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A thin-film limit in the Landau–Lifshitz–Gilbert equation relevant for the formation of Néel walls

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To our Professor Haïm Brezis on his 70th anniversary with esteem

Abstract. We consider an asymptotic regime for two-dimensional ferromagnetic films that is consistent with the formation of transition layers, called Néel walls. We first establish compactness of $\mathbb{S}^2$-valued magnetizations in the energetic regime of Néel walls and characterize the set of accumulation points. We then prove that Néel walls are asymptotically the unique energy-minimizing configurations. We finally study the corresponding dynamical issues, namely the compactness properties of the magnetizations under the flow of the Landau–Lifshitz–Gilbert equation.

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1. Introduction and main results

The purpose of this paper is to study an asymptotic regime for two-dimensional ferromagnetic thin films allowing for the occurrence and persistence of special transition layers called Néel walls. We prove compactness, optimality and energy concentration of Néel walls, together with dynamical properties driven by the Landau–Lifshitz–Gilbert equation.

1.1. A two-dimensional model for thin-film micromagnetics

We focus on the following two-dimensional model for thin ferromagnetic films. For that, let

$$\Omega = \mathbb{R} \times \mathbb{T} \quad \text{with} \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

be a two-dimensional horizontal section of a magnetic sample that is infinite in x_1-direction and periodic in x_2-direction. The admissible magnetizations are vector fields

$$m = (m', m_3) : \Omega \to \mathbb{S}^2, \quad m' = (m_1, m_2),$$

that are periodic in x_2-direction (this condition is imposed in order to rule out lateral surface charges) and connect two mesoscopic directions forming an angle, i.e., for a fixed $m_{1,\infty} \in [0, 1)$,

$$m(x_1, x_2) = \begin{cases} m_{-\infty} & \text{for } x_1 \le -1, \\ m_{+\infty} & \text{for } x_1 \ge 1, \end{cases} \tag{1.1}$$

where

$$m_{\pm\infty} = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1 - {m_{1,\infty}}^2} \\ 0 \end{pmatrix}.$$

We consider the following micromagnetic energy approximation in a thin-film regime that is written in the absence of crystalline anisotropy and external magnetic fields (see, e.g., [5, 14]):

$$E_\delta(m) = \int_\Omega \left(|\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2 \right) dx + \frac{1}{\delta} \int_{\Omega\times\mathbb{R}} |h(m')|^2 \, dx\, dz, \tag{1.2}$$

where $\delta > 0$ and $\varepsilon = \varepsilon(\delta) > 0$ are two small parameters. The first term in (1.2) is called the exchange energy, while the other two terms stand for the stray field energy created by the surface charges m_3 at the top and bottom of the sample and by the volume charges $\nabla \cdot m'$ in the interior of the sample. More precisely, the stray field $h(m') : \Omega \times \mathbb{R} \to \mathbb{R}^3$ generated only by the volume charges is defined as the unique $L^2(\Omega \times \mathbb{R}, \mathbb{R}^3)$-gradient field

$$h(m') = \left(\nabla, \frac{\partial}{\partial z} \right) U(m')$$

that is x_2-periodic and is determined by static Maxwell's equation in the weak sense:[1] For all $\zeta \in C_c^\infty(\Omega \times \mathbb{R})$,

$$\int_{\Omega\times\mathbb{R}} \left(\nabla, \frac{\partial}{\partial z} \right) U(m') \cdot \left(\nabla, \frac{\partial}{\partial z} \right) \zeta \, dx\, dz = \int_\Omega m' \cdot \nabla\zeta \, dx. \tag{1.3}$$

By explicitly solving (1.3) using[2] the Fourier transform $\mathcal{F}(\cdot)$, the stray-field

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[1] In other words, $h(m')$ is the Helmholtz projection of the vector measure $m'\mathcal{H}^2 \llcorner \Omega \times \{0\}$ onto the $L^2(\Omega \times \mathbb{R})$ space of gradient fields.

[2] Given a function $\zeta : \Omega \to \mathbb{R}$ which is 1-periodic in x_2, we introduce the combination of Fourier transformation in x_1 and Fourier series in x_2 by $\mathcal{F}(\zeta)(\xi) = \frac{1}{\sqrt{2\pi}} \int_\Omega e^{-i\xi\cdot x} \zeta(x)\, dx$, where $\xi \in \mathbb{R} \times 2\pi\mathbb{Z}$.

energy can be equivalently expressed in terms of the homogeneous $\dot{H}^{-1/2}$-norm of $\nabla \cdot m'$ (see, e.g., [10]):[3]

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} |h(m')|^2\, dx\, dz &= \frac{1}{2}\int_{\mathbb{R}\times 2\pi\mathbb{Z}} \frac{1}{|\xi|}|\mathcal{F}(\nabla\cdot m')(\xi)|^2\, d\xi \\ &= \frac{1}{2}\int_{\Omega} \left||\nabla|^{1/2}\mathcal{H}(m')\right|^2 dx,\end{aligned} \tag{1.4}$$

where

$$\mathcal{H}(m') = -\nabla(-\Delta)^{-1}\nabla\cdot m',$$

that is,

$$\mathcal{F}(\mathcal{H}(\cdot))(\xi) = \frac{\xi\otimes\xi}{|\xi|^2}, \quad \xi\in\mathbb{R}\times 2\pi\mathbb{Z}\setminus\{(0,0)\},$$

so that the gradient of the energy $E_\delta(m)$ is given by

$$\nabla E_\delta(m) = -2\Delta m + \left(\frac{1}{\delta}(-\Delta)^{1/2}\,\mathcal{H}(m'), \frac{2m_3}{\varepsilon^2}\right). \tag{1.5}$$

Here and in the following, we denote the planar coordinates by

$$x = (x_1, x_2) \quad \text{and} \quad (x_1, x_2)^\perp = (-x_2, x_1),$$

the vertical coordinate by z and, furthermore, we write

$$\left(\nabla, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

In this model, we expect two types of singular patterns: Néel walls and vortices (the so-called Bloch lines in micromagnetic jargon). These patterns result from the competition between the different contributions in the total energy $E_\delta(m)$ and the nonconvex constraint $|m| = 1$. We explain these structures in the following and compare their respective energies (for more details, see DeSimone et al. [6]).

Néel walls. The Néel wall is a dominant transition layer in thin ferromagnetic films. It is characterized by a one-dimensional in-plane rotation connecting two directions (1.1) of the magnetization. More precisely, it is a one-dimensional transition $m = (m_1, m_2) : \mathbb{R} \to \mathbb{S}^1$ that minimizes the energy under the boundary constraint (1.1):

$$E_\delta(m) = \int_{\mathbb{R}} \left|\frac{dm}{dx_1}\right|^2 dx_1 + \frac{1}{2\delta}\int_{\mathbb{R}} \left|\left|\frac{d}{dx_1}\right|^{1/2} m_1\right|^2 dx_1.$$

It follows that the minimizer is a two-length scale object: it has a small core with fast varying rotation and two logarithmically decaying tails.[4] As $\delta \to 0$, the scale of the Néel core is given by

$$|x_1| \lesssim w_{\text{core}} = O(\delta)$$

[3] One computes that $\mathcal{F}(U(m')(\cdot,z))(\xi) = -\frac{1}{2|\xi|}e^{-|\xi|\,|z|}\mathcal{F}(\nabla\cdot m')(\xi)$ for $\xi \neq 0$ and $z \in \mathbb{R}$.

[4] In our model, the tails are contained in the system, thanks to the confining mechanism of steric interaction with the sample edges placed at $x_1 = \pm 1$.

(up to a logarithmic scale in δ) while the two logarithmic decaying tails scale as

$$w_{\text{core}} \lesssim |x_1| \lesssim w_{\text{tail}} = O(1).$$

The energetic cost (by unit length) of a Néel wall is given by

$$E_\delta(\text{Néel wall}) = \frac{\pi(1-m_{1,\infty})^2 + o(1)}{2\delta|\log\delta|} \quad \text{as } \delta \to 0$$

(see, e.g., [6, 8]).

Micromagnetic vortex. A vortex point corresponds in our model to a topological singularity at the microscopic level where the magnetization points out-of-plane. The prototype of a vortex configuration is given by a vector field $m : B^2 \to \mathbb{S}^2$ defined in a unit disk $\Omega = B^2$ of a thin film that satisfies

$$\nabla \cdot m' = 0 \text{ in } B^2 \quad \text{and} \quad m'(x) = x^\perp \text{ on } \partial B^2$$

and minimizes the energy (1.2):[5]

$$E_\delta(m) = \int_{B^2} |\nabla m|^2 \, dx + \frac{1}{\varepsilon^2}\int_{B^2} m_3^2 \, dx.$$

Since the magnetization turns in-plane at the boundary of the disk B^2 (so, $\deg(m', \partial\Omega) = 1$), a localized region is created, that is the core of the vortex of size ε, where the magnetization becomes indeed perpendicular to the horizontal plane. Remark that the energy E_δ controls the Ginzburg–Landau energy; i.e.,

$$E_\delta(m) \geq \int_{B^2} \mathbf{e}_\varepsilon(m') \, dx \quad \text{with } \mathbf{e}_\varepsilon(m') = |\nabla m'|^2 + \frac{1}{\varepsilon^2}\left(1 - |m'|^2\right)^2$$

since $|\nabla(m', m_3)|^2 \geq |\nabla m'|^2$ and $m_3^2 \geq m_3^4 = (1-|m'|^2)^2$. Due to the similarity with vortex points in Ginzburg–Landau-type functionals (see the seminal book of Bethuel, Brezis and Hélein [3]), the energetic cost of a micromagnetic vortex is given by

$$E_\delta(\text{Vortex}) = 2\pi|\log\varepsilon| + O(1)$$

(see, e.g., [9]).

Regime. We focus on an energetic regime allowing for Néel walls, but excluding vortices. More precisely, we assume that $\delta \to 0$ and $\varepsilon = \varepsilon(\delta) \to 0$ such that

$$\frac{1}{\delta|\log\delta|} = o\left(|\log\varepsilon|\right) \tag{1.6}$$

and we consider families of magnetization $\{m_\delta\}_{0<\delta<1/2}$ satisfying the energy bound

$$\sup_{\delta\to 0} \delta|\log\delta| E_\delta(m_\delta) < +\infty. \tag{1.7}$$

In particular, (1.6) implies that the size ε of the vortex core is exponentially smaller than the size of the Néel wall core δ, i.e.,

$$\varepsilon = O\left(e^{-\frac{1}{\delta|\log\delta|}}\right).$$

[5] In our model, the parameter $\varepsilon = \varepsilon(\delta) > 0$ is related to δ by (1.6).

Compactness of Néel walls. We first show that the energetic regime (1.7) is indeed favorable for the formation of Néel walls. We start by proving a compactness result for $\mathbb{S}^2$-valued magnetizations in (1.6) and (1.7) that is reminiscent to the compactness results of Ignat and Otto in [12, 13].

Theorem 1.1. *Let $\delta > 0$ and $\varepsilon(\delta) > 0$ be small parameters satisfying the regime* (1.6)*. Let $m_\delta \in H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$ satisfy* (1.1) *and* (1.7)*. Then $\{m_\delta\}_{\delta\to 0}$ is relatively compact in $L^2_{\mathrm{loc}}(\Omega)$ and any limit $m : \Omega \to \mathbb{S}^2$ satisfies the constraints* (1.1) *and*

$$|m'| = 1, \quad m_3 = 0, \quad \nabla \cdot m' = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

The proof of compactness is based on an argument of approximating $\mathbb{S}^2$-valued magnetizations by $\mathbb{S}^1$-valued magnetizations having the same level of energy (see Theorem 2.1). Such an approximation is possible due to our regime (1.6) and (1.7) that excludes existence of topological point defects.

Optimality of the Néel wall. Our second result proves the optimality of the Néel wall, namely that the Néel wall is the unique asymptotic minimizer of E_δ over $\mathbb{S}^2$-valued magnetizations within the boundary condition (1.1). For every magnetization $m : \Omega \to \mathbb{S}^2$, we associate the energy density $\mu_\delta(m)$ as a nonnegative x_2-periodic measure on $\Omega \times \mathbb{R}$ via

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \zeta \, d\mu_\delta(m) \\ := \frac{2}{\pi}\delta|\log\delta| \left(\int_\Omega \zeta(x,0)\left(|\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2 \right) dx + \frac{1}{\delta}\int_{\Omega\times\mathbb{R}} \zeta |h(m')|^2 \, dx\, dz \right)\end{aligned} \tag{1.8}$$

for every $\zeta = \zeta(x,z) \in C_c(\Omega \times \mathbb{R})$. Recall that $h(m')$ denotes the x_2-periodic stray field associated with m' via (1.3). We now show that the straight walls (1.10) are the unique minimizers of E_δ as $\delta \to 0$, in which case the energy density μ_δ is concentrated on a straight line in x_2-direction.

