then it is an s-relaxed$_p$ metric space.*

Corollary 2.7. *Let $\{p_n\}$ be a sequence in a strong quasi-metric space and suppose that*

$$\sum_{i=1}^{\infty} d(p_i, p_{i+1}) < \infty.$$

Then $\{p_n\}$ is a Cauchy sequence.

We give a slight extension of these facts below.

Definition 2.8 (See [2]). Let (X, d) be a semimetric space. A mapping

$$\Phi : \overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$$

is said to be a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, $\Phi(0,0) = 0$, and for all $x, y, z \in X$,

$$d(x,y) \leq \Phi(d(x,z), d(z,y)).$$

Notice that by taking $z = x$ we conclude that for any triangle function, for $x, y \in X$,

$$d(x,y) \leq \Phi(0, d(x,y)).$$

It is shown in [2] that every semimetric space (X, d) has a *basic triangle function* Φ_d which has the property that if Φ is any other triangle function for d, then $\Phi_d \leq \Phi$. Only those semimetric spaces for which the basic triangle function is continuous at $(0,0)$ are considered in [2]. Such spaces are called *regular.*

A monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *comparison function* if $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \in \mathbb{R}^+$. If (X, d) is a semimetric space, a mapping $T : X \to X$ is called a *φ-contraction* if

$$d(T(x), T(y)) \leq \varphi(d(x,y)) \quad \text{for each } x, y \in X.$$

These concepts are due to Matkowski [20]. The following is the main result of [2].

Theorem 2.9. *If (X, d) is a complete regular semimetric space and φ is a comparison function, then every φ-contraction has a unique fixed point.*

We now introduce a strengthening of a triangle function definition analogous to the strengthening of a quasi-metric space to a strong quasi-metric space.

Definition 2.10. Let (X, d) be a semimetric space and let Φ be a triangle function. We say that Φ is a *strong triangle function for d* if Φ is monotone increasing in each of its variables, $\Phi(0,0) = 0$ and for all $x, y, z, w \in X$,

$$|d(x,y) - d(z,w)| \leq \Phi(d(x,z), d(y,w)). \tag{2.4}$$

Upon taking $z = w$ we immediately see that a strong triangle function for d is a triangle function for d. Also, in view of Proposition 2.5, if $\Phi(u,v) = s(u+v)$, then (X, d) is a strong quasi-metric space. In view of this, the following is an extension of Proposition 2.6.

Proposition 2.11. *If a semimetric space (X, d) has a strong triangle function Φ, then for each $x, y, x_1, \ldots, x_{n-1} \in X$,*

$$d(x,y) \leq \Phi(0, d(x,x_1)) + \Phi(0, d(x_1,x_2)) + \cdots + \Phi(0, d(x_{n-1},y)).$$

Proof. Let $\{p_n\} \subset X$. First we assert that for any $n, j \in \mathbb{N}$,

$$d(p_n, p_{n+j}) \leq d(p_n, p_{n+1}) + \sum_{i=n+1}^{n+j-1} \Phi(0, d(p_i, p_{i+1})). \tag{2.5}$$

The proof is by induction on j. Taking $j = 2$ we have by (2.4)

$$\begin{aligned} d(p_n, p_{n+2}) - d(p_n, p_{n+1}) &\le \Phi(d(p_n, p_n), d(p_{n+2}, p_{n+1})) \\ &= \Phi(0, d(p_{n+2}, p_{n+1})). \end{aligned}$$

Thus

$$d(p_n, p_{n+2}) \le d(p_n, p_{n+1}) + \Phi(0, d(p_{n+2}, p_{n+1})).$$

Assume that for $j \ge 2$,

$$d(p_n, p_{n+j}) \le d(p_n, p_{n+1}) + \sum_{i=n+1}^{n+j-1} \Phi(0, d(p_i, p_{i+1})).$$

