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The Ran–Reurings fixed point theorem without partial order: A simple proof

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To Professor Andrzej Granas

Abstract. The purpose of this note is to generalize the celebrated Ran–Reurings fixed point theorem to the setting of a space with a binary relation that is only transitive (and not necessarily a partial order) and a relation-complete metric. The arguments presented here are simple and straightforward. It is also shown that extensions by Rakotch and by Hu and Kirk of Edelstein's generalization of the Banach contraction principle to local contractions on chainable complete metric spaces are derived from the Ran–Reurings theorem.

Mathematics Subject Classification. 47H10.

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1. Preliminaries

In 1961, Edelstein [2] extended the Banach contraction principle by establishing that every uniform local contraction $f : X \to X$ of an ϵ-chainable complete metric space (X, d) has a unique fixed point. In 1962, Rakotch [8] refined Edelstein's result to a local contraction f of a complete metric space containing some rectifiable path (i.e., a path of finite length[1]) joining a given point x_0 to $f(x_0)$.

Recall that a metric space (X, d) is said to be *ϵ-chainable* for some $\epsilon > 0$ if for all $x, y \in X$ there exists a finite sequence $\{u_i\}_{i=0}^{m}$ in X such that

$$x = u_0, \quad u_m = y, \quad d(u_{i-1}, u_i) < \epsilon \quad \text{for all } i = 1, \dots, m.$$

[1]The *length* of a (continuous) path $\gamma : [0, 1] \to X$ is $l(\gamma) := \sup\{L(P) : P \in \mathcal{P}[0, 1]\}$, where $\mathcal{P}[0, 1]$ is the collection of all finite partitions $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0, 1]$, and $L(P) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))$.

It is readily seen that a connected metric space is ϵ-chainable. Thus, if a metric space X is rectifiably path-connected (i.e., any two points in X are joined by a rectifiable path), then it is ϵ-chainable.

A mapping f of a metric space (X,d) onto itself is a *local contraction* (with constant $0<k<1$) at a given point $x\in X$, if there exists $\epsilon_x>0$ such that (see [8])

$$y,z\in B(x,\epsilon_x)\Longrightarrow d\big(f(y),f(z)\big)<kd(y,z).$$

The mapping f is a local contraction on X if it so at every point of X.

Improving on results of Holmes [5], Hu and Kirk [4] established in 1978 a unique fixed point for a local radial contraction f of a complete metric space containing an element x_0 joined to $f(x_0)$ by a rectifiable path. Recall that a self-mapping f of a metric space (X,d) is said to be a *local radial contraction* at x if the weaker condition

$$d(x,y)<\epsilon_x\Longrightarrow d\big(f(x),f(y)\big)<kd(x,y)\quad\text{for } x,y\in X$$

holds (see [5]). The mapping f is a local radial contraction on X if it so at every point of X.

The authors in [4] insightfully noted that for a local radial contraction $f:X\to X$, Rakotch's result readily reduces to the Banach contraction principle applied to the restriction of f on a meaningful subspace of X, namely the set $\tilde{X}$ consisting of those points of X that can be joined from x_0 by a rectifiable path. It turns out that $\tilde{X}$ is kept invariant by f and that f is a contraction for the *path metric*

$$\tilde{d}(x,y)=\inf_{\gamma\in\Gamma(x;y)} l(\gamma)$$

on $\tilde{X}$, where $\Gamma(x;y)$ is the collection of all rectifiable paths joining x to y. As the completeness of (X,d) implies that of $(\tilde{X},\tilde{d})$, the Banach contraction principle thus applies to f on $(\tilde{X},\tilde{d})$, yielding a fixed point for f.

Before going any further let us recall the original result of Ran and Reurings, which can be seen as a combination of the Banach contraction principle and the Tarski's fixed point theorem (see, e.g., [3] for the two celebrated seminal results). A *partial order* on a set X is a binary relation $\preccurlyeq$ that is reflexive, antisymmetric, and transitive; the pair $(X,\preccurlyeq)$ consisting of a set with a partial order is a *poset*.