Theorem 1.2. *Let $\delta > 0$ and $\varepsilon(\delta) > 0$ be small parameters satisfying the regime* (1.6)*. Let $m_\delta \in H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$ satisfy* (1.1) *and*

$$\limsup_{\delta\to 0} \delta|\log\delta| E_\delta(m_\delta) \le \frac{\pi}{2}(1 - m_{1,\infty})^2. \tag{1.9}$$

Then there exists a subsequence $\delta_n \to 0$ such that $m_{\delta_n} \to m^*$ in $L^2_{\mathrm{loc}}(\Omega)$, where m^* is a straight wall given by

$$m^*(x_1, x_2) = \begin{cases} m_{-\infty} & \text{for } x_1 < x_1^*, \\ m_{+\infty} & \text{for } x_1 > x_1^*, \end{cases} \quad \text{for some } x_1^* \in [-1, 1]. \tag{1.10}$$

In this case we have the concentration of the measures defined at (1.8) *on the jump line of m^*:*

$$\mu_{\delta_n}(m_{\delta_n}) \rightharpoonup (1 - m_{1,\infty})^2 \, \mathcal{H}^1 \llcorner \{x_1^*\} \times \mathbb{T} \times \{0\} \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega \times \mathbb{R}).$$

The energy bound (1.9) is relevant for Néel walls (see, e.g., [8]). A similar result in the case of $\mathbb{S}^1$-valued magnetizations was previously proved by Ignat and Otto in [12]. Theorem 1.2 represents the extension of that result to the case of $\mathbb{S}^2$-valued magnetizations. An immediate consequence of Theorem 1.2 is the following lower bound of the energy E_δ within the boundary condition (1.1).

Corollary 1.3. *Let $\delta > 0$ and $\varepsilon(\delta) > 0$ be small parameters satisfying the regime* (1.6). *Let $m_\delta \in H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$ satisfy* (1.1). *Then*

$$\liminf_{\delta\to 0} \delta|\log\delta| E_\delta(m_\delta) \geq \frac{\pi}{2}(1 - m_{1,\infty})^2. \tag{1.11}$$

1.2. Dynamics. The Landau–Lifshitz–Gilbert equation

The dynamics in ferromagnetism is governed by a torque balance which gives rise to a damped gyromagnetic precession of the magnetization around the effective field defined through the micromagnetic energy. The resulting system is the Landau–Lifshitz–Gilbert (LLG) equations which is neither a Hamiltonian system nor a gradient flow.

Let us present the setting of LLG equations. As condition (1.1) is not preserved by the LLG flow, we *impose* the boundary condition (1.1) at each time $t \geq 0$, and we look for solutions of LLG equations in the space domain

$$x \in \omega := (-1, 1) \times \mathbb{T}.$$

In order to define the micromagnetic energy and its gradient on ω, we introduce the functional calculus derived from the Laplace operator on ω with Dirichlet boundary conditions. More precisely, for $f \in H^{-1}(\omega)$, we define $g := (-\Delta)^{-1} f$ as the solution of

$$\begin{cases} -\Delta g = f & \text{in } \omega, \\ g(x_1, x_2) = 0 & \text{on } \partial\omega, \text{ i.e., for } |x_1| = 1, x_2 \in \mathbb{T}. \end{cases} \tag{1.12}$$

Then $(-\Delta)^{-1}$ is a bounded operator $H^{-1}(\omega) \to H^1_0(\omega)$ and a compact self-adjoint operator $L^2(\omega) \to L^2(\omega)$. We can therefore construct a functional calculus based on it and denote as usual

$$|\nabla|^{-2s} := \left[(-\Delta)^{-1}\right]^s \quad \text{for } s = 1/2,\ 1/4.$$

The dynamics of the state of the thin ferromagnetic sample is described by the time-dependent magnetization

$$m = m(t, x) : [0, +\infty) \times \omega \to \mathbb{S}^2$$

that solves the following equation (see [7, 17]):

$$\partial_t m + \alpha m \times \partial_t m + \beta m \times \nabla \tilde{E}_\delta(m) = 0 \quad \text{on } [0, \infty) \times \omega. \tag{LLG$_0$}$$

Here, $\times$ denotes the cross product in $\mathbb{R}^3$, while $\alpha > 0$ is the Gilbert damping factor characterizing the dissipation form of (LLG_0) and $\beta > 0$ is the gyromagnetic ratio characterizing the precession. The micromagnetic energy $\tilde{E}_\delta$ corresponding to the domain ω is defined via (1.12):

$$\tilde{E}_\delta(m) = \int_\omega \left(|\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2 \right) dx + \frac{1}{2\delta} \int_\omega \left| |\nabla|^{-1/2} \nabla \cdot m' \right|^2 dx, \tag{1.13}$$

so that the gradient of the energy $\tilde{E}_\delta(m)$ is given as

$$\nabla \tilde{E}_\delta(m) = -2\Delta m + \left(\frac{1}{\delta} \mathcal{P}(m'), \frac{2m_3}{\varepsilon^2} \right), \tag{1.14}$$

where we have introduced[6] the operator $\mathcal{P}$ acting on $m' \in H^1(\omega, \mathbb{R}^2)$ via (1.12):

$$\mathcal{P}(m') := -\nabla |\nabla|^{-1} \nabla \cdot m'.$$

Observe that as in (1.4), we have

$$\int_\omega \left| |\nabla|^{-1/2} \mathcal{P}(m') \right|^2 dx = \int_\omega \left| |\nabla|^{-1/2} \nabla \cdot m' \right|^2 dx.$$

Remark 1.4. (i) We highlight that Theorems 1.1 and 1.2 remain valid in the context of the micromagnetic energy $\tilde{E}_\delta$ on ω within the boundary conditions (1.1), i.e., $m(x_1, x_2) = m_{\pm\infty}$ for $x_1 = \pm 1$ and every $x_2 \in \mathbb{T}$.

(ii) Note that for a map $m : \omega \to \mathbb{S}^2$, one has $\tilde{E}_\delta(m) < \infty$ if and only if $m \in H^1(\omega)$.

In this paper, we consider a more general form of the LLG equation including additional drift terms, which has been derived in a related setting in [21, 22] (see also [15]):

$$\begin{aligned} \partial_t m + \alpha m \times \partial_t m + \beta m \times \nabla \tilde{E}_\delta(m) + (v \cdot \nabla) m \\ = m \times (v \cdot \nabla) m \quad \text{on } [0, +\infty) \times \omega, \end{aligned} \tag{LLG}$$

where $v : [0, +\infty) \times \omega \to \mathbb{R}^2$ represents the direction of an applied spin-polarized current.[7]

Regime. We analyze the dynamics of the magnetization through (LLG) in the asymptotics $\delta \to 0$, $\varepsilon(\delta) \to 0$ in the regime (1.6), while

$$\alpha = \nu\varepsilon, \quad \beta = \lambda\varepsilon, \tag{1.15}$$

where $\nu > 0$ is kept fixed and

$$\lambda(\delta) = o\left(\sqrt{\delta |\log \delta|} \right). \tag{1.16}$$

The dynamics of the magnetization for equation (LLG_0) has been derived by Capella, Melcher and Otto [4] (see also Melcher [18]) in the asymptotics $\varepsilon \to 0$ with *fixed* δ (see [4, Theorem 1]). The more general equation (LLG) (*in the absence of the nonlocal energy term*) was studied by

[6]Observe that our original nonlocal operator appearing in the energy gradient (1.5) can be written as $(-\Delta)^{1/2} \mathcal{H}(m') = -\nabla |\nabla|^{-1} \nabla \cdot m'$.

[7]By definition $(v \cdot \nabla) m = v_1 \partial_1 m + v_2 \partial_2 m$.

Kurzke, Melcher and Moser in [15] where they derived rigorously the motion law of point vortices in a different regime, namely $\varepsilon \to 0$ and $\delta = +\infty$. We highlight that in those papers, the parameter $\delta > 0$ is kept *fixed or large* yielding a uniform H^1 bound via the energy; it is far beyond the grasp of (1.6). Therefore, in the analysis developed below, we will have to deal with the loss of the uniform H^1 bound; our strategy relies on the fine qualitative behavior of the magnetization presented in Theorems 1.1 and 1.2 (that remain valid in the context of the micromagnetic energy $\tilde{E}_\delta$ on ω within the boundary condition (1.1)).

In the present paper we consider initial data with finite energy at $\delta > 0$ fixed.[8] We first have to solve the corresponding Cauchy problem for (LLG) imposing the boundary condition (1.1) at each time $t \geq 0$. Naturally, we understand that here the boundary condition (1.1) reads as

$$m(t, x_1, x_2) = m_{\pm\infty} \quad \text{for } x_1 = \pm 1,\, x_2 \in \mathbb{T},\, t \geq 0.$$

Moreover, these solutions have finite energy for all time $t \geq 0$. We insist on the fact that the energy can possibly increase in time, unlike for (LLG_0) which is dissipative.

Definition 1.5. We say that m is a global weak solution to (LLG) if

$$m \in L^\infty_{\mathrm{loc}}\big([0,+\infty), H^1\big(\omega, \mathbb{S}^2\big)\big) \cap \dot{H}^1_{\mathrm{loc}}\big([0,+\infty), L^2(\omega)\big), \tag{1.17}$$

and m solves equation (LLG) in the distributional sense $\mathcal{D}'((0,+\infty) \times \omega)$.

Observe that the regularity assumption (1.17) of this definition allows to make all terms in (LLG) meaningful in the distributional sense, this gives its relevance to the definition.

Indeed, (1.17) first gives (due to Remark 1.4 (ii)) that $\tilde{E}_\delta(m(t))$ is finite for all $t \geq 0$. Also, $\nabla \tilde{E}_\delta(m) \in L^\infty_{\mathrm{loc}}([0,+\infty), H^{-1}(\omega))$ since for $\nabla m(t) \in L^2(\omega)$, $\Delta m(t) \in H^{-1}(\omega)$, while $\mathcal{P}(m'(t)) \in L^2(\omega)$. From there, we infer that $m \times \nabla \tilde{E}_\delta(m) \in L^\infty_{\mathrm{loc}}([0,+\infty), H^{-1}(\omega))$. Indeed, by setting

$$\langle m(t) \times \Delta m(t), \phi \rangle_{H^{-1}(\omega), H^1_0(\omega)} := -\sum_{j=1}^{2} \int_\omega \big(m(t) \times \partial_j m(t)\big) \cdot \partial_j \phi \, dx$$

and by noticing that $\mathcal{P}(m')$ and m_3 belong to $L^\infty_{\mathrm{loc}}([0,+\infty), L^2(\omega))$, we get for $0 < \varepsilon \leq \delta$ small (see (4.4)),

$$\|m(t) \times \nabla \tilde{E}_\delta(m(t))\|_{H^{-1}(\omega)} \leq \frac{C}{\varepsilon} \tilde{E}_\delta(m(t))^{1/2}.$$

All the other terms in (LLG) belong to $L^2_{\mathrm{loc}}([0,+\infty) \times \omega)$.

We construct global weak solutions for (LLG) in the following theorem.

Theorem 1.6. *Let $\delta \in (0, 1/2)$ be fixed, $m^0 \in H^1(\omega, \mathbb{S}^2)$ an initial data and the spin current $v \in L^\infty([0,+\infty) \times \omega, \mathbb{R}^2)$.*

[8] Recall that in the regime (1.7) the initial energy blows up in the limit $\delta \to 0$.

Then there exists a global weak solution m *to* (LLG) *(in the sense of Definition* 1.5*), which satisfies the boundary conditions*

$$m(t=0,\cdot) = m^0 \qquad \text{in } \omega, \tag{1.18}$$

$$m(t,x_1,x_2) = m^0(x_1,x_2) \quad \text{if } x_1 = \pm 1 \text{ and for every } x_2 \in \mathbb{T},\ t \geq 0. \tag{1.19}$$

Furthermore, m *satisfies the following energy bound: for all* $t \geq 0$*,*

$$\begin{aligned} &\tilde{E}_\delta(m(t)) + \frac{\alpha}{2\beta}\int_0^t \|\partial_t m(s)\|^2_{L^2(\omega)}\,ds \\ &\quad \leq \tilde{E}_\delta(m^0)\exp\left(\frac{4}{\alpha\beta}\int_0^t \|v(s)\|^2_{L^\infty(\omega)}\,ds\right). \end{aligned} \tag{1.20}$$

The proof of Theorem 1.6 takes its roots in [1] via a space discretization. To the best of our knowledge, however, there is no such result taking into account the nonlocal term in $\nabla\tilde{E}_\delta$ (see (1.14)). One needs to carry on the computations carefully, specially as it comes together with the constraint of $\mathbb{S}^2$-valued maps. For the convenience of the reader we provide a full proof in Section 5 below.