Then by (2.4)

$$\begin{aligned} d(p_n, p_{n+j+1}) - d(p_n, p_{n+j}) &\le \Phi(d(p_n, p_n), d(p_{n+j+1}, p_{n+j})) \\ &= \Phi(0, d(p_{n+j+1}, p_{n+j})), \end{aligned}$$

so by the inductive assumption

$$\begin{aligned} d(p_n, p_{n+j+1}) &\le d(p_n, p_{n+j}) + \Phi(0, d(p_{n+j+1}, p_{n+j})) \\ &\le d(p_n, p_{n+1}) + \sum_{i=n+1}^{n+j-1} \Phi(0, d(p_i, p_{i+1})) \\ &\quad + \Phi(0, d(p_{n+j+1}, p_{n+j})) \\ &= d(p_n, p_{n+1}) + \sum_{i=n+1}^{n+j} \Phi(0, d(p_i, p_{i+1})). \end{aligned}$$

This completes the induction. On the other hand, since

$$d(p_n, p_{n+1}) \le \Phi(0, d(p_n, p_{n+1})),$$

inequality (2.5) now implies

$$d(p_n, p_{n+j}) \le \sum_{i=n}^{n+j-1} \Phi(0, d(p_i, p_{i+1})).$$

Taking $p_n = x$, $p_{n+j} = y$, and $x_k = p_{n+k}$ the conclusion follows. □

In view of the above, if

$$\sum_{i=n}^{\infty} \Phi(0, d(p_i, p_{i+1})) < \infty,$$

then $\{p_n\}$ is a Cauchy sequence. The following is Theorem 2.4 of [4]. It is derived from a version of Ekeland's variational principle in b-metric spaces. (Recall that in a semimetric space (X, d), the distance function d is said to be *continuous* if for any sequences $\{p_n\}, \{q_n\} \subseteq X$, $\lim_{n\to\infty} d(p_n, p) = 0$ and $\lim_{n\to\infty} d(q_n, q) = 0$ imply that $\lim_{n\to\infty} d(p_n, q_n) = d(p, q)$.)

Theorem 2.12. *Let (X,d) be a complete b-metric space (with $s>1$) such that the b-metric d is continuous, and let $\psi : X \to \mathbb{R}$ be lower semicontinuous and bounded from below. Suppose that $f : X \to X$ satisfies*

$$d(u,v) + sd(u,f(u)) \geq d(f(u),v) \tag{2.6}$$

and

$$\frac{s^2}{s-1}\, d(u,f(u)) \leq \psi(u) - \psi(f(u)) \tag{2.7}$$

for all $u,v \in X$. Then f has a fixed point.

This quickly yields Caristi's theorem for strong quasi-metric spaces. For convenience we state Caristi's theorem in its original form. (There have, of course, been numerous extensions and generalizations of this result over the years.) (Recall that if M is a metric space, a mapping $\varphi : M \to \mathbb{R}$ is said to be (sequentially) *lower semicontinuous* (l.s.c.) if, given $x \in X$ and a sequence $\{x_n\}$ in M, the conditions $\lim_{n\to\infty} x_n \to x$ and $\lim_{n\to\infty} \varphi(x_n) \to r$ imply that $\varphi(x) \leq r$.)

Theorem 2.13 (See [7]). *Let (X,d) be a complete metric space. Let $f : X \to X$ be a mapping, and $\varphi : X \to \mathbb{R}^+$ a lower semicontinuous function. Suppose that*

$$d(x,f(x)) \leq \varphi(x) - \varphi(f(x)), \quad x \in X. \tag{C}$$

Then f has a fixed point.

Theorem 2.14 (See [19]). *Let (X,d) be a complete strong quasi-metric space (with $s>1$) and let $\varphi : X \to \mathbb{R}$ be lower semicontinuous and bounded from below. Suppose that $f : X \to X$ satisfies*

$$d(x,f(x)) \leq \varphi(x) - \varphi(f(x))$$

for all $x \in X$. Then f has a fixed point.

Proof. Continuity of the distance functions comes from the fact that d is a strong quasi-metric. Also, taking $q=t$ in (2.3) we obtain

$$|d(p,t) - d(r,t)| \leq sd(p,r)$$

for each $p,r,t \in X$. Thus

$$d(r,t) + sd(p,r) \geq d(p,t)$$

for each $p,r,t \in X$, and it follows upon taking $p = f(u)$, $t = v$, and $r = u$, that

$$d(u,v) + sd(u,f(u)) \geq d(f(u),v)$$

for each $u,v \in X$, so (2.6) holds. Finally, taking $\psi = \frac{s^2}{s-1}\varphi$, we obtain (2.7). □

Let (X,d) be a strong quasi-metric space, and let $\mathcal{CB}(X)$ be the collection of all nonempty bounded closed subsets of X. Define the Hausdorff distance H on $\mathcal{CB}(X)$ in the usual way. The following is a generalization of Nadler's set-valued contraction mapping theorem in metric spaces. With the aid of Corollary 2.7, Nadler's original proof of [24] carries over with only minor change. See [19] for the details.