Theorem 1.1 (See [9]). *Let $(X,\preccurlyeq)$ be a poset where every pair $x,y\in X$ has an upper bound and a lower bound. Furthermore, let d be a metric on X such that (X,d) is a complete metric space. If $f:X\to X$ is a continuous and monotonic (i.e., either order-preserving or order-reversing) mapping such that*

(i) *there exists $0<k<1$ with*

$$d\big(f(x),f(y)\big)\le kd(x,y)\quad\textit{for all } x\preceq y,$$

(ii) *there exists $x_0\in X$ such that x_0 and $f(x_0)$ are comparable.*[2]

[2]Two elements x,y in a poset $(X,\preccurlyeq)$ are said to be *comparable* if either $x\preccurlyeq y$ or $y\preccurlyeq x$.

Then f has a unique fixed point $x^* \in X$ with

$$\lim_{n\to\infty} f^n(x) = x^* \quad \textit{for all } x \in X.$$

Nieto and Rodríguez-López [7] noted in 2005 that the continuity of the mapping f in Theorem 1.1 can be replaced by the following condition.

If a monotonic sequence $\{x_n\}_{n\in\mathbb{N}} \to x^*$ in X, then x_n and x^* are consistently comparable for all $n \in \mathbb{N}$ (i.e., $x_n \preccurlyeq x^*$ for a nondecreasing sequence).

The aim of this note is to extend the Ran–Reurings theorem in [9], but by considering a space X equipped with a merely *transitive* binary relation $\preccurlyeq$ (not a partial order) and with a so-called relation-complete metric d, and in which, suitable comparable pairs can be joined by what we call ϵ-monotonic chains.

This is a significant departure from Ran–Reurings' theorem (Theorem 1.1) and from its extensions by Nieto and Rodríguez-López. Interestingly, we also show that the theorem of Hu and Kirk can easily be derived.

Set-valued formulations of the results below are easily written and left to the reader. Applications of the main theorems will be discussed in a subsequent work.

2. Fixed point for a uniform local contraction on comparable elements

In the remainder of this section, $(X, \preccurlyeq, d)$ is a triple consisting of a set X together with a transitive binary relation $\preccurlyeq$ and a metric d on X. It should be kept in mind that the relation $\preccurlyeq$ *is not necessarily a partial order on X.* Expediency imposes the occasional use of X to designate $(X, \preccurlyeq, d)$ in the absence of any confusion.

We introduce natural concepts of relation-chainability and relation-completeness.

Definition 2.1. (i) Two elements $x, y \in X$ are said to be *comparable* if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

(ii) A mapping $f : X \to X$ is said to be *monotonic* if it is either always *relation-preserving*, i.e., $x \preccurlyeq y \Longrightarrow f(x) \preccurlyeq f(y)$ or always *relation-reversing*, i.e., $x \preccurlyeq y \Longrightarrow f(y) \preccurlyeq f(x)$ for any given $x, y \in X$.

(iii) Analogously, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is *monotonic* if $x_n \preccurlyeq x_{n+1}$ for all n or $x_{n+1} \preccurlyeq x_n$ for all n.

(iv) Two elements $x, y \in X$ are joined by an ϵ-monotonic chain for some $\epsilon > 0$ if there exists a monotonic sequence $\{u_i\}_{i=0}^m$ in X such that

$$x = u_0, \quad u_m = y, \quad d\big(u_{i-1}, u_i\big) < \epsilon \quad \text{for all } i = 1, \ldots, m.$$

(Note that, by transitivity, x and y must be comparable.)

(v) The space $(X, \preccurlyeq, d)$ is said to be ϵ-monotonic chainable for some $\epsilon > 0$, if any two comparable elements $x, y \in X$ are joined by an ϵ-monotonic chain.

(vi) The metric d is monotonic complete if and only if every monotonic Cauchy sequence converges in X.