We next specify our set of assumptions for the dynamics in the asymptotics $\delta, \varepsilon(\delta) \to 0$.

(A1) The initial data $m^0_\delta \in H^1(\omega,\mathbb{S}^2)$ satisfy (1.1) and

$$\sup_{\delta\to 0}\delta|\log\delta|\tilde{E}_\delta\big(m^0_\delta\big) < +\infty.$$

(A2) The regime (1.6) holds as $\delta \to 0$ and the parameters α and β satisfy (1.15) and (1.16).

(A3) The spin-polarized current satisfies

$$\|v_\delta\|^2_{L^\infty([0,+\infty)\times\omega)} \leq \alpha\beta. \tag{1.21}$$

In particular, we have $v_\delta \to 0$ in $L^\infty([0,+\infty)\times\omega)$.

Due to the energy estimate (1.20), the energetic regime in (A1) holds for all times $t \geq 0$ (with no uniformity in t though). In particular, Theorem 1.1 implies that for all $t > 0$, the magnetizations $\{m_\delta(t)\}_\delta$ admit a subsequence converging in $L^2(\omega)$ to a limiting magnetization $(m'(t), 0)$ as $\delta \to 0$. Our main result is that the subsequence does not depend on t and that the limiting configuration is stationary.

Theorem 1.7. *Let* $\{m^0_\delta\}_{0<\delta<1/2}$ *be a family of initial data in* $H^1(\omega,\mathbb{S}^2)$*. Suppose that assumptions* (A1)–(A3) *are satisfied. Let* $\{m_\delta\}_{0<\delta<1/2}$ *denote any family of global weak solutions to* (LLG) *satisfying* (1.18)*,* (1.19) *and the energy estimate* (1.20)*.*

Then there exists a subsequence $\delta_n \to 0$ *such that* $m_{\delta_n}(t) \to m(t)$ *in* $L^2(\omega)$ *for all* $t \in [0,+\infty)$ *as* $n \to \infty$ *where the accumulation point*

$$m = (m', 0) \in C\big([0,+\infty), L^2\big(\omega,\mathbb{S}^2\big)\big)$$

satisfies

$$|m'(t)| = 1, \quad \nabla\cdot m'(t) = 0 \quad \text{in } \mathcal{D}'(\omega) \quad \forall t \in [0,+\infty).$$

Moreover, the limit m is stationary, i.e.,

$$\partial_t m' = 0 \quad \text{in } \mathcal{D}'([0,+\infty)\times\omega).$$

In particular, it follows immediately from Theorems 1.2 and 1.7 (and Remark 1.4 (i)) that for well-prepared initial data, the asymptotic magnetization is a static straight wall for all $t \geq 0$.

Corollary 1.8. *Under the same assumptions as in Theorem* 1.7*, assume moreover that the initial data are well prepared:*

$$\limsup_{\delta\to 0} \delta|\log\delta|\tilde{E}_\delta(m_\delta^0) \leq \frac{\pi}{2}(1-m_{1,\infty})^2.$$

Let $\delta_n \to 0$ and let $x_1^* \in [-1,1]$ be such that $m_{\delta_n}^0 \to m^*$ in $L^2(\omega)$, where m^* is a straight wall defined by (1.10)*. Then we have $m_{\delta_n}(t) \to m^*$ in $L^2(\omega)$ for all $t \geq 0$.*

The paper is organized as follows. In Sections 2 and 3, we focus on the stationary results and prove Theorems 1.1 and 1.2. In Section 4, we prove Theorem 1.7, assuming Theorem 1.6, which is proved in Section 5. Finally, we prove in the appendix a uniform estimate in the context of the Ginzburg–Landau energy, which is needed in the proof of Theorem 1.1.

In all the following, C denotes an absolute constant (independent of the parameters of the system) which can possibly change from one line to another.

2. Approximation and compactness

This section is devoted to the proof of Theorem 1.1. A similar compactness result to Theorem 1.1 has been already established by Ignat and Otto in [12, Theorem 4] for $\mathbb{S}^1$-valued magnetizations. In order to establish compactness for $\mathbb{S}^2$-valued magnetizations, we will use an argument consisting in approximating $\mathbb{S}^2$-valued maps by $\mathbb{S}^1$-valued maps with quantitative bounds given in terms of the energy, which is stated as follows.

Theorem 2.1. *Let $\beta \in (0,1)$. Let $\delta > 0$ and $\varepsilon(\delta) > 0$ satisfy the regime* (1.6)*, i.e.,*

$$\frac{1}{\delta|\log\delta|\,|\log\varepsilon|} \to 0 \quad \text{as } \delta \to 0,$$

and let $m_\delta = (m'_\delta, m_{3,\delta}) \in H^1_{\mathrm{loc}}(\Omega,\mathbb{S}^2)$ satisfy (1.1) *and* (1.7)*. Then there exists $M_\delta \in H^1_{\mathrm{loc}}(\Omega,\mathbb{S}^1)$ that satisfies* (1.1) *such that*

$$\begin{aligned} \int_\Omega |M_\delta - m'_\delta|^2\,dx &\leq C\varepsilon^{2\beta}E_\delta(m_\delta),\\ \int_\Omega |\nabla(M_\delta - m'_\delta)|^2\,dx &\leq CE_\delta(m_\delta), \end{aligned} \tag{2.1}$$

$$\int_{\Omega\times\mathbb{R}} |h(M_\delta) - h(m'_\delta)|^2\,dx\,dz \leq C\varepsilon^\beta E_\delta(m_\delta), \tag{2.2}$$

and

$$E_\delta(M_\delta) \le E_\delta(m_\delta)\,(1+o(1))\,, \tag{2.3}$$

where

$$o(1) = O\left(\left(\frac{1}{\delta|\log\delta|\,|\log\varepsilon|}\right)^{1/6-}\right)$$

and $1/6-$ *is any fixed positive number less than* $1/6$. *Moreover, for every full square* $T(x,r)$ *centered at* x *of side of length* $2r$ *with* $\varepsilon^\beta/r \to 0$ *as* $\delta \to 0$, *we have*[9]

$$\int_{T(x,r-2\varepsilon^\beta)} |\nabla M_\delta|^2\,dx \le (1+o(1)) \int_{T(x,r)} \left(|\nabla m'_\delta|^2 + \frac{1}{\varepsilon^2} m_{3,\delta}^2\right) dx. \tag{2.4}$$

Theorem 2.1 is reminiscent of the argument developed by Ignat and Otto [13] with a major improvement given by (2.3); i.e., the approximating $\mathbb{S}^1$-map M_δ has lower energy than the $\mathbb{S}^2$-map m_δ (up to $o(1)$ error).

Proof. To simplify notation, we will often omit the index δ in the following. We introduce a Ginzburg–Landau-type energy density:

$$\mathbf{e}_\varepsilon(m') = |\nabla m'|^2 + \frac{1}{\varepsilon^2}\left(1-|m'|^2\right)^2. \tag{2.5}$$

The approximation scheme is inspired by [13].

Step 1. *Construction of a squared grid.* For each shift $t \in (0,\varepsilon^\beta)$, we consider the set

$$H_t = \left\{x = (x_1,x_2) \in \mathbb{R}\times(0,1) : x_2 \in \left(\varepsilon^\beta, 1-\varepsilon^\beta\right),\ x_2 \equiv t \left(\operatorname{mod} \varepsilon^\beta\right)\right\}$$

and we repeat it 1-periodically in x_2 to obtain a net of horizontal lines at a distance ε^β in Ω. By the mean value theorem, there exists $t \in (0,\varepsilon^\beta)$ such that

$$\int_{H_t} \mathbf{e}_\varepsilon(m')\,d\mathcal{H}^1 \le \frac{1}{\varepsilon^\beta}\int_\Omega \mathbf{e}_\varepsilon(m')\,dx.$$

If one repeats the above argument for the net of vertical lines at distance ε^β in Ω, we get a shift $s \in (0,\varepsilon^\beta)$ such that the net

$$V_s := \left\{x \in \Omega : x_1 \in \left(-1+\varepsilon^\beta, 1-\varepsilon^\beta\right),\ x_1 \equiv s \left(\operatorname{mod} \varepsilon^\beta\right)\right\}$$

satisfies

$$\int_{V_s} \mathbf{e}_\varepsilon(m')\,d\mathcal{H}^1 \le \frac{1}{\varepsilon^\beta}\int_\Omega \mathbf{e}_\varepsilon(m')\,dx.$$

Set

$$\tilde{V}_s := V_s \cup \{(x_1,x_2) : x_1 \in \{\pm 1\}, x_2 \in [0,1)\}$$

and remark that

$$\int_{V_s} \mathbf{e}_\varepsilon(m')\,d\mathcal{H}^1 = \int_{\tilde{V}_s} \mathbf{e}_\varepsilon(m')\,d\mathcal{H}^1$$

[9]In (2.4), $o(1)$ is the same as in (2.3).

since m satisfies (1.1). Therefore, we obtain an x_2-periodic squared grid $\mathcal{R} = H_t \cup \tilde{V}_s$ of size more than ε^β such that

$$\int_{\mathcal{R}} \mathbf{e}_\varepsilon(m')\, d\mathcal{H}^1 \le \frac{2}{\varepsilon^\beta} \int_\Omega \mathbf{e}_\varepsilon(m')\, dx \le \frac{2E_\delta(m)}{\varepsilon^\beta} \le \frac{C}{\varepsilon^\beta \delta |\log \delta|}. \tag{2.6}$$

Due to periodicity, one may assume that $\mathcal{R}$ includes the horizontal line $\mathbb{R} \times \{0\}$.

Step 2. *Vanishing degree on the cells of the grid* $\mathcal{R}$. In order to approximate m' in Ω by $\mathbb{S}^1$-valued vector fields, it is necessary for m' to have zero degree on each cell of the grid $\mathcal{R}$. Let us prove this property. For that, let $\mathcal{C}$ be a full squared cell of $\mathcal{R}$ having all four sides of the cell of length $\in [\varepsilon^\beta, 4\varepsilon^\beta]$. We know that (2.6) holds (in particular, for $\mathbf{e}_\varepsilon$ on $\mathcal{C}$). Set

$$\kappa := \frac{1}{\delta |\log \delta|} = o\left(|\log \varepsilon|\right).$$

By Theorem A.1 given in the appendix, we deduce that $|m'| \ge 1/2$ on $\mathcal{R}$ and $\deg(m', \partial\mathcal{C}) = 0$ for small $\varepsilon > 0$.

Step 3. *Construction of an approximating* $\mathbb{S}^1$*-valued vector field* M *of* m'. On each *full* squared cell $\mathcal{C}$ of $\mathcal{R}$ of side of length of order ε^β, we define $u = u_\delta \in H^1(\mathcal{C}, \mathbb{R}^2)$ to be a minimizer of

$$\min\left\{ \int_{\mathcal{C}} \mathbf{e}_\varepsilon(u)\, dx : u = m' \text{ on } \partial\mathcal{C} \right\}.$$

Putting together all the cells, u is now defined in the whole hull$(\mathcal{R})$ (which is $[-1,1] \times \mathbb{T}$) and satisfies (1.1). Extend u by $m^\pm$ for $\pm x_1 \ge 1$ so that u is defined now in Ω and is periodic in x_2. Moreover, by construction,

$$\int_\Omega \mathbf{e}_\varepsilon(u)\, dx \le \int_\Omega \mathbf{e}_\varepsilon(m')\, dx.$$

By Theorem A.1 given in the appendix, we have

$$\eta := \sup_\Omega \left| |u|^2 - 1 \right| \le C \left(\frac{1}{\delta |\log \delta|\, |\log \varepsilon|} \right)^{1/6-} = o(1).$$

In particular,

$$|u|^2 \ge 1 - \eta \quad \text{in } \Omega.$$

Therefore, we define $M \in H^1(\Omega, \mathbb{S}^1)$ by

$$M := \frac{u}{|u|} \quad \text{in } \Omega.$$

So, M satisfies (1.1). We deduce that

$$|\nabla u|^2 \ge |u|^2 |\nabla M|^2 \ge (1-\eta)|\nabla M|^2 \quad \text{in } \Omega$$

and

$$\begin{aligned}(1-\eta)\int_\Omega |\nabla M|^2\, dx &\le \int_\Omega |\nabla u|^2\, dx \le \int_\Omega \mathbf{e}_\varepsilon(u)\, dx \\ &\le \int_\Omega \mathbf{e}_\varepsilon(m')\, dx \le E_\delta(m).\end{aligned} \tag{2.7}$$

We prove now (2.4) which is a local version of (2.7). Using the above constructed grid, we cover $T(x, r-2\varepsilon^\beta) \cap ([-1,1]\times\mathbb{T})$ by a subgrid $\bigcup_{k\in K}\mathcal{C}_k$, with K finite, of *full* cells of $\mathcal{R}$ such that $\bigcup_{k\in K}\mathcal{C}_k \subset T(x,r)\cap([-1,1]\times\mathbb{T})$. Therefore, we have

$$\begin{aligned}(1-\eta)&\int_{T(x,r-2\varepsilon^\beta)} |\nabla M|^2\,dx\\ &= (1-\eta)\int_{T(x,r-2\varepsilon^\beta)\cap([-1,1]\times\mathbb{T})} |\nabla M|^2\,dx\\ &\le (1-\eta)\int_{\bigcup_{k\in K}\mathcal{C}_k} |\nabla M|^2\,dx \le \int_{\bigcup_{k\in K}\mathcal{C}_k} |\nabla u|^2\,dx\\ &\le \int_{\bigcup_{k\in K}\mathcal{C}_k} \mathbf{e}_\varepsilon(u)\,dx \le \int_{\bigcup_{k\in K}\mathcal{C}_k} \mathbf{e}_\varepsilon(m')\,dx \le \int_{T(x,r)} \mathbf{e}_\varepsilon(m')\,dx.\end{aligned}$$

The goal is now to prove that the $\mathbb{S}^1$-valued vector field M approximates m' in $L^2(\Omega,\mathbb{R}^2)$ and the $\dot{H}^1$-seminorm of M is comparable with the one of m'.