Theorem 2.15. *Let (X, d) be a complete strong quasi-metric space, and let $\mathcal{CB}(X)$ be the collection of all nonempty bounded closed subsets of X endowed with the Hausdorff strong quasi-metric H. Let $k \in (0, 1)$ and suppose that $T : X \to \mathcal{CB}(X)$ satisfies*

$$H(T(x), T(y)) \leq kd(x, y) \tag{2.8}$$

for all $x, y \in X$. Then there exists $x \in X$ such that $x \in T(x)$.

The preceding discussion suggests the following fundamental questions.

Question 1. Does Caristi's theorem hold in a complete regular semimetric space which has a strong triangle function?

Question 2. Does the analogue of Nadler's theorem hold in a complete regular semimetric space which has a strong triangle function?

3. Generalized metric spaces

We now turn to the concept of generalized metric space. This section is a sequel to [17]. In an effort to generalize Banach's contraction mapping principle, which holds in all complete metric spaces, to a broader class of spaces, Branciari [6] conceived of the notion to replace triangle inequality with a weaker assumption he called the quadrilateral inequality. This concept was introduced by Branciari almost fifteen years ago. However, it is by no means clear whether this concept has any utilitarian value aside from its curious intrinsic interest because, as we note below, *generalized metric spaces in the sense of Branciari are invariably metric if all points are limits of nontrivial Cauchy sequences.* Thus generalized metric spaces which are not metric would seem to be somewhat esoteric. He called these spaces "generalized metric spaces." These spaces retain the fundamental notion of distance. However, as we shall see, the quadrilateral inequality, while useful in some sense, ignores the importance of such things as the continuity of the distance function, uniqueness of limits, etc. In fact it has been asserted (see, e.g., [25]) that for an accurate generalization of Banach's fixed point theorem along the lines envisioned by Branciari, one needs the quadrilateral inequality in conjunction with the assumption that the space is Hausdorff.

Throughout this section we shall refer to the following as the Cauchy summation (CS) criterion.

Definition 3.1. A semimetric space (X, d) is said to satisfy the (CS) *criterion* if $\sum_{i=1}^{\infty} d(p_i, p_{i+1}) < \infty$ for $\{p_n\} \subset X$ implies that $\{p_n\}$ is a Cauchy sequence.

Definition 3.2 (See [6]). Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).
Then X is called a *generalized metric space* (g.m.s).

The following observation shows that the quadrilateral inequality implies a weaker but useful form of distance continuity. (This is a special case of [27, Proposition 1].)

Proposition 3.3 (See [17, 19]**).** *Suppose that* $\{q_n\}$ *is a nontrivial (i.e., an infinite) Cauchy sequence in a generalized metric space* X *and suppose that*

$$\lim_{n\to\infty} d(q_n, q) = 0.$$

Then $\lim_{n\to\infty} d(p, q_n) = d(p, q)$ *for all* $p \in X$*. In particular,* $\{q_n\}$ *does not converge to* p *if* $p \neq q$*.*

Proof. We may assume that $p \neq q$. If $q_n = p$ for arbitrarily large n, it must be the case that $p = q$. So we may also assume that $p \neq q_n$ for all n. Also $q_n \neq q$ for infinitely many n; otherwise the result is trivial. So we may assume that $q_n \neq q_m \neq q$ and $q_n \neq q_m \neq p$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Then by the quadrilateral inequality,

$$d(p, q) \leq d(p, q_n) + d(q_n, q_{n+1}) + d(q_{n+1}, q)$$

and

$$d(p, q_n) \leq d(p, q) + d(q, q_{n+1}) + d(q_{n+1}, q_n).$$

Since $\{q_n\}$ is a Cauchy sequence, $\lim_{n\to\infty} d(q_n, q_{n+1}) = 0$. Therefore, letting $n \to \infty$ in the above inequalities,

$$\limsup_{n\to\infty} d(p, q_n) \leq d(p, q) \leq \liminf_{n\to\infty} d(p, q_n).$$ □

The above proposition shows that nontrivial Cauchy sequences in a generalized metric space have unique limits. However, even more can be said. (Cf. [15, Lemma 1.10] and [16, Lemma 3.1].)