We start with fixed point results for a uniform local contraction on comparable elements of $(X, \preccurlyeq, d)$ where d is a relation-complete metric.

Theorem 2.2. *Let $(X, \preccurlyeq, d)$ be a triple consisting of a metric space (X, d) and a transitive binary relation $\preccurlyeq$ on X, let $f : X \to X$ be a mapping, and let $\epsilon > 0$ be such that*

(a) *there exists $x_0 \in X$ such that x_0 and $f(x_0)$ are joined by an ϵ-monotonic chain;*
(b) *f is monotonic;*
(c) *if $\lim_{n\to\infty} f^n(x_0) = x^* \in X$, then $f^n(x_0)$ and x^* are comparable (consistent with the monotonicity of f) for all n;*
(d) *there exists $0 < k < 1$ such that for any comparable elements $x, y \in X$, $d(x, y) < \epsilon$ implies $d(f(x), f(y)) \leq kd(x, y)$.*

Then, f has a fixed point $x^* = \lim_{n\to\infty} f^n(x_0)$ provided that the metric d is monotonic complete.

Proof. By hypothesis, there exists a finite sequence $\{u_i\}_0^m$ with

$$d(u_{i-1}, u_i) < \epsilon$$

and, without loss of generality,

$$\begin{aligned} x_0 &= u_0 \preccurlyeq u_1 \preccurlyeq \cdots \preccurlyeq u_m \\ &= f(x_0) \preccurlyeq f(u_1) \preccurlyeq \cdots \preccurlyeq f(u_m) \\ &= f^2(x_0) \preccurlyeq f^2(u_1) \preccurlyeq \cdots . \end{aligned}$$

Thus,

$$\begin{aligned} d\big(x_0, f(x_0)\big) &\leq \sum_{i=1}^{m} d\big(u_{i-1}, u_i\big) < m\epsilon, \\ d\big(f(x_0), f^2(x_0)\big) &\leq \sum_{i=1}^{m} d\big(f(u_{i-1}), f(u_i)\big) \leq k \sum_{i=1}^{m} d(u_{i-1}, u_i) < mk\epsilon, \\ &\vdots \\ d\big(f^n(x_0), f^{n+1}(x_0)\big) &\leq k \sum_{i=1}^{m} d\big(f^{n-1}(u_{i-1}), f^n(u_i)\big) \leq k \sum_{i=1}^{m} d(u_{i-1}, u_i) \\ &< mk^n\epsilon \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Surely, there exists $n_0 \in \mathbb{N}$ such that $0 < mk^{n_0} < 1$. We show that the monotonic sequence $\{x_n = f^{n_0+n}(x_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Indeed, given $n' > n$,

$$\begin{aligned} d(x_n, x_{n'}) &\le d(x_n, x_{n+1}) + \cdots + d(x_{n'-1}, x_{n'}) \\ &\le k^{n_0}\left(k^n + k^{n+1} + \cdots + k^{n'-1}\right) m\epsilon \\ &= k^n\left(1 + k + \cdots + k^{n'-n-1}\right)\epsilon \\ &= k^n\left(\frac{1-k^{n'-n}}{1-k}\right)\epsilon \\ &< \frac{k^n}{1-k}\,\epsilon. \end{aligned}$$

Thus, $d(x_n, x_{n'}) \to 0$ as $n \to \infty$. By monotonic completeness, the sequence $\{x_n\}_1^\infty$ converges to some $x^* \in X$ which, by assumption (c), verifies $x_n \preccurlyeq x^*$ for all n.

We conclude the proof by showing that $x^* = f(x^*)$.