Step 4. *Estimate* $\|\nabla(M-m')\|_{L^2(\Omega)}$. Indeed, by (2.7), we have

$$(1-\eta)\int_\Omega |\nabla M|^2\,dx \le E_\delta(m) \quad \text{and} \quad \int_\Omega |\nabla m|^2\,dx \le E_\delta(m).$$

Thus, the second estimate in (2.1) holds.

Step 5. *Estimate* $\|M-m'\|_{L^2(\Omega)}$. By Poincaré's inequality, we have for each *full* cell $\mathcal{C}$ of $\mathcal{R}$,

$$\int_{\mathcal{C}} \left| M - \fint_{\partial\mathcal{C}} M\right|^2 dx \le C\varepsilon^{2\beta}\int_{\mathcal{C}} |\nabla M|^2\,dx \tag{2.8}$$

and

$$\int_{\mathcal{C}} \left| m' - \fint_{\partial\mathcal{C}} m'\right|^2 dx \le C\varepsilon^{2\beta}\int_{\mathcal{C}} |\nabla m'|^2\,dx. \tag{2.9}$$

Writing $m' = \rho v'$ with $\rho \ge \frac{1}{2}$ on $\mathcal{R}$ (by Theorem A.1), we have $v' = M$ on $\mathcal{R}$ and by Jensen's inequality, we also compute

$$\begin{aligned}\int_{\mathcal{C}} \left|\fint_{\partial\mathcal{C}}(M-m')\right|^2 dx &= \int_{\mathcal{C}} \left|\fint_{\partial\mathcal{C}}(v'-m')\right|^2 dx = \mathcal{H}^2(\mathcal{C})\fint_{\partial\mathcal{C}}(1-\rho)^2\,d\mathcal{H}^1\\ &\le C\varepsilon^\beta\int_{\partial\mathcal{C}}(1-\rho^2)^2\,d\mathcal{H}^1 \le C\varepsilon^{\beta+2}\int_{\partial\mathcal{C}}\mathbf{e}_\varepsilon(m')\,d\mathcal{H}^1.\end{aligned} \tag{2.10}$$

Summing up (2.8), (2.9) and (2.10) over all the cells $\mathcal{C}$ of the grid $\mathcal{R}$, by (2.6) and (2.7), we obtain

$$\int_\Omega |M-m'|^2\,dx' \le C\varepsilon^{2\beta}E_\delta(m).$$

Step 6. *Proof of* (2.2). Let $h(m') = \nabla U(m')$ and $h(M) = \nabla U(M)$ be the unique minimal stray fields given by (1.3). By uniqueness and linearity of the stray field, we deduce that $h(m') - h(M)$ is the minimal stray field associated with $m' - M$, i.e.,

$$h(m') - h(M) = h(m' - M).$$

Therefore, we have by interpolation,

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} |h(M) - h(m')|^2\, dx\, dz
&\overset{(1.4)}{=} \frac{1}{2}\int_\Omega \Big| |\nabla|^{-1/2}\nabla\cdot(M - m')\Big|^2\, dx \\
&\le C\int_\Omega \Big| |\nabla|^{1/2}(M - m')\Big|^2\, dx \\
&\le C\Big(\int_\Omega |M - m'|^2\, dx\Big)^{1/2}\Big(\int_\Omega |\nabla(M - m')|^2\, dx\Big)^{1/2} \\
&\overset{(2.1)}{\le} C\varepsilon^\beta E_\delta(m).
\end{aligned}$$

Step 7. *End of the proof.* It remains to prove (2.3). Indeed, by (2.7) and Step 6, we have

$$\begin{aligned}
E_\delta(M) &= \int_\Omega |\nabla M|^2\, dx + \frac{1}{\delta}\int_{\Omega\times\mathbb{R}} |h(M)|^2\, dx\, dz \\
&\le \frac{1}{1-\eta}\int_\Omega \mathbf{e}_\varepsilon(m')\, dx + \frac{1}{\delta}\int_{\Omega\times\mathbb{R}} |h(m')|^2\, dx\, dz + C\Big(\frac{\varepsilon^\beta}{\delta}\Big)^{1/2} E_\delta(m) \\
&\le (1 + C\eta)\, E_\delta(m)
\end{aligned}$$

because $(\varepsilon^\beta/\delta)^{1/2} \le \eta$ by (1.6). □

Observe that Theorem 2.1 remains true in the context of the energy $\tilde{E}_\delta$ on the domain ω.

Proof of Theorem 1.1. It is a direct consequence of the approximation result in Theorem 2.1 and of the compactness result in [12] (see [12, Theorem 4], and also [13, Theorem 2]). □

3. Optimality of the Néel wall

We present now the proof of Theorem 1.2. A similar result in the case of $\mathbb{S}^1$-valued magnetizations was proved by Ignat and Otto in [12] (see [12, Theorem 1]). Theorem 1.2 represents the extension to the case of $\mathbb{S}^2$-valued magnetizations.

Proof of Theorem 1.2. Let M_δ be the approximating $\mathbb{S}^1$-map of m_δ constructed in Theorem 2.1. By (1.9) and (2.3), we deduce that

$$\limsup_{\delta\to 0} \delta|\log\delta| E_\delta(M_\delta) \le \frac{\pi}{2}(1-m_{1,\infty})^2.$$

Then [12, Theorem 1] implies the existence of a sequence $\delta = \delta_n$ and $x_1^* \in [-1,1]$ such that

$$M_\delta - m^* \to 0 \quad \text{in } L^2(\Omega),$$

which by (2.1) entails $m_\delta - m^* \to 0$ in $L^2(\Omega)$. Moreover, the x_2-periodic uniformly bounded sequence of measures $\mu_\delta(M_\delta)$ has the property that

$$\mu_\delta(M_\delta) \rightharpoonup \mu_0 \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega\times\mathbb{R}),$$

where μ_0 is a nonnegative x_2-periodic measure in $\Omega\times\mathbb{R}$. Our first aim is to prove that

$$\mu_0 = (1-m_{1,\infty})^2\, \mathcal{H}^1 \llcorner \{x_1^*\}\times\mathbb{T}\times\{0\}. \tag{3.1}$$

Indeed, let us define the function $\chi : \Omega \to \mathbb{R}$ by

$$\chi = \pm\frac{1}{2} \quad \text{if } \pm x_1 \ge \pm x_1^*.$$

Then, by Step 3 of the proof of Theorem 1 (and Remark 4) in [12], it follows that

$$\frac{1}{4\alpha}\int_{\Omega\times\mathbb{R}} \zeta\, d\mu_0 = \int_\Omega \nabla\zeta\cdot m^*\chi\, dx + (1-\alpha)\int_\Omega \zeta|D\chi|$$

for every $\alpha\in(0,1)$ and for every smooth test function $\zeta:\mathbb{R}^3\to\mathbb{R}$ which is 1-periodic in x_2 with compact support in x_1 and x_3. Then we compute

$$\int_\Omega \nabla\zeta\cdot m^*\chi\, dx = -m_{1,\infty}\int_0^1 \zeta(x_1^*, x_2, 0)\, dx_2$$

so that by setting

$$\alpha := \frac{1-m_{1,\infty}}{2},$$

we conclude that

$$\int_{\Omega\times\mathbb{R}} \zeta\, d\mu_0 = 4\alpha^2 \int_{\{x_1^*\}\times[0,1)\times\{0\}} \zeta\, d\mathcal{H}^1,$$

i.e., $\mu_0 = (1-m_{1,\infty})^2\mathcal{H}^1\llcorner\{x_1^*\}\times\mathbb{T}\times\{0\}$.

It remains to show that $\mu_\delta(m_\delta)\rightharpoonup\mu_0$ in $\mathcal{M}(\Omega\times\mathbb{R})$. Indeed, by (1.9), there exists an x_2-periodic nonnegative measure $\mu\in\mathcal{M}(\Omega\times\mathbb{R})$ such that up to a subsequence,

$$\mu_\delta(m_\delta) \rightharpoonup \mu \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega\times\mathbb{R}). \tag{3.2}$$

The aim is to show that

$$\mu = \mu_0.$$

Indeed, let $r>0$ and $x=(x_1^*, x_2)\in\Omega$ with $x_2\in[0,1)$. We consider an arbitrary smooth nonnegative test function $\zeta:\mathbb{R}^3\to[0,+\infty)$ that is x_2-periodic with compact support in x_1 and x_3 such that $\zeta\equiv 1$ on $T(x,r)\times(-\gamma,\gamma)$ for some fixed $\gamma>0$ (recall that $T(x,r)$ is the full closed square centered at x of side of length $2r$). Within the notation (2.5), by Theorem 2.1, for $\beta = 1/2$ and

$$\eta = \left(\frac{1}{\delta|\log\delta|\,|\log\varepsilon|}\right)^{1/6-},$$

we have

$$\delta|\log\delta| \int_{T(x,r-2\varepsilon^\beta)} |\nabla M_\delta|^2 \overset{(2.4)}{\leq} (1+C\eta)\delta|\log\delta| \int_\Omega \mathbf{e}_\varepsilon(m'_\delta)\zeta(x,0)\,dx$$

and

$$\begin{aligned} |\log\delta| &\int_{T(x,r-2\varepsilon^\beta)\times(-\gamma,\gamma)} |h(M_\delta)|^2\,dx\,dz \\ &\leq |\log\delta| \int_{\Omega\times\mathbb{R}} |h(M_\delta)|^2\zeta(x,z)\,dx\,dz \\ &\overset{(2.2)}{\leq} |\log\delta| \int_{\Omega\times\mathbb{R}} |h(m'_\delta)|^2\zeta(x,z)\,dx\,dz + \|\zeta\|_{L^\infty} O(\delta\varepsilon^\beta)^{1/2}|\log\delta| E_\delta(m_\delta). \end{aligned}$$

Therefore, by (1.6), we obtain

$$\begin{aligned} \liminf_{\delta\to 0} \int_{T(x,r-2\varepsilon^\beta)\times(-\gamma,\gamma)} d\mu_\delta(M_\delta) &\leq \liminf_{\delta\to 0} \int_{\Omega\times\mathbb{R}} \zeta(x,z)\,d\mu_\delta(m_\delta) \\ &\overset{(3.2)}{=} \int_{\Omega\times\mathbb{R}} \zeta(x,z)\,d\mu. \end{aligned}$$

On the other hand, by (3.1), one has

$$2r(1-m_{1,\infty})^2 = \mu_0\big(\dot{T}(x,r)\times\{0\}\big) \leq \liminf_{\delta\to 0} \int_{T(x,r-2\varepsilon^\beta)\times(-\gamma,\gamma)} d\mu_\delta(M_\delta),$$

where $\dot{T}(x,r)$ is the interior of $T(x,r)$. Thus, we conclude that

$$2r(1-m_{1,\infty})^2 \leq \int_{\Omega\times\mathbb{R}} \zeta(x,z)\,d\mu.$$

Taking infimum over all test functions ζ and then infimum over $\gamma \to 0$, we deduce that

$$2r(1-m_{1,\infty})^2 \leq \mu\big(T(x,r)\times\{0\}\big).$$

Setting

$$\text{Line} := \{x_1^*\}\times\mathbb{T}\times\{0\} \quad \text{and} \quad \mu_L := \mu \llcorner \text{Line},$$

we deduce that $\mu_L(S) \geq (1-m_{1,\infty})^2\mathcal{H}^1(S) = \mu_0(S)$ for every (closed) segment $S \subset$ Line; therefore, $\mu_0 \leq \mu_L \leq \mu$ as measures in $\mathcal{M}(\Omega\times\mathbb{R})$. In particular,

$$\mu_0(\text{Line}) \leq \mu_L(\text{Line}) \leq \mu(\Omega\times\mathbb{R}) \leq \liminf_{\delta\to 0} \int_{\Omega\times\mathbb{R}} d\mu_\delta(m_\delta) \overset{(1.9)}{\leq} \mu_0(\text{Line}),$$

thus

$$\mu = \mu_L = \mu_0 \quad \text{in } \mathcal{M}(\Omega\times\mathbb{R}).$$

Now (1.11) is straightforward. □

4. Asymptotics of the Landau–Lifshitz–Gilbert equation

We start now the study of the dynamics of the magnetization. We assume Theorem 1.6 holds and postpone its proof to the next section; our goal here is to establish Theorem 1.7. Let $\{m_\delta^0\}_{0<\delta<1/2}$ be a family of initial data as in Theorem 1.7 and let

$$m_\delta = (m'_\delta, m_{3,\delta}) : [0,+\infty) \times \omega \to \mathbb{S}^2$$

be any family of global weak solutions to (LLG) satisfying (1.18), (1.19) and the energy estimate (1.20). Throughout this section we assume that assumptions (A1)–(A3) are satisfied.