Proposition 3.4. *In a generalized metric space, any Cauchy sequence has at most one limit.*

Proof. Let (X, d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X which converges to $x \in X$. Suppose that there exists $y \in X$, $y \neq x$, such that $\lim_{n\to\infty} x_n = y$. In view of Proposition 3.3 it must be the case that $\{x_n\}$ is eventually constant, and since $\{x_n\}$ converges to x, it must be the case that $x_n \equiv x$ for n sufficiently large. However, since $y \neq x$, this contradicts $\lim_{n\to\infty} x_n = y$. □

The following result shows that many complete generalized metric spaces have subspaces that are complete metric spaces.

Theorem 3.5. *Let* (X, d) *be a complete generalized metric space and let*

$$X_C = \{x \in X : x \text{ is the limit of a nontrivial Cauchy sequence in } X\}.$$

Then (X_C, d) *is a complete metric space.*

Proof. Let x, y, and z be three distinct points in X_C and let $\{z_n\}$ be a nontrivial Cauchy sequence converging to z. Since $\{z_n\}$ is infinite we may suppose that for sufficiently large n, $z_n \neq z_{n+1}$ and moreover, $z_n \neq x \neq z_{n+1}$ and $z_n \neq y \neq z_{n+1}$. Then, by the quadrilateral inequality,

$$d(x,y) \leq d(x,z_n) + d(z_n,z_{n+1}) + d(z_{n+1},y).$$

Upon letting $n \to \infty$ and applying Proposition 3.3,

$$d(x,y) \leq d(x,z) + d(z,y).$$

This proves that (X_C, d) is a metric space.

To see that (X_C, d) is complete, let $\{x_n\}$ be a Cauchy sequence in X_C. Then, since (X, d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. If $\{x_n\}$ is infinite, $x \in X_C$ and there is nothing to prove. Otherwise $\{x_n\}$ is finite, and since $\lim_{n\to\infty} x_n = x$, it follows that $x_n = x$ for some $n \in \mathbb{N}$. Since $x_n \in X_C$, again $x \in X_C$. This proves that (X_C, d) is complete. □

Caristi's theorem

The assertion in [17] that Caristi's theorem holds in generalized metric spaces is based, among other things, on the assertion that if $\{p_n\}$ is a sequence in a generalized metric space (X, d), and if $\{p_n\}$ satisfies

$$\sum_{i=1}^{\infty} d(p_i, p_{i+1}) < \infty,$$

then $\{p_n\}$ is a Cauchy sequence. However, the authors of [17] later gave an example in [18] which shows that this assertion is false, and it is also shown in [18] that in fact Caristi's theorem fails in such spaces. It is noteworthy that in the space of the counterexample, there are no nontrivial Cauchy sequences. Here we show that if this assumption holds in a generalized metric space, then not only does Caristi's theorem hold, but in fact, Caristi's theorem almost immediately reduces to its metric counterpart under this assumption.

We now prove the following. The approach given here was alluded to in [17] but not explained in detail. As already mentioned, the significance of generalized metric spaces remains unclear to the authors, and the additional assumption that the space satisfies the (CS) criterion would seem to be a strong further restriction.

Theorem 3.6 (Cf. Caristi [7]). *Let (X, d) be a complete generalized metric space which satisfies the (CS) criterion. Let $f : X \to X$ be a mapping, and let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Suppose that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad x \in X.$$

Then f has a fixed point.