For any $\epsilon' \in (0, \epsilon)$, there exists $n_{\epsilon'} \in \mathbb{N}$ such that

$$d(x_n, x^*) < \frac{\epsilon}{2} \quad \text{for all } n \ge n_{\epsilon'}.$$

For all $n > n_{\epsilon'}$ and since $x_n \preccurlyeq x^*$, it follows that

$$d\big(f(x^*), f(x_{n-1})\big) \le kd(x^*, x_{n-1}).$$

Now, as $x_n = f(x_{n-1})$,

$$\begin{aligned} d\big(f(x^*), x^*\big) &\le d\big(f(x^*), f(x_{n-1})\big) + d(x_n, x^*) \\ &\le kd(x^*, x_{n-1}) + d(x_n, x^*) < k\frac{\epsilon'}{2} + \frac{\epsilon'}{2} < \epsilon'. \end{aligned}$$

As $0 < \epsilon' < \epsilon$ is arbitrary,

$$f(x^*) = x^* = \lim_{n\to\infty} f^{n_0+n}(x_0) = \lim_{n\to\infty} f^n(x_0). \qquad \square$$

Remark 2.3. (1) Clearly, if the monotonic mapping $f : X \to X$ globally contracts comparable elements; i.e., there exists $0 < k < 1$ with

$$d\big(f(x), f(y)\big) \le kd(x, y)$$

for any comparable pair $x, y \in X$, and if there exists $x_0 \in X$ comparable to $f(x_0)$, then, given any $\epsilon > 0$, there exists $u_0 = f^n(x_0)$ with n large such that u_0 and $f(u_0) = f^{n+1}(x_0)$ are comparable, and $d(u_0, f(u_0)) < \epsilon$, i.e., u_0 and $f(u_0)$ are joined by a two-element ϵ-monotonic chain. Theorem 2.2 thus applies to the pair $u_0, f(u_0)$ to immediately obtain Nieto–Rodríguez-López's version of Ran–Reurings' theorem. But here, we again point out that the relation $\preccurlyeq$ is merely transitive and not an order relation.

(2) The existence of a fixed point holds if hypothesis (c) of Theorem 2.2 is replaced by the less general assumption:

(c′) f is sequentially continuous along the sequence $f^n(x_0)$, or more generally if f is monotonic-sequentially continuous, i.e.,

$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n)$$

for any monotonic converging sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.

(3) It is worth mentioning that hypothesis (d) of Theorem 2.2 implies local uniqueness of comparable fixed points; i.e., there are no other fixed points comparable to x^* in the ball $B(x^*, \epsilon)$ (the author is indebted to an anonymous referee for pointing this out).

To secure uniqueness of the fixed point, we require global ϵ-monotonic chainability of the space as well as the existence, for any given pair of elements $x, y \in X$, of a third element $z \in X$ similarly comparable to both x and y (i.e., $z \preccurlyeq x$ and $z \preccurlyeq y$ or $x \preccurlyeq z$ and $y \preccurlyeq z$).

Theorem 2.4. *If $(X, \preccurlyeq, d)$, where $\preccurlyeq$ is a transitive relation and d is a metric, is ϵ-monotonic chainable for some $\epsilon > 0$ and $f : X \to X$ is a mapping satisfying*

(a) *there exists $x_0 \in X$ such that x_0 and $f(x_0)$ are comparable;*
(b) *f is monotonic;*
(c) *if $\lim_{n\to\infty} f^n(x_0) = x^* \in X$, then $f^n(x_0)$ and x^* are comparable (consistent with the monotonicity of f) for all n;*
(d) *there exists $0 < k < 1$ such that if $x, y \in X$ are comparable, $d(x, y) < \epsilon$ implies*
$$d\big(f(x), f(y)\big) \le kd(x, y);$$
(e) *every pair of elements of X admits a third element similarly comparable to both.*

Then, f has a unique fixed point $x^* = \lim_{n\to\infty} f^n(x)$ for any initial point $x \in X$, provided that the metric d is monotonic complete.