Let us also recall the energy inequality (1.20), on which we will crucially rely:

$$\begin{aligned}\tilde{E}_\delta(m_\delta(t)) &+ \frac{\alpha}{2\beta}\int_0^t \|\partial_t m_\delta(s)\|^2_{L^2(\omega)}\,ds\\ &\le \tilde{E}_\delta(m_\delta^0)\exp\left(\frac{4}{\alpha\beta}\int_0^t \|v_\delta(s)\|^2_{L^\infty(\omega)}\,ds\right).\end{aligned}$$

In particular, it follows from (1.20) and the assumption (A3) on v_δ that

$$\begin{aligned}\tilde{E}_\delta(m_\delta(t)) &+ \frac{\nu}{2\lambda}\int_0^t \|\partial_t m_\delta(s)\|^2_{L^2(\omega)}\,ds\\ &\le \tilde{E}_\delta\left(m_\delta^0\right)\exp\left(CT\right), \quad 0<t\le T,\end{aligned} \tag{4.1}$$

and therefore it follows from the energy bound (A1) on the initial data that

$$\sup_{0<\delta<1/2} \delta|\log\delta|\tilde{E}_\delta(m_\delta(T)) < +\infty \quad \forall T>0. \tag{4.2}$$

Also, we infer the following bound on the time derivative in $L^2_{\mathrm{loc}}([0,+\infty)\times\omega)$:

$$\|\partial_t m_\delta\|_{L^2([0,T],L^2(\omega))} \le C\exp(CT)\,\frac{\sqrt{\lambda}}{\sqrt{\delta|\log(\delta)|}} \quad \forall T>0. \tag{4.3}$$

This is however not a uniform bound on $\lambda/(\delta|\log(\delta)|)$ as $\delta\to 0$ in the regime (1.16). Nevertheless, in the next proposition, we will establish a uniform bound of $\{\partial_t m_\delta\}$ in the weaker space $L^2_{\mathrm{loc}}(H^{-1})$.

Proposition 4.1. *Under the assumptions of Theorem* 1.7, *we have*

$$\|\partial_t m_\delta\|_{L^2([0,T],H^{-1}(\omega))} \le \frac{C(T)}{\sqrt{\delta|\log(\delta)|}}\left(\lambda + \frac{\lambda\varepsilon}{\delta} + \varepsilon^2\right) \quad \forall T>0.$$

Proof. Let $T>0$. By (LLG) we have, on $[0,+\infty)\times\omega$,

$$\partial_t m_\delta = -\alpha m_\delta \times \partial_t m_\delta - \beta m_\delta \times \nabla\tilde{E}_\delta(m_\delta) - (v_\delta\cdot\nabla)m_\delta + m_\delta \times (v_\delta\cdot\nabla)m_\delta.$$

First, inequality (4.3) yields

$$\begin{aligned}\|\alpha m_\delta \times \partial_t m_\delta\|_{L^2([0,T],L^2(\omega))} &\leq C\exp(CT)\,\frac{\varepsilon\sqrt{\lambda}}{\sqrt{\delta|\log(\delta)|}}\\ &\leq C\exp(CT)\left(\frac{\varepsilon^2}{\sqrt{\delta|\log(\delta)|}}+\frac{\lambda}{\sqrt{\delta|\log(\delta)|}}\right).\end{aligned}$$

Next, by (A3) we have

$$\begin{aligned}&\|(v_\delta\cdot\nabla)m_\delta\|_{L^2([0,T],L^2(\omega))}+\|m_\delta\times(v_\delta\cdot\nabla)m_\delta\|_{L^2([0,T],L^2(\omega))}\\ &\qquad\leq C\sqrt{T}\exp(CT)\,\frac{\varepsilon\sqrt{\lambda}}{\sqrt{\delta|\log(\delta)|}}\\ &\qquad\leq C\sqrt{T}\exp(CT)\left(\frac{\varepsilon^2}{\sqrt{\delta|\log(\delta)|}}+\frac{\lambda}{\sqrt{\delta|\log(\delta)|}}\right).\end{aligned}$$

Finally, recalling (1.14), we have

$$\begin{aligned}&\beta\left\|m_\delta(t)\times\nabla\tilde{E}_\delta(m_\delta)(t)\right\|_{H^{-1}(\omega)}\\ &\leq C\left(\lambda\varepsilon\,\|\nabla m_\delta(t)\|_{L^2(\omega)}+\frac{\lambda\varepsilon}{\delta}\|\nabla m_\delta(t)\|_{L^2(\omega)}+\lambda\left\|\frac{m_{3,\delta}(t)}{\varepsilon}\right\|_{L^2(\omega)}\right) \qquad (4.4)\\ &\leq C\exp(CT)\,\frac{\lambda}{\sqrt{\delta|\log(\delta)|}}\left(1+\frac{\varepsilon}{\delta}\right).\end{aligned}$$

Combining the previous estimates, we obtain the estimate of the proposition. □

We now prove Theorem 1.7.

Proof of Theorem 1.7. Let $T>0$. By Proposition 4.1 and assumptions (1.6) and (1.16) on ε, δ and λ, the family

$$\{\partial_t m_\delta\}_{0<\delta<1/2}$$

is bounded in $L^2([0,T],H^{-1}(\omega))$. On the other hand, $\{m_\delta\}_{0<\delta<1/2}$ is bounded in $L^\infty([0,T],L^2(\omega))$. Therefore, by the Aubin–Lions lemma (see, e.g., [20, Corollary 1]), it is relatively compact in $\mathcal{C}([0,T],H^{-1}(\omega))$. Thus by a diagonal argument, there exist $\delta_n\to 0$ and $m\in\mathcal{C}([0,+\infty),H^{-1}(\omega))$ such that $m_{\delta_n}\to m$ in $\mathcal{C}([0,T],H^{-1}(\omega))$ for all $T>0$ as $n\to\infty$.

On the other hand, let $t\in[0,+\infty)$. In view of the bound (4.2) we conclude from Theorem 1.1 that any subsequence of $(m_{\delta_n}(t))_{n\in\mathbb{N}}$ is relatively compact in $L^2(\omega)$. Since

$$m_{\delta_n}(t)\to m(t)\quad\text{in } H^{-1}(\omega),$$

we infer that the full sequence $m_{\delta_n}(t)\to m(t)=(m'(t),0)$ strongly in $L^2(\omega)$ as $n\to\infty$, where $|m'(t)|=1$, $m_3(t)=0$ almost everywhere and $\nabla\cdot m'(t)=0$ in the sense of distributions. In particular,

$$t\mapsto\|m(t)\|_{L^2(\omega)}=|\omega|^{1/2}\in\mathcal{C}([0,+\infty),\mathbb{R}).$$

Let us now prove that $m \in \mathcal{C}([0,+\infty), L^2(\omega))$. Indeed, consider a sequence of times $t_n \geq 0$ converging to $t \geq 0$. As $m \in \mathcal{C}([0,T], H^{-1}(\omega))$ and $m(t_n)$ is bounded in $L^2(\omega)$, we infer that $m(t_n) \rightharpoonup m(t)$ weakly in $L^2(\omega)$. But we just saw that $\|m(t_n)\|_{L^2(\omega)} \to \|m(t)\|_{L^2(\omega)}$, so that in fact $m(t_n) \to m(t)$ strongly in $L^2(\omega)$. This is the desired continuity.

Finally, Proposition 4.1 and (1.16) imply that

$$\partial_t m_{\delta_n} \to 0 = (\partial_t m', 0) \quad \text{in } \mathcal{D}'([0,+\infty) \times \omega),$$

which concludes the proof. □

5. The Cauchy problem for the Landau–Lifshitz–Gilbert equation

In this section we handle the Cauchy problem for the LLG equation in the energy space.

Proof of Theorem 1.6. We use an approximation scheme by discretizing in space. We first introduce some notation.

Notation and discrete calculus

Let $n \geq 1$ be an integer, $h = 1/n$ and $\omega_h = h\mathbb{Z}^2 \cap \overline{\omega}$. For a vector field $m^h : \omega_h \to \mathbb{R}^3$, we will always assume x_2-periodicity in the following sense:

$$m^h(x_1, 1+eh) = m^h(x_1, eh) \quad \forall x_1 \in h\,\mathbb{Z} \cap [-1,1],\ e \in \mathbb{Z}.$$

We then define the differentiation operators as follows: for $x = (x_1, x_2) \in \omega_h$,

$$\partial_1^h m^h(x) = \begin{cases} \frac{1}{2h}\big(m^h(x_1+h, x_2) - m^h(x_1-h, x_2)\big) & \text{if } |x_1| < 1, \\ \pm\frac{1}{2h}\big(m^h(x_1, x_2) - m^h(x_1 \mp h, x_2)\big) & \text{if } x_1 = \pm 1, \end{cases}$$

$$\partial_2^h m^h(x) = \frac{1}{2h}\big(m^h(x_1, x_2+h) - m^h(x_1, x_2-h)\big).$$

Observe that ∂_1^h is the half sum of the usual operators ∂_{1+}^h and ∂_{1-}^h vanishing at the boundary $x_1 = 1$ and $x_1 = -1$, respectively. Also we define the discrete gradient and Laplacian: denoting $(\hat{e}_1, \hat{e}_2)$ the canonical base of $\mathbb{R}^2$, we let

$$\nabla^h m^h = \sum_{k=1}^{2} \partial_k^h m^h \otimes \hat{e}_k, \quad \Delta^h m^h = \sum_{k=1}^{2} \partial_k^h \partial_k^h m^h.$$

We introduce the scalar product

$$\langle m^h, \tilde{m}^h \rangle_h = h^2 \sum_{x \in \omega_h} m^h(x) \cdot \tilde{m}^h(x)$$

and the L_h^2-norm and $\dot{H}_h^1$-seminorm:

$$|m^h|_{L_h^2}^2 := \langle m^h, m^h \rangle_h, \quad |m^h|_{\dot{H}_h^1}^2 := \langle \nabla^h m^h, \nabla^h m^h \rangle_h.$$

Then we have the integration-by-parts formulas:

$$\begin{aligned}\langle \partial_1^h m^h, \tilde m^h\rangle_h &= -\langle m^h, \partial_1^h \tilde m^h\rangle_h + h \sum_{x\in\omega_h,\, x_1=1} m^h(x)\tilde m^h(x)\\ &\quad - h \sum_{x\in\omega_h,\, x_1=-1} m^h(x)\tilde m^h(x),\\ \langle \partial_2^h m^h, \tilde m^h\rangle_h &= -\langle m^h, \partial_2^h \tilde m^h\rangle_h,\end{aligned}$$

where we used the above boundary conditions and periodicity.