Proof. First observe that if $f^i(x) = f^j(x)$ for some $x \in X$ and $i, j \in \mathbb{N}$, $i > j$, then f has a fixed point. Indeed,

$$\begin{aligned} 0 &= \varphi\left(f^i(x)\right) - \varphi\left(f^j(x)\right) \\ &= \left[\varphi\left(f^i(x)\right) - \varphi\left(f^{i+1}(x)\right)\right] + \left[\varphi\left(f^{i+1}(x)\right) - \varphi\left(f^{i+2}(x)\right)\right] \\ &\quad + \cdots + \left[\varphi\left(f^{j-1}(x)\right) - \varphi\left(f^j(x)\right)\right] \\ &\geq d\left(f^i(x), f^{i+1}(x)\right) + d\left(f^{i+1}(x), f^{i+2}(x)\right) \\ &\quad + \cdots + d\left(f^{j-1}(x), f^j(x)\right). \end{aligned}$$

It follows that each of the terms on the right side of the inequality is 0, so it must be the case that $f^i(x)$ is a fixed point of f.

We therefore assume that $f^i(x) \neq f^j(x)$ for each $x \in X$ and $i, j \in \mathbb{N}$, $i \neq j$.

Now let $x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \varphi(x) - \varphi(f^n(x)) &= \varphi(x) - \varphi(f(x)) + \varphi(f(x)) - \varphi\left(f^2(x)\right) \\ &\quad + \cdots + \varphi\left(f^{n-1}(x)\right) - \varphi(f^n(x)) \\ &\geq d(x, f(x)) + d\left(f(x), f^2(x)\right) \\ &\quad + \cdots + d\left(f^{n-1}(x), f^n(x)\right). \end{aligned}$$

Hence

$$\sum_{i=0}^{n-1} d\left(f^i(x), f^{i+1}(x)\right) \leq \varphi(x) - \varphi(f^n(x)) \leq \varphi(x),$$

so

$$\sum_{i=0}^{\infty} d\left(f^i(x), f^{i+1}(x)\right) < \infty.$$

Since (X, d) satisfies the (CS) criterion, this proves that $\{f^n(x)\}$ is an infinite Cauchy sequence, and hence it converges to some point $F(x) \in X_C$. We assert that for each $x \in X_C$,

$$d(x, F(x)) \leq \varphi(x) - \varphi(F(x)). \tag{3.1}$$

First we show that for each $n \in \mathbb{N}$,

$$d\left(x, f^{2n+1}(x)\right) \leq \varphi(x) - \varphi\left(f^{2n+1}(x)\right). \tag{3.2}$$

The proof is by induction. By the quadrilateral inequality,

$$\begin{aligned} d\left(x, f^3(x)\right) &\leq d(x, f(x)) + d\left(f(x), f^2(x)\right) + d\left(f^2(x), f^3(x)\right) \\ &\leq \varphi(x) - \varphi\left(f^3(x)\right). \end{aligned}$$

Now assume that

$$d\left(x, f^{2n+1}(x)\right) \leq \varphi(x) - \varphi\left(f^{2n+1}(x)\right).$$

Then again by the quadrilateral inequality,

$$\begin{aligned} d\left(x, f^{2n+3}(x)\right) \leq d\left(x, f^{2n+1}(x)\right) &+ d\left(f^{2n+1}(x), f^{2n+2}(x)\right) \\ &+ d\left(f^{2n+2}(x), f^{2n+3}(x)\right) \\ &\leq \varphi(x) - \varphi\left(f^{2n+1}(x)\right) + \varphi\left(f^{2n+1}(x)\right) - \varphi\left(f^{2n+2}(x)\right) \\ &+ \varphi\left(f^{2n+2}(x)\right) - \varphi\left(f^{2n+3}(x)\right) \\ &= \varphi(x) - \varphi\left(f^{2n+3}(x)\right). \end{aligned}$$

This completes the induction and establishes (3.2). Now, using (3.2), Theorem 3.5, and lower semicontinuity of φ, we have

$$\begin{aligned} d(x, F(x)) &= \lim_{n\to\infty} d\left(x, f^{2n+1}(x)\right) \\ &\leq \lim_{n\to\infty} \left[\varphi(x) - \varphi\left(f^{2n+1}(x)\right)\right] \\ &= \varphi(x) - \lim_{n\to\infty} \varphi\left(f^{2n+1}(x)\right) \\ &\leq \varphi(x) - \varphi\left(\lim_{n\to\infty} f^{2n+1}(x)\right) \\ &= \varphi(x) - \varphi(F(x)). \end{aligned}$$