Proof. Proceeding along the lines of Ran and Reurings [9], given an arbitrary element $x \in X$, we consider first the case where x and x_0 are comparable, say $x \preccurlyeq x_0$. By hypothesis, x and x_0 can be joined by an ϵ-monotonic chain

$$x = v_0 \preccurlyeq \cdots \preccurlyeq v_p = x_0.$$

Arguing as in the preceding proof, it is easy to see that

$$d\big(f^n(v_{i-1}), f^n(v_i)\big) \le k^n \epsilon$$

for all $n = 0, 1, \dots$ and all $i = 1, \dots, p$. It follows that for any given $\delta > 0$, there exists $n_\delta \in \mathbb{N}$ such that for $n \ge n_\delta$,

$$d\big(f^n(x), f^n(x_0)\big) \le k^n p\,\epsilon < \frac{\delta}{2} \quad \text{and} \quad d\big(f^n(x_0), x^*\big) < \frac{\delta}{2}.$$

Hence,

$$d\big(f^n(x), x^*\big) \le d\big(f^n(x), f^n(x_0)\big) + d\big(f^n(x_0), x^*\big) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

that is,

$$\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(x_0) = x^*.$$

To complete the proof, let $x \in X$ be arbitrary and let $z \in X$ be similarly comparable to both x and x_0, say,

$$z \preccurlyeq x \quad \text{and} \quad z \preccurlyeq x_0.$$

From the first part of the argument, we have

$$\lim_{n\to\infty} f^n(z) = \lim_{n\to\infty} f^n(x_0) = x^*.$$

Also, z and x are joinable by an ϵ-monotonic chain, and as above, for n large enough, $d(f^n(z), f^n(x))$ can be made arbitrarily small. Thus,

$$\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(z) = x^*.$$

This completes the proof. □

Quite interestingly, the theorems of Rakotch [8] and Hu and Kirk [4] can be obtained from Theorem 2.2 (in fact from Ran–Reurings' theorem). The proof makes crucial use of the following key observations.

Proposition 2.5 (See [4]**).** *Let $f : X \to X$ be a local radial contraction with constant $0 < k < 1$ on a metric space (X, d). Then*

$$d\big(f\big(\gamma(0)\big), f\big(\gamma(1)\big)\big) \le kl(\gamma) \quad \textit{and} \quad l\big(f(\gamma)\big) \le kl(\gamma)$$

for any rectifiable path $\gamma : [0, 1] \to X$.

The reader is referred to [4] for the proof.

Corollary 2.6 (See [4]**).** *Let (X, d) be a complete metric space and $f : X \to X$ a local radial contraction with constant $k \in (0, 1)$. Suppose that there exists $x_0 \in X$ such that x_0 and $f(x_0)$ are joined by a rectifiable path. Then f has a fixed point.*

Proof. By hypothesis, there exists a rectifiable path γ_0 joining x_0 to $f(x_0)$. By Proposition 2.5, each path $f^n(\gamma_0)$ has length smaller than $k^n l(\gamma_0)$ and joins the element $f^n(x_0)$ to $f^{n+1}(x_0)$. Let

$$X_0 = \{f^n(x_0)\}_{n=0}^{\infty}$$

(with $f^0(x_0) = x_0$). Obviously,

$$f(X_0) = \{f^n(x_0)\}_{n=1}^{\infty} \subset X_0.$$

Define a total order on X_0 as follows:

$$f^n(x_0) \preccurlyeq f^m(x_0) \Longleftrightarrow n \le m.$$

Clearly, $x_0 \preccurlyeq f(x_0)$ and f is obviously monotonic on X_0. Define a metric d_0 on X_0 as

$$d_0\big(f^n(x_0), f^m(x_0)\big) = d_0\big(f^m(x_0), f^n(x_0)\big) = \sum_{i=n}^{m-1} l\big(f^i(\gamma_0)\big) \quad \text{for } n < m,$$

$$d_0(x, y) = 0 \Longleftrightarrow x = y = f^n(x_0) \quad \text{for some } n \in \{0, 1, 2, \ldots\}.$$

Note that the initial metric d, the path metric[3] $\tilde{d}$, and the metric d_0 verify $d \leq \tilde{d} \leq d_0$ on X_0.