We now define the sampling and interpolating operators S^h and I^h. We discretize a map $m:\omega\to\mathbb{R}^3$ by defining $S^h m:\omega_h\to\mathbb{R}^3$ as follows:

$$S^h m(x) = \begin{cases} \dfrac{1}{h^2}\displaystyle\int_{C_x^h} m(y)dy & \text{if } x_1<1,\\ m(x) & \text{if } x_1 = 1,\end{cases}$$

where $C_x^h = \{y\in\omega \mid x_k \le y_k < x_k + h,\ k=1,2\}$. We will also identify $S^h m$ with the function $\overline{\omega}\to\mathbb{R}^3$ which is constant on each cell C_x^h for $x\in\omega_h$ with value $S^h m(x)$. With this convention, $S^h m$ is the orthogonal projection onto piecewise constant functions on each cell C_x^h in $L^2(\omega)$. Also we have

$$|S^h m|_{L_h^2} = \|S^h m\|_{L^2(\omega)},$$

and

$$|S^h m|_{L_h^2} \le \|m\|_{L^2(\omega)}, \quad |\nabla^h S^h m|_{L_h^2} \le \|\nabla m\|_{L^2(\omega)}, \quad |S^h m|_{L_h^\infty} \le \|m\|_{L^\infty(\omega)}. \tag{5.1}$$

We interpolate a discrete map $m^h:\omega_h\to\mathbb{R}^3$ to $I^h m^h:\omega\to\mathbb{R}^3$ by a quadratic approximation as follows: if $x\in C_y^h$ with $y\in\omega_h$, we set

$$I^h m^h(x) = m^h(y) + \sum_{k=1}^{2} \partial_k^{h+} m^h(y)(x_k - y_k) + \partial_1^{h+}\partial_2^{h+} m^h(y)(x_1-y_1)(x_2-y_2),$$

where

$$\partial_k^{h+} m^h(y) = \begin{cases} \dfrac{1}{h}\big(m^h(y+h\hat e_k) - m^h(y)\big) & \text{if } k=2 \text{ or } (k=1 \text{ and } y_1<1),\\ 0 & \text{if } k=1 \text{ and } y_1 = 1.\end{cases}$$

One can check that $I^h m^h \in H^1(\omega)$ is continuous (it is linear in each variable x_1 and x_2 and coincides with m^h at every point of ω_h), quadratic on each cell C_y^h, and

$$\begin{aligned}|m^h|_{L_h^2} &\sim \|I^h m^h\|_{L^2(\omega)},\\ |\nabla^h m^h|_{L_h^2} &\sim \|\nabla I^h m^h\|_{L^2(\omega)},\\ |m^h|_{L_h^\infty} &= \|I^h m^h\|_{L^\infty(\omega)}\end{aligned} \tag{5.2}$$

(we refer the reader, for example, to [19]).

We discretize the nonlocal operator $\mathcal{P}$ so as to preserve the structure of a discrete form of $\int ||\nabla|^{-1/2}\nabla\cdot m'|^2$. For this, notice that $|\nabla|^{-1}$ and $|\nabla|^{-1/2}$

naturally act as compact operators on $L^2(\omega)$, and hence if $m^h : \omega_h \to \mathbb{R}^3$, $|\nabla|^{-1} m^h \in L^2(\omega)$ has a meaning. Also observe that due to Dirichlet boundary conditions, d/dt commutes with $(-\Delta)^{-1}$, and hence with any operator of the functional calculus: in particular,

$$\frac{d}{dt}|\nabla|^{-1} m = |\nabla|^{-1} \frac{dm}{dt}.$$

Therefore, we define for $m^h : \omega_h \to \mathbb{R}^3$ the discrete operator

$$\mathcal{P}^h m^{h\prime} := -\nabla^h S^h\big(|\nabla|^{-1}\nabla^h \cdot m^{h\prime}\big).$$

Then as $\|\nabla^h S^h m\|_{L^2_h} \leq C\|\nabla m\|_{L^2(\omega)}$, we have

$$\begin{aligned} \|\mathcal{P}^h m^{h\prime}\|_{L^2_h} &\leq C\||\nabla|^{-1}\nabla^h \cdot m^{h\prime}\|_{\dot{H}^1(\omega)} \\ &\leq C\|\nabla^h \cdot m^{h\prime}\|_{L^2(\omega)} \leq C\|\nabla^h m^{h\prime}\|_{L^2_h}. \end{aligned} \tag{5.3}$$

Step 1*: Discretized solution and uniform energy estimate.* Let

$$v^h(t) := S^h v(t) : \omega_h \to \mathbb{R}^3 \quad \text{and} \quad m_0^h(x) := \frac{1}{|S^h(m_0)(x)|} S^h(m^0)(x).$$

We consider the solution $m^h(t) : \omega_h \to \mathbb{R}^3$ to the following discrete ordinary differential equation (ODE) system: for $x = (x_1, x_2) \in \omega_h$ such that $|x_1| < 1$, we have

$$\begin{cases} \dfrac{dm^h}{dt} + m^h \times \left(\alpha \dfrac{dm^h}{dt} + \beta\left(-2\Delta^h m^h + \left(\dfrac{1}{\delta}\mathcal{P}^h\big(m^{h\prime}\big), \dfrac{2}{\varepsilon^2} m_3^h\right)\right)\right. \\ \qquad \left. - \big(v^h \cdot \nabla^h\big) m^h - m^h \times \big(v^h \cdot \nabla^h\big) m^h\right) = 0, \\ \qquad m^h(0, x) = m_0^h(x), \end{cases} \tag{5.4}$$

and at the boundary

$$m^h(t, -1, x_2) = m_0^h(-1, x_2), \quad m^h(t, 1, x_2) = m_0^h(1, x_2). \tag{5.5}$$

As the operator $A(m^h) : \mu \mapsto \mu + \alpha m^h \times \mu$ is (linear and) invertible, this ODE takes the form

$$\frac{dm^h}{dt} = A\big(m^h\big)^{-1}\big(\Phi\big(m^h\big)\big),$$

where

$$\begin{aligned} \Phi\big(m^h\big) = m^h \times \Bigg(\beta\left(-2\Delta^h m^h + \left(\frac{1}{\delta}\mathcal{P}^h\big(m^{h\prime}\big), \frac{2}{\varepsilon^2}\, m_3^h\right)\right) \\ - \big(v^h \cdot \nabla^h\big) m^h - m^h \times \big(v^h \cdot \nabla^h\big) m^h\Bigg) \end{aligned}$$

is $\mathcal{C}^\infty$. Hence the Cauchy–Lipschitz theorem applies and guarantees the existence of a maximal solution. Furthermore, we see that for all $x \in \omega_h$,

$$\frac{d}{dt}|m^h(t,x)|^2 = 2\left(m^h(t,x), \frac{d}{dt}m^h(t,x)\right)$$
$$= \left(m^h(t,x), m^h(t,x) \times \left(\alpha\frac{d}{dt}m^h(t,x) + \Phi\big(m^h\big)(t,x)\right)\right) = 0.$$

This shows that for all $x \in \omega_h$, $|m^h(t,x)| = 1$ remains bounded, and hence m^h is defined for all times $t \in \mathbb{R}$.

We now derive an energy inequality for m^h. For this we take the L^2_h scalar product of (5.4) with $m^h \times (dm^h/dt)$. Recall that if $a, b, c \in \mathbb{R}^3$, then

$$(a \times b) \times c = (a \cdot c)b - (a \cdot b)c,$$

hence

$$(c \times a) \cdot (c \times b) = ((c \times a) \times c) \cdot b = (a \cdot b)|c|^2 - (c \cdot a)(c \cdot b),$$

so that for any $\tilde{m} \in \mathbb{R}^3$, and pointwise $(t,x) \in [0,+\infty) \times \omega_h$,

$$\left(m^h(t,x) \times \frac{dm^h}{dt}(t,x)\right) \cdot \big(m^h(t,x) \times \tilde{m}\big) = \frac{dm^h}{dt}(t,x) \cdot \tilde{m}.$$

Hence we have the pointwise equalities for $x \in \omega_h$ with $|x_1| < 1$:

$$\left(m^h \times \frac{dm^h}{dt}\right) \cdot \left(m^h \times \alpha\frac{dm^h}{dt}\right) = \alpha\left|\frac{dm^h}{dt}\right|^2,$$
$$\left(m^h \times \frac{dm^h}{dt}\right) \cdot \left(m^h \times 2\big(0,0,m_3^h\big)^T\right) = \frac{d}{dt}|m_3^h|^2.$$

If $x \in \omega_h$ is on the boundary, that is $|x_1| = 1$, then

$$\frac{dm^h}{dt}(0,x) = 0$$

due to (5.5), and the previous identities also hold: we can therefore sum over $x \in \omega_h$. Now consider the term involving the discrete Laplacian. The discrete integration by parts yields no boundary term due to dm^h/dt; and of course d/dt commutes with ∇^h. Therefore,

$$\left\langle m^h \times \frac{dm^h}{dt}, m^h \times \big(-2\Delta^h m^h\big)\right\rangle_h$$
$$= -2\left\langle \frac{dm^h}{dt}, \Delta^h m^h\right\rangle_h = 2\left\langle \nabla^h\frac{dm^h}{dt}, \nabla^h m^h\right\rangle_h = \frac{d}{dt}\left\|\nabla^h m^h\right\|_h^2.$$

For the nonlocal term, $|\nabla|^{-1/2}$ is a self-adjoint operator on L^2 due to the Dirichlet boundary conditions: the integration by parts yields no boundary term either. More precisely, as d/dt commutes with all space operators, we have

$$\begin{aligned}\left\langle \frac{dm^{h\prime}}{dt}, \mathcal{P}^h m^{h\prime} \right\rangle_h &= -\left\langle \nabla^h \frac{dm^{h\prime}}{dt}, S^h |\nabla|^{-1} \nabla^h \cdot m^{h\prime} \right\rangle_h \\ &= \int \nabla^h \frac{dm^{h\prime}}{dt} \cdot \left(|\nabla|^{-1} \nabla^h \cdot m^{h\prime}\right) \\ &= -\int \left(|\nabla|^{-1/2} \nabla^h \cdot \frac{dm^{h\prime}}{dt}\right) \left(|\nabla|^{-1/2} \nabla^h \cdot m^{h\prime}\right) \\ &= -\frac{1}{2} \frac{d}{dt} \big\| |\nabla|^{-1/2} \nabla^h \cdot m^{h\prime} \big\|_{L^2}^2.\end{aligned}$$

Thus we get

$$\begin{aligned}&\alpha \left\| \frac{dm^h}{dt} \right\|_{L_h^2}^2 + \beta \frac{d}{dt} \left(\|\nabla^h m^h\|_{L_h^2}^2 + \frac{1}{\delta} \big\| |\nabla|^{-1/2} (\nabla^h \cdot m^{h\prime}) \big\|_{L^2}^2 + \frac{1}{\varepsilon^2} \|m_3^h\|_{L_h^2}^2 \right) \\ &\quad = \left\langle \left(v^h \cdot \nabla^h\right) m^h + m^h \times \left(v^h \cdot \nabla^h\right) m^h, \frac{dm^h}{dt} \right\rangle_{L_h^2}.\end{aligned}$$

Denote

$$E^h(m^h) = \|\nabla^h m^h\|_{L_h^2}^2 + \frac{1}{2\delta} \big\| |\nabla|^{-1/2} (\nabla^h \cdot m^{h\prime}) \big\|_{L^2}^2 + \frac{1}{\varepsilon^2} \|m_3^h\|_{L_h^2}^2.$$

Now we have

$$\begin{aligned}&\left| \left\langle \left(v^h \cdot \nabla^h\right) m^h + m^h \times \left(v^h \cdot \nabla^h\right) m^h, \frac{dm^h}{dt} \right\rangle_{L_h^2} \right| \\ &\qquad \le \sqrt{2} \|v^h\|_{L_h^\infty} \|\nabla^h m^h\|_{L^2} \left\| \frac{dm^h}{dt} \right\|_{L_h^2} \\ &\qquad \le \sqrt{2} \|v\|_{L^\infty} E^h\left(m^h\right)^{1/2} \left\| \frac{dm^h}{dt} \right\|_{L_h^2} \\ &\qquad \le \frac{\alpha}{2} \left\| \frac{dm^h}{dt} \right\|_{L_h^2}^2 + \frac{4}{\alpha} \|v\|_{L^\infty}^2 E^h(m^h).\end{aligned}$$

Thus we obtain

$$\frac{\alpha}{2\beta} \left\| \frac{dm^h}{dt} \right\|_{L_h^2}^2 + \frac{d}{dt} E^h\left(m^h\right) \le \frac{4}{\alpha\beta} \|v\|_{L^\infty}^2 E^h\left(m^h\right).$$

By Gronwall's inequality, we deduce that

$$\begin{aligned}&E^h\left(m^h(t)\right) + \frac{\alpha}{2\beta} \int_0^t \left\| \frac{dm^h}{dt}(s) \right\|_{L_h^2}^2 ds \\ &\qquad \le E^h\left(m^h(0)\right) \exp\left(\frac{4}{\alpha\beta} \int_0^t \|v(s)\|_{L^\infty}^2 ds \right).\end{aligned} \tag{5.6}$$

Step 2: Continuous limit of the discretized solution. Notice that $\|v^h\|_{L^\infty} \leq \|v\|_{L^\infty}$. Also, as $m_0 \in H^1(\omega)$, then $m^h(0) \to m_0$ in $H^1(\omega)$ and

$$E^h\big(m^h(0)\big) \to \tilde{E}_\delta(m_0).$$

Fix $T > 0$. It follows from (5.6) and (5.2) that the sequence $I^h m^h$ is bounded in

$$L^\infty([0,T], H^1(\omega)) \quad \text{and} \quad \dot{H}^1([0,T], L^2(\omega))$$