Therefore, $F : X_C \to X_C$ is a mapping of the complete metric space (X_C, d) into itself which satisfies (3.1). In view of this, Caristi's original theorem implies that there exists $x \in X_C$ such that $x = F(x)$. To see that this implies $f(x) = x$ recall that

$$\sum_{i=0}^{n-1} d\left(f^i(x), f^{i+1}(x)\right) \leq \varphi(x) - \varphi(f^n(x)).$$

Letting $n \to \infty$,

$$\begin{aligned} \sum_{i=0}^{\infty} d\left(f^i(x), f^{i+1}(x)\right) &\leq \varphi(x) - \lim_{n\to\infty} \varphi(f^n(x)) \\ &\leq \varphi(x) - \varphi\left(\lim_{n\to\infty} f^n(x)\right) \\ &= \varphi(x) - \varphi(F(x)) \\ &= 0. \end{aligned}$$

This implies that $d(x, f(x)) = 0$, so $x = f(x)$. □

It might be interesting to note that the above observations yield a quick proof of the Banach–Caccioppoli theorem in generalized metric spaces.

Theorem 3.7 (See [6]**).** *Let (X, d) be a complete generalized metric space, $c \in [0, 1)$, and $f : X \to X$ a mapping such that*

$$d(f(x), f(y)) \leq cd(x, y) \quad \text{for each } x, y \in X. \tag{3.3}$$

Then f has a fixed unique point.

Let $x \in X$. As above, we may assume $f^i(x) \neq f^j(x)$ if $i \neq j$. Defining

$$\varphi(x) = \frac{1}{1-c} d\left(x, f\left(x\right)\right),$$

inequality (3.3) reduces to

$$d\left(x, f\left(x\right)\right) \leq \varphi\left(x\right) - \varphi\left(f\left(x\right)\right).$$

We now have

$$\sum_{i=0}^{k-1} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) \leq \varphi\left(x\right) - \varphi\left(f^k\left(x\right)\right).$$

In particular,

$$\sum_{i=0}^{\infty} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) < \infty,$$

and in general,

$$\sum_{i=n}^{n+k-1} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) \leq \varphi\left(f^n\left(x\right)\right) - \varphi\left(f^{n+k}\left(x\right)\right).$$

Thus, given $\varepsilon > 0$, for n sufficiently large,

$$\sum_{i=n}^{\infty} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) < \frac{\varepsilon}{2}$$

and

$$c^n d\left(x, f^2\left(x\right)\right) < \frac{\varepsilon}{2}.$$

In order to show that $\{f^n(x)\}$ is a Cauchy sequence, we need to compare $d(f^n(x), f^{n+k}(x))$ with $\sum_{i=n}^{\infty} d(f^i(x), f^{i+1}(x))$. This can be done by essentially following Branciari's original inductive argument. If k is odd,

$$d\left(f^n\left(x\right), f^{n+k}\left(x\right)\right) \leq \sum_{i=n}^{n+k-1} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) < \frac{\varepsilon}{2}.$$

On the other hand, if $k \geq 1$ is even,

$$\begin{aligned} & d\left(f^n\left(x\right), f^{n+k}\left(x\right)\right) \\ & \quad \leq \sum_{i=n}^{n+k-1} d\left(f^i\left(x\right), f^{i+1}\left(x\right)\right) + d\left(f^{n+k-1}\left(x\right), f^{n+k+1}\left(x\right)\right) \\ & \qquad + d\left(f^{n+k+1}\left(x\right), f^{n+k}\left(x\right)\right) \\ & \quad \leq \frac{\varepsilon}{2} + d\left(f^{n+k-1}\left(x\right), f^{n+k+1}\left(x\right)\right) \\ & \quad \leq \frac{\varepsilon}{2} + c^{n+k-1} d\left(x, f^2\left(x\right)\right) \\ & \quad \leq \frac{\varepsilon}{2} + c^n d\left(x, f^2\left(x\right)\right) \\ & \quad < \varepsilon. \end{aligned}$$

Therefore $\{f^n(x)\}$ is a Cauchy sequence and, since X is complete and f is continuous, there exists $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0 = f(x_0)$. Uniqueness follows in the usual way.

Acknowledgments

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors acknowledge with thanks DSR for financial support. The authors are grateful to the reviewer for his useful comments.

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