For any given pair $x, y \in X_0$, say $x = f^n(x_0)$ and $y = f^m(x_0)$ with $x \preccurlyeq y$, it follows from Proposition 2.5 that

$$\begin{aligned} d_0\big(f(x), f(y)\big) &= d_0\big(f^{n+1}(x_0), f^{m+1}(x_0)\big) \\ &= \sum_{i=n+1}^{m} l\big(f^i(\gamma_0)\big) \\ &\leq k \sum_{i=n}^{m-1} l\big(f^i(\gamma_0)\big) = kd_0(x,y), \end{aligned}$$

i.e., f is a contraction on X_0 relative to the metric d_0.

Given an arbitrary but fixed $\epsilon > 0$, one may assume, without loss of generality, that

$$d_0\big(f^n(x_0), f^{n+1}(x_0)\big) < \epsilon \quad \text{for } n = 0, 1, 2, \ldots.$$

Indeed, since $k_n \downarrow 0^+$ as $n \to \infty$, there exists a positive integer n_ϵ large enough such that

$$k^n d_0\big(x_0, f(x_0)\big) < \epsilon \quad \text{for all } n \geq n_\epsilon.$$

One could then replace the full sequence of iterates $\{f^n(x_0)\}_{n=0}^{\infty}$ by its tail $\{f^n(x_0)\}_{n \geq n\epsilon}$ which verifies

$$d_0\big(f^n(x_0), f^{n+1}(x_0)\big) \leq k^n d_0\big(x_0, f(x_0)\big) < \epsilon \quad \text{for all } n \geq n_\epsilon,$$

and view $f^{n_\epsilon}(x_0)$ as the initial point instead of x_0. Therefore, every two elements in X_0 can be joined by an ϵ-monotonic chain.

It was established in [4] that if the original metric d is complete on X, then the path metric $\tilde{d}$ is complete on the space $\tilde{X}$ of points joinable from x_0 by a rectifiable path. Naturally, the closure of X_0 for the metric d_0 must also be complete. Indeed, let $\{x_r\}$ be a Cauchy sequence in (X_0, d_0). Since $d \leq d_0$, the sequence $\{x_m\}$ is also Cauchy in (X, d), implying $\lim_{m\to\infty} d(x_m, x^*) = 0$ for some $x^* \in X$.

We establish first that every element x_m of the Cauchy sequence can be joined to x^* by a rectifiable path γ. Indeed, let $\{\epsilon_i\}$ be a sequence of summable positive real numbers, i.e., $\sum_{i=1}^{\infty} \epsilon_i < \infty$. For each i, choose m_i large enough such that

$$l(\gamma_i) = d_0(x_{m_i}, x_{m_{i+1}}) < \epsilon_i,$$

where γ_i is the rectifiable path joining $x_{m_i} = f^{n_{m_i}}(x_0)$ to $x_{m_{i+1}} = f^{n_{m_{i+1}}}(x_0)$ consisting of finite union

$$f^{n_{m_i}}(\gamma_0) \cup \cdots \cup f^{n_{m_i}-1}(\gamma_0).$$

[3]Note that X_0 is a subset of the space $\tilde{X} := \{x \in X; \Gamma(x_0; x) \neq \emptyset\}$ equipped with the path metric $\tilde{d}(x,y) = \inf_{\gamma \in \Gamma(x;y)} l(\gamma)$ defined in [4] and mentioned in the Preliminaries above.

Each path γ_i can be rescaled as a path

$$\gamma : \left[\frac{1}{i+1}, \frac{1}{i}\right] \longrightarrow X.$$

Define a path $\gamma : [0,1] \to X$ by putting

$$\gamma(t) = \gamma_i(t)$$

for $t \in [\frac{1}{i+1}, \frac{1}{i}]$ and $\gamma(0) = x^*$. By construction, the path γ is continuous on $(0,1]$. To ascertain continuity at $t = 0$, let $t_k \downarrow 0^+$. Observe that each t_k is in some interval $[\frac{1}{i+1}, \frac{1}{i}]$ and, for all k large enough,

$$\begin{aligned} d\big(\gamma(t_k), x^*\big) &\leq d\big(\gamma(t_k), x_{m_i}\big) + d(x_{m_i}, x^*) \\ &\leq d(x_{m_i}, x_{m_{i+1}}) + d(x_{m_i}, x^*) \\ &\leq d_0(x_{m_i}, x_{m_{i+1}}) + d(x_{m_i}, x^*) \\ &< \epsilon_i + d(x_{m_i}, x^*). \end{aligned}$$