(observe that $\partial_t(I^h m^h) = I^h(dm^h/dt)$):

$$\sup_h \left(\sup_{t\in[0,T]} \left\|\nabla\big(I^h m^h\big)(t)\right\|^2_{L^2(\omega)} + \int_0^T \left\|\partial_t\big(I^h m^h\big)(s)\right\|_{L^2(\omega)} ds \right) < +\infty.$$

As this is valid for all $T \geq 0$, we can extract via a diagonal argument a weak limit $m \in L^\infty_{\rm loc}([0,+\infty), H^1(\omega)) \cap \dot{H}^1_{\rm loc}([0,+\infty), L^2(\omega))$ (up to a subsequence that we still denote by m^h) in the following sense:

$$I^h m^h \overset{*}{\rightharpoonup} m \qquad \text{weakly}^* \text{ in } L^\infty_{\rm loc}([0,+\infty), H^1(\omega)), \tag{5.7}$$

$$\partial_t\big(I^h m^h\big) \rightharpoonup \partial_t m \quad \text{weakly in } L^2_{\rm loc}([0,\infty), L^2(\omega)), \tag{5.8}$$

$$I^h m^h \to m \qquad \text{a.e.} \tag{5.9}$$

By compact embedding, the following strong convergence also holds:

$$I^h m^h \to m \quad \text{strongly in } L^2_{\rm loc}([0,+\infty), L^2(\omega)).$$

Then it follows that for all $t \geq 0$,

$$\begin{aligned} \tilde{E}_\delta(m(t)) &\leq \liminf_{h\to 0} \tilde{E}_\delta\big(I^h m^h(t)\big) = \liminf_{h\to 0} E^h\big(m^h(t)\big) \\ &\leq \liminf_{h\to 0} E^h\big(m^h(0)\big) \exp\left(\frac{4}{\alpha\beta}\int_0^t \|v(s)\|^2_{L^\infty} ds\right) \\ &\leq \tilde{E}_\delta(m_0) \exp\left(\frac{4}{\alpha\beta}\int_0^t \|v(s)\|^2_{L^\infty} ds\right). \end{aligned}$$

This is the energy dissipation inequality.

Observe that if φ is a test function, then $\nabla^h\varphi \to \nabla\varphi$ in L^2 (strongly). Using (5.9), it follows classically (cf. [16, p. 224]) that

$$m^h \to m \quad \text{strongly in } L^2_{\rm loc}([0,+\infty), L^2(\omega)).$$

Therefore, $|m| = 1$ a.e and

$$\nabla^h m^h \rightharpoonup \nabla m \quad \text{weakly in } L^2_{\rm loc}([0,+\infty, L^2(\omega)).$$

From there, arguing in the same way, it follows that

$$\begin{aligned} m^h \times \Delta^h m^h &= \nabla^h \cdot \big(m^h \times \nabla^h m^h\big) \\ &\rightharpoonup \nabla\cdot(m\times\nabla m) = (m\times\Delta m) \quad \text{weakly in } \mathcal{D}'((0,+\infty)\times\omega). \end{aligned}$$

Also notice that

$$\partial_t\big(I^h m^h\big) = I^h\frac{dm^h}{dt}.$$

Hence

$$\partial_t m^h \rightharpoonup \partial_t m \quad \text{weakly in } L^2_{\rm loc}([0,\infty), L^2(\omega)).$$

We can now deduce the convergences of the other nonlinear terms in the distributional sense:

$$m^h \times \partial_t m^h \rightharpoonup m \times \partial_t m \qquad \text{weakly in } \mathcal{D}'((0,+\infty)\times\omega),$$
$$m^h \times \big(v \cdot \nabla^h\big) m^h \rightharpoonup m \times (v\cdot\nabla) m \quad \text{weakly in } \mathcal{D}'((0,+\infty)\times\omega)$$

and

$$m^h \times \big(m^h \times (v\cdot\nabla^h)m^h\big) = -\big(v^h\cdot\nabla^h\big)m^h$$
$$\rightharpoonup -(v\cdot\nabla)m = m\times(m\times(v\cdot\nabla)m) \quad \text{weakly in } \mathcal{D}'((0,+\infty)\times\omega).$$

It remains to consider the nonlocal term. As

$$\nabla^h m^h \rightharpoonup \nabla m \quad \text{weakly in } L^2_{\text{loc}}([0,+\infty), L^2(\omega)),$$

we have

$$|\nabla|^{-1}\nabla^h\cdot m^{h\prime} \rightharpoonup |\nabla|^{-1}\nabla\cdot m' \quad \text{weakly in } L^2_{\text{loc}}([0,\infty), H^1(\omega)).$$

But from (5.1), and noticing that $S^h\varphi \to \varphi$ in H^1 strongly for any test function φ and as S^h is L^2-self-adjoint, we infer that

$$S^h|\nabla|^{-1}\nabla^h\cdot m^{h\prime} \rightharpoonup |\nabla|^{-1}\nabla\cdot m' \quad \text{weakly in } L^2_{\text{loc}}([0,\infty), H^1(\omega)),$$

and similarly,

$$\mathcal{P}^h\big(m^{h\prime}\big) = -\nabla^h S^h\big(|\nabla|^{-1}\nabla^h\cdot m^{h\prime}\big)$$
$$\rightharpoonup -\nabla|\nabla|^{-1}\nabla\cdot m = \mathcal{P}(m') \quad \text{weakly in } L^2_{\text{loc}}([0,\infty), L^2(\omega)).$$

Recalling that $m^h \to m$ strongly in $L^2_{\text{loc}}([0,\infty), L^2(\omega))$, we deduce by weak-strong convergence that

$$m^h \times \mathcal{H}^h(m^h) \rightharpoonup m \times \mathcal{H}(m), \quad \mathcal{D}'([0,\infty)\times\omega).$$

This shows that m satisfies (LLG) on $[0,\infty)\times\omega$ in the sense of Definition 1.5. □

Appendix A. A uniform estimate

For $\varepsilon > 0$ small, we consider the *full* cell $\mathcal{C} = (0,\varepsilon^\beta)^2 \subset \mathbb{R}^2$ with ν (resp., τ) the unit outer normal vector (resp., the tangent vector) at $\partial\mathcal{C}$ and a boundary data $g_\varepsilon \in H^1(\partial\mathcal{C},\mathbb{R}^2)$ with $|g_\varepsilon| \le 1$ on $\partial\mathcal{C}$. We recall the definition of the Ginzburg–Landau energy density

$$\mathbf{e}_\varepsilon(u) = |\nabla u|^2 + \frac{1}{\varepsilon^2}\big(1-|u|^2\big)^2 \quad \text{for } u \in H^1(\mathcal{C},\mathbb{R}^2).$$

Let $u_\varepsilon \in H^1(\mathcal{C},\mathbb{R}^2)$ be a minimizer of the variational problem

$$\min\left\{\int_{\mathcal{C}} \mathbf{e}_\varepsilon(u)\,dx : u = g_\varepsilon \text{ on } \partial\mathcal{C}\right\}.$$

In the spirit of Bethuel, Brezis and Hélein [2], we will prove that $|u_\varepsilon|$ is uniformly close to 1 as $\varepsilon \to 0$ under certain energetic conditions. The same argument is used in [11].[10]

Theorem A.1. *Let $\beta \in (0,1)$. Let $\kappa = \kappa(\varepsilon) > 0$ be such that $\kappa = o(|\log \varepsilon|)$ as $\varepsilon \to 0$. Assume that there exists $K_0 > 0$ such that*

$$\int_{\partial\mathcal{C}} \left(|\partial_\tau g_\varepsilon|^2 + \frac{1}{\varepsilon^2}\big(1-|g_\varepsilon|^2\big)^2 \right) d\mathcal{H}^1 \le \frac{K_0\kappa}{\varepsilon^\beta} \quad \text{and} \quad \int_{\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,dx \le K_0\kappa \tag{A.1}$$

for all $\varepsilon \in (0,1/2)$. Then there exist $\varepsilon_0(\beta) > 0$ and $C(K_0) > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ we have

$$\sup_{\mathcal{C}} \big||u_\varepsilon| - 1\big| \le C \left(\frac{\kappa}{|\log\varepsilon|} \right)^{1/6-},$$

where $1/6-$ is any fixed positive number less that $1/6$. In particular, $|g_\varepsilon| \ge 1/2$ on $\partial\mathcal{C}$ and $\deg(g_\varepsilon, \partial\mathcal{C}) = 0$.

Remark A.2. In the setting of the proof of Theorem 1.1 we take

$$\kappa = \frac{1}{\delta|\log\delta|}.$$

The proof of Theorem A.1 is done by using the following results.

Lemma A.3. *Under the hypothesis of Theorem A.1, we have*

$$\int_{\partial\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,dx \le \frac{C\,K_0\kappa}{\varepsilon^\beta},$$

where $C > 0$ is some universal constant. Up to a change of K_0 in Theorem A.1, we will always assume that the above $C = 1$.

Proof. Since u_ε is a minimizer of $\mathbf{e}_\varepsilon$, then u_ε is a solution of

$$-\Delta u_\varepsilon = \frac{2}{\varepsilon^2} u_\varepsilon\big(1-|u_\varepsilon|^2\big) \quad \text{in } \mathcal{C}. \tag{A.2}$$

We use the Pohozaev identity for u_ε. More precisely, multiplying the equation by $(x-x_0)\cdot\nabla u_\varepsilon$ and integrating by parts, we deduce that

$$\begin{aligned} &\left| \frac{1}{\varepsilon^2} \int_{\mathcal{C}} u_\varepsilon\big(1-|u_\varepsilon|^2\big)\cdot\big((x-x_0)\cdot\nabla u_\varepsilon\big)\,dx \right| \\ &\quad = \left| \frac{1}{2\varepsilon^2}\int_{\mathcal{C}} \big(1-|u_\varepsilon|^2\big)^2 dx - \frac{1}{4\varepsilon^2}\int_{\partial\mathcal{C}} (x-x_0)\cdot\nu\big(1-|g_\varepsilon|^2\big)^2 d\mathcal{H}^1 \right| \\ &\quad \overset{(A.1)}{\le} CK_0\kappa, \end{aligned} \tag{A.3}$$

$$\begin{aligned} &\int_{\mathcal{C}} \Delta u_\varepsilon\cdot\big((x-x_0)\cdot\nabla u_\varepsilon\big)\,dx \\ &\quad = \int_{\partial\mathcal{C}} \left(-\frac{1}{2}(x-x_0)\cdot\nu|\nabla u_\varepsilon|^2 + \frac{\partial u_\varepsilon}{\partial\nu}\cdot\frac{\partial u_\varepsilon}{\partial(x-x_0)} \right) d\mathcal{H}^1, \end{aligned} \tag{A.4}$$

[10] Theorem A.1 is an improvement of the results in [2] in the case where the energy of the boundary data g_ε is no longer uniformly bounded.

where

$$\frac{\partial u_\varepsilon}{\partial(x-x_0)} = \nabla u_\varepsilon \cdot (x-x_0).$$

For $x \in \partial\mathcal{C}$, we have $x - x_0 = \varepsilon^\beta(\nu + s\tau)$ with $s \in (-1,1)$, $u_\varepsilon(x) = g_\varepsilon(x)$ and we write (as complex numbers)

$$\nabla u_\varepsilon = \nabla u_{1,\varepsilon} + i\nabla u_{2,\varepsilon} = \frac{\partial u_\varepsilon}{\partial\nu}\,\nu + \frac{\partial g_\varepsilon}{\partial\tau}\,\tau \quad \text{on } \partial\mathcal{C}.$$

By (A.2), (A.3) and (A.4), it follows by Young's inequality that

$$\begin{aligned}\frac{CK_0\kappa}{\varepsilon^\beta} &\geq \int_{\partial\mathcal{C}} \left(\frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 - \frac{1}{2}\left|\frac{\partial g_\varepsilon}{\partial\tau}\right|^2 + s\frac{\partial u_\varepsilon}{\partial\nu}\cdot\frac{\partial g_\varepsilon}{\partial\tau}\right) d\mathcal{H}^1 \\ &\geq \int_{\partial\mathcal{C}} \left(\frac{1}{4}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 - \frac{3}{2}\left|\frac{\partial g_\varepsilon}{\partial\tau}\right|^2\right) d\mathcal{H}^1.\end{aligned}$$

Therefore, by (A.1), we deduce that

$$\int_{\partial\mathcal{C}} \left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 d\mathcal{H}^1 \leq \frac{CK_0\kappa}{\varepsilon^\beta}$$

and the conclusion follows. □

In the following, we denote by $T(x,r)$ the square centered at x of side of length $2r$.

Lemma A.4. *Fix* $1 > \beta_1 > \beta_2 > \beta > 0$. *Under the hypothesis of Theorem* A.1, *there exist* $\varepsilon_0 = \varepsilon_0(\beta_2, \beta) > 0$ *and* $C = C(K_0) > 0$ *such that for every* $x_0 \in \mathcal{C}$ *and all* $0 < \varepsilon \leq \varepsilon_0$, *we can find* $r_0 = r_0(\varepsilon) \in (\varepsilon^{\beta_1}, \varepsilon^{\beta_2})$ *such that*

$$\int_{\partial(T(x_0,r_0)\cap\mathcal{C})} \mathbf{e}_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \leq \frac{C\kappa}{r_0|\log\varepsilon|}. \tag{A.5}$$

Moreover, we have

$$\frac{1}{\varepsilon^2}\int_{T(x_0,r_0)\cap\mathcal{C}} \left(1-|u_\varepsilon|^2\right)^2 dx \leq \frac{\tilde{C}\kappa}{|\log\varepsilon|} \tag{A.6}$$

for some $\tilde{C} > 0$ *depending on* K_0.

Proof. We distinguish two steps.

Step 1. *Proof of* (A.5). Fix $\varepsilon_0 \in (0, 1/2)$ (depending on $\beta_2 - \beta$) such that $\varepsilon_0^{\beta_2-\beta}|\log\varepsilon_0| \leq 1/2$. Assume by contradiction that for every $C \geq K_0$ there exist $x \in \mathcal{C}$ and $\varepsilon \in (0, \varepsilon_0)$ such that for every $r \in (\varepsilon^{\beta_1}, \varepsilon^{\beta_2})$, we have

$$\int_{\partial(T(x_0,r)\cap\mathcal{C})} \mathbf{e}_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \geq \frac{C\kappa}{r|\log\varepsilon|}.$$

By Lemma A.3, we have

$$\int_{\partial\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \leq \frac{K_0\kappa}{\varepsilon^\beta} \leq \frac{K_0\kappa}{2\varepsilon^{\beta_2}|\log\varepsilon|} \leq \frac{C\kappa}{2r|\log\varepsilon|} \quad \forall r \in \left(\varepsilon^{\beta_1}, \varepsilon^{\beta_2}\right).$$

Therefore, we deduce that

$$\int_{\partial T(x_0,r)\cap\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,d\mathcal{H}^1 \geq \int_{\partial(T(x_0,r)\cap\mathcal{C})} \mathbf{e}_\varepsilon(u_\varepsilon)\,d\mathcal{H}^1 - \int_{\partial\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,d\mathcal{H}^1 \geq \frac{C\kappa}{2r|\log\varepsilon|}.$$

Integrating with respect to $r \in (\varepsilon^{\beta_1}, \varepsilon^{\beta_2})$, we obtain

$$K_0\kappa \overset{(A.1)}{\geq} \int_{\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,dx \geq \int_{T(x_0,\varepsilon^{\beta_2})\cap\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,dx \geq \int_{\varepsilon^{\beta_1}}^{\varepsilon^{\beta_2}} dr \int_{\partial T(x_0,r)\cap\mathcal{C}} \mathbf{e}_\varepsilon(u_\varepsilon)\,d\mathcal{H}^1 \geq \frac{C(\beta_1-\beta_2)\kappa}{2},$$

which is a contradiction with the fact that C can be arbitrary large.

Step 2. *Proof of* (A.6). Let $x_0 \in \mathcal{C}$. We use the same argument as in Lemma A.3 involving a Pohozaev identity for the solution u_ε of (A.2) in the domain

$$\mathcal{D} := T(x_0, r_0) \cap \mathcal{C},$$

where r_0 is given in (A.5). Multiplying the equation by $(x-x_0)\cdot\nabla u_\varepsilon$ and integrating by parts, we deduce that

$$\int_{\mathcal{D}} -\Delta u_\varepsilon \cdot \big((x-x_0)\cdot\nabla u_\varepsilon\big)\,dx = \int_{\partial\mathcal{D}} \left(\frac{1}{2}(x-x_0)\cdot\nu|\nabla u_\varepsilon|^2 - \frac{\partial u_\varepsilon}{\partial\nu}\cdot\frac{\partial u_\varepsilon}{\partial(x-x_0)}\right) d\mathcal{H}^1,$$

$$\frac{1}{\varepsilon^2}\int_{\mathcal{D}} u_\varepsilon\big(1-|u_\varepsilon|^2\big)\cdot\big((x-x_0)\cdot\nabla u_\varepsilon\big)\,dx = \frac{1}{2\varepsilon^2}\int_{\mathcal{D}}\big(1-|u_\varepsilon|^2\big)^2\,dx - \frac{1}{4\varepsilon^2}\int_{\partial\mathcal{D}}(x-x_0)\cdot\nu\big(1-|u_\varepsilon|^2\big)^2\,d\mathcal{H}^1.$$

Since $|x-x_0| \leq \sqrt{2}r_0$ on $\partial\mathcal{D}$, by (A.5), we deduce that (A.6) holds true. □

Lemma A.5. *Under the hypothesis of Theorem* A.1*, we have* $\|u_\varepsilon\|_{L^\infty(\mathcal{C})} \leq 1$ *and*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\left(\frac{|x-y|}{\varepsilon} + \frac{|x-y|^{1/2-}}{\varepsilon^{1/2-}}\right) \quad \forall x,y \in \mathcal{C},$$

where $C \geq 1$ *is a universal constant (independent of* K_0*) and* $1/2-$ *is some positive number less than* $1/2$.

Remark A.6. Unlike [2], the estimate $\|\nabla u_\varepsilon\|_{L^\infty(\mathcal{C})} \leq C/\varepsilon$ does not hold in general here since it might already fail for the boundary data g_ε (due to (A.1)). Therefore, the estimate given by Lemma A.5 is the natural one in our situation.

Proof. Let $\rho = 1-|u_\varepsilon|^2$. Then (A.2) implies that

$$-\Delta\rho + \frac{4}{\varepsilon^2}|u_\varepsilon|^2\rho \geq 0 \quad \text{in } \mathcal{C}$$

and

$$\rho = 1 - |g_\varepsilon|^2 \geq 0 \quad \text{on } \partial\mathcal{C}.$$

Thus, the maximal principle implies that $\rho \geq 0$, i.e., $|u_\varepsilon| \leq 1$ on $\partial\mathcal{C}$. For the second estimate, we do the rescaling $U(x) = u_\varepsilon(\varepsilon^\beta x)$ for $x \in \Omega_0 := (0,1)^2$ and $G(x) = g_\varepsilon(\varepsilon^\beta x)$ for $x \in \partial\Omega_0$ and get the equation

$$-\Delta U = \frac{2}{\varepsilon^{2(1-\beta)}} U\big(1 - |U|^2\big)$$

in Ω_0 with $U = G$ on $\partial\Omega_0$. Then we write $U = V + W$ with

$$\begin{cases} -\Delta V = \dfrac{2}{\varepsilon^{2(1-\beta)}}\, U\big(1 - |U|^2\big) & \text{in } \Omega_0,\\ V = 0 & \text{on } \partial\Omega_0, \end{cases}$$

$$\begin{cases} \Delta W = 0 & \text{in } \Omega_0,\\ W = G & \text{on } \partial\Omega_0. \end{cases}$$

In particular, $-\Delta|W|^2 = -2|\nabla W|^2 \leq 0$ in Ω_0; since $|W| \leq 1$ on $\partial\Omega_0$, the maximal principle implies that $|W| \leq 1$ in Ω_0. Because $|U| \leq 1$, we deduce that $|V| \leq 2$ in Ω_0. Using the Gagliardo–Nirenberg inequality, we have

$$\|\nabla V\|_{L^\infty(\Omega_0)} \leq C_0 \|V\|^{1/2}_{L^\infty(\Omega_0)} \|\Delta V\|^{1/2}_{L^\infty(\Omega_0)},$$

so that we obtain

$$\|\nabla V\|_{L^\infty(\Omega_0)} \leq \frac{C}{\varepsilon^{1-\beta}}.$$

In order to have the $C^{0,1/2-}$ estimate for W, we start by noting that

$$\int_{\partial\Omega_0} |\partial_\tau G|^2 \, d\mathcal{H}^1 = \varepsilon^\beta \int_{\partial\mathcal{C}} |\partial_\tau g_\varepsilon|^2 \, d\mathcal{H}^1 \overset{(A.1)}{\leq} K_0\kappa.$$

So, by regularity theory for harmonic functions, we deduce that[11]

$$\|W\|_{\dot{H}^{3/2-}(\Omega_0)} \leq C_0 \|G\|_{\dot{H}^{1-}(\partial\Omega_0)} \leq C(K_0\kappa)^{1/2}.$$

By Sobolev embedding $H^{3/2-}(\Omega_0) \subset C^{0,1/2-}(\Omega_0)$, it follows that

$$\begin{aligned} |W(x) - W(y)| &\leq C|x-y|^{1/2-} \|W\|_{\dot{H}^{3/2-}(\Omega_0)} \\ &\leq C(K_0\kappa)^{1/2} |x-y|^{1/2-} \quad \forall (x,y) \in \Omega_0^2. \end{aligned}$$

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[11] Let us consider for simplicity the following two-dimensional situation: $\Delta W = 0$ for $x_2 \neq 0$ and $W = G$ for $x_2 = 0$. Passing in Fourier transform in x_1, we obtain that $\mathcal{F}(W)(\xi_1, x_2) = e^{-|\xi_1|\,|x_2|}\mathcal{F}(G)(\xi_1)$. Therefore, the Fourier transform in both variables of $\mathbb{R}^2$ of W is given by $\hat{W}(\xi) = \mathcal{F}(G)(\xi_1) \int_{\mathbb{R}} e^{-i\xi_2 x_2} e^{-|\xi_1||x_2|}\, dx_2 = \mathcal{F}(G)(\xi_1)(|\xi_1|/|\xi|^2)$ because $\mathcal{F}\big(x_1 \mapsto 1/\big(1 + x_1^2\big)\big)(\xi_1) = e^{-|x_1|}$. Therefore, $\|W\|_{\dot{H}^{3/2-}(\mathbb{R}^2)} \sim \|G\|_{\dot{H}^{1-}(\mathbb{R})}$.

Therefore, we obtain

$$\begin{aligned}|U(x)-U(y)| &\le |V(x)-V(y)|+|W(x)-W(y)|\\ &\le C\left(\frac{|x-y|}{\varepsilon^{1-\beta}}+\frac{|x-y|^{1/2-}}{\varepsilon^{\frac{1-\beta}{2}-}}(K_0\kappa)^{1/2}\varepsilon^{\frac{1-\beta}{2}-}\right) \quad \forall (x,y)\in\Omega_0^2.\end{aligned}$$

Scaling back, we obtain the desired estimate for u_ε in $\mathcal{C}$ since

$$(K_0\kappa)^{1/2}\varepsilon^{\frac{1-\beta}{2}-}=o(1). \qquad \square$$

Proof of Theorem A.1. We will show that

$$\||u_\varepsilon|^2-1\|_{L^\infty(\mathcal{C})}\le C\left(\frac{\kappa}{|\log\varepsilon|}\right)^{1/6-}.$$

Let $x_0\in\mathcal{C}$ such that $|u_\varepsilon(x_0)|<1$. Set $0<A<1$ such that

$$2C\big(2A+(2A)^{1/2-}\big)=\frac{\big(1-|u_\varepsilon(x_0)|^2\big)}{2}>0,$$

where C is given by Lemma A.5. In particular,

$$A^{1/2-}\ge A\ge C_1\big(1-|u_\varepsilon(x_0)|^2\big)^{2+}.$$

By Lemma A.5, we obtain for any $y\in T(x_0,A\varepsilon)\cap\mathcal{C}$: $|y-x_0|\le 2A\varepsilon$ and

$$1-|u_\varepsilon(y)|^2\ge 1-|u_\varepsilon(x_0)|^2-2C\left(2A+(2A)^{1/2-}\right)=\frac{1-|u_\varepsilon(x_0)|^2}{2}.$$

Hence, for small ε, we have $A\varepsilon<\varepsilon\le\varepsilon^{\beta_1}\le r_0$ (with r_0 given in (A.5)) and

$$\begin{aligned}\frac{\tilde{C}\kappa\varepsilon^2}{|\log\varepsilon|} &\overset{(A.6)}{\ge} \int_{T(x_0,A\varepsilon)\cap\mathcal{C}}\big(1-|u_\varepsilon(y)|^2\big)^2\,dy\\ &\ge \frac{1}{16}A^2\varepsilon^2\big(1-|u_\varepsilon(x_0)|^2\big)^2\\ &= \frac{1}{16}C_1^2\varepsilon^2\big(1-|u_\varepsilon(x_0)|^2\big)^{6+}.\end{aligned}$$

Thus, we conclude that

$$\big(1-|u_\varepsilon(x_0)|^2\big)^{6+}\le\hat{C}\frac{\kappa}{|\log\varepsilon|}. \qquad \square$$

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