As $t_k \to 0, i \to \infty, m_i \to \infty$, and $\epsilon_i \to 0$, thus $d(\gamma(t_k), x^*) \to 0$, i.e., $\gamma(t_k) \to \gamma(0)$. It should be noted, in addition, that the continuous path γ joining x^* and x_{m_1} verifies

$$l(\gamma) \leq \sum_{i=1}^{\infty} l(\gamma_i) \leq \sum_{i=1}^{\infty} \epsilon_i < \infty.$$

Now, define $d_0(x_{m_i}, x^*) = l(\gamma|_{[0,\frac{1}{i}]})$ and note that

$$d_0(x_{m_i}, x^*) = l\big(\gamma|_{[0,\frac{1}{i}]}\big) \leq \sum_{j=i}^{\infty} \epsilon_j \to 0 \quad \text{as } i \to \infty.$$

Since $\{x_{m_i}\}$ is a subsequence of the Cauchy sequence $\{x_m\}$ in X_0, it follows that

$$\lim_{m\to\infty} d_0(x_m, x^*) = 0,$$

i.e., $x^* \in \overline{X_0}^{d_0}$, which means that $\overline{X_0}^{d_0}$ is d_0-complete. Let us extend the binary relation $\preccurlyeq$ to $\overline{X_0}^{d_0}$ by putting

$$x \preccurlyeq z \quad \text{for all } x \in X_0 \text{ and all } z \in \overline{X_0}^{d_0} \setminus X_0.$$

To conclude the proof, it remains to note that the mapping f (a d_0-contraction on comparable elements of X_0) naturally extends to a contraction on comparable elements of $\overline{X_0}^{d_0}$, thus verifying all hypotheses of Theorem 2.2 on $\overline{X_0}^{d_0}$. □

Remark 2.7. Of course, it is much simpler to prove Corollary 2.6 by observing that $X_0 = \{f^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, d), hence convergent to a fixed point of f. Indeed, for all $m > n \geq n_\epsilon$, we do have

$$\begin{aligned} d\big(f^m(x_0), f^n(x_0)\big) &\leq \sum_{i=n}^{m-1} d\big(f^{i+1}(x_0), f^i(x_0)\big) \\ &\leq \sum_{i=n}^{m-1} k^{i+1}\, l(\gamma_0) \\ &= k^{n+1}\big(1 + \cdots + k^{m-(n+1)}\big)\, l(\gamma_0) \\ &= k^{n+1}\left(\frac{1-k^{m-n}}{1-k}\right) l(\gamma_0) \\ &< \frac{k^n}{1-k}\, l(\gamma_0) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

But our point here is to show that the results of Hu and Kirk and Rakotch follow also from Theorem 2.2 (and indeed from the Ran–Reurings theorem).

Acknowledgments

The author is indebted to Mohamed A. Khamsi for bringing to his attention the remarkable paper of Jacek Jachymski [6] where the Ran–Reurings fixed point theorem is significantly extended to complete metric spaces endowed with a directed graph, as well as to reference [1] for the extension of Caristi's fixed point theorem to such spaces. The use of the language and concepts from graph theory in [6] allows for the unification of the main results in [2, 4, 7, 8, 9] as well as for the consideration of a quasi-order (a reflexive and transitive) relation instead of a partial order. Theorem 2.2 is to be compared with Theorem 3.4 in [6]; it does not require the completeness of the metric over the whole space X but rather over chains. In addition, while keeping the Ran and Reurings' perspective of a compatibility between a metric and a (merely transitive) binary relation, the arguments used here are simple and straightforward.

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