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Abstract weak Harnack inequality, multiple fixed points and p-Laplace equations

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Abstract. This paper presents an abstract theory for the existence, localization and multiplicity of fixed points in a cone. The key assumption is the property of the nonlinear operator of satisfying an inequality of Harnack type. In particular, the theory offers a completely new approach to the problem of positive solutions of quasilinear elliptic equations with p-Laplace operator.

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1. Introduction

The main goal of this paper is to investigate by a completely new method the existence of multiple positive weak solutions of the Dirichlet problem for quasilinear elliptic equations involving the p-Laplace operator, namely

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$, $1 < p < n$ and

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$$

stands for the usual p-Laplacian. This problem has been investigated in a number of papers mainly using variational principles, fixed point and continuation methods or upper and lower solution techniques (see [1, 3, 4, 7, 8, 16, 17, 19, 23, 24, 25] and the references therein). Our new approach is based on fixed point index and compression-expansion-type properties with respect to the norm and a seminorm in $L^\infty(\Omega)$. Compression-expansion arguments, like those in the popular Krasnoselskii's fixed point theorems in cones [11], have been extensively used in the literature in order to obtain existence, localization and multiplicity results for numerous classes of problems involving integral and ordinary differential equations (see, e.g., [5, 12, 14, 18]). However, less have been obtained this way for partial differential equations (except the cases when problems reduce to ordinary differential equations, for instance, the radial solution case [27]). As pointed out in [20], the reason is the bad representation of the "integral" inverse mappings associated with the partial differential operators, in contrast to the case of ordinary differential equations, where the corresponding inverse mappings admit integral representations in terms of Green functions with good properties of bilateral estimation. In [20], these good properties of the Green functions have been put in connection with global weak Harnack inequalities. To be more explicit, assume that Lu is an ordinary differential operator subjected to some boundary conditions on a bounded open interval (a,b) and that S is its inverse, i.e., $S=L^{-1}$. If S can be represented as an integral mapping

$$(Sf)(x)=\int_a^b G(x,y)\,f(y)\,dy,\quad x\in(a,b),$$

where $G\geq 0$ and G enjoys bilateral estimations

$$\begin{aligned} MG(y,y)&\leq G(x,y)\quad \text{for } x\in(c,d),\ y\in(a,b),\\ G(x,y)&\leq G(y,y)\quad \text{for all } x,y\in(a,b)\end{aligned}$$

for some positive M and subinterval $(c,d)\subset(a,b)$, then for every $f\geq 0$, $x\in(c,d)$ and $x^*\in(a,b)$, we have

$$\begin{aligned}(Sf)(x)&=\int_a^b G(x,y)f(y)dy\geq M\int_a^b G(y,y)f(y)dy\\ &\geq M\int_a^b G(x^*,y)f(y)dy=M(Sf)(x^*).\end{aligned}$$

This gives the global Harnack-type inequality on $[c,d]$,

$$(Sf)(x)\geq M\sup_{[a,b]}Sf,\quad x\in[c,d], \tag{1.2}$$

which is crucial for the applicability of compression-expansion theorems. Notice that (1.2) can be written in the form

$$Sf\geq|Sf|_\infty\,\chi\quad\text{on }[a,b],$$

where $|Sf|_\infty=\sup_{[a,b]}Sf$ and $\chi=M\chi_{[c,d]}$, $\chi_{[c,d]}$ being the characteristic function of subinterval $[c,d]$. The inequality is said to be *global* since the sup in the right-hand side is taken over the whole interval $[a,b]$. Note that inequality (1.2) refers to the *supersolutions* of the equation $Lu=0$, that is, to functions u with $Lu\geq 0$. Unfortunately, in several dimensions, such properties of the Green functions, and consequently global Harnack inequalities for supersolutions, are not known making Krasnoselskii's theorems commonly unused for partial differential equations. Fortunately, as we have recently shown in [21] for elliptic problems with common Laplacian (see also [22]), local weak Harnack inequalities (see, e.g., [9] and [10])—also known in the literature as Moser–Harnack inequalities—are enough to be used together with variants of Krasnoselskii's theorems in a conical "annular" domain jointly defined by norm and a seminorm. Once such a fundamental remark has been made, one may expect that the same strategy could be applied for quasilinear elliptic equations with p-Laplace operator, and furthermore, to other classes of equations for which a local weak Harnack inequality holds. Thus, instead of iterating the same strategy to more and more classes of problems, an abstract unified theory appears to be useful. This is the goal of Section 2. The theory is then illustrated in Section 3 to the case of p-Laplace equations.

2. Main abstract results

Let X be a Banach space with norm $|\cdot|$, ordered by a cone K_0, and let $\|\cdot\|$ be a continuous seminorm on X. Denote by $\leq$ the partial order relation associated with K_0, given by $u \leq v$ if and only if $v - u \in K_0$ and assume that both norm and seminorm are monotone; i.e.,

$$0 \leq u \leq v \quad \text{implies} \quad |u| \leq |v| \text{ and } \|u\| \leq \|v\|.$$

Consider an operator equation

$$u = Nu, \quad u \in K_0,$$

where the operator $N : K_0 \to K_0$ is completely continuous. Assume that there are elements $\phi, \chi \in K_0$, $\chi \neq 0$, such that

$$|\phi| = 1, \quad \phi \geq \|\phi\| \, \chi. \tag{2.1}$$

Also assume that for N, the following *abstract weak Harnack inequality* holds:

$$Nu \geq \|Nu\| \, \chi \quad \text{for every } u \in K_0. \tag{2.2}$$

Relating to inequality (2.2), define a cone K, smaller than K_0, by

$$K := \{u \in K_0 : u \geq \|u\| \, \chi\}.$$

Also, for any numbers $R_0 \geq 0$ and $R_1 > 0$, consider the conical "annular" region

$$K_{R_0 R_1} := \{u \in K : R_0 \leq \|u\| \text{ and } |u| \leq R_1\}.$$

Theorem 2.1. *Assume that the above conditions are satisfied and in addition that there are numbers $R_0 \geq 0$ and $R_1 > 0$ with*

$$R_0 < \|\phi\| \, \|\chi\| \, R_1 \tag{2.3}$$

such that

$$\inf_{\substack{u \in K_{R_0 R_1} \\ \|u\| = R_0}} \|N(u)\| \geq \frac{R_0}{\|\chi\|} \tag{2.4}$$

and

$$\sup_{|u| \leq R_1} |N(u)| \leq R_1. \tag{2.5}$$

Then N has at least one fixed point $u \in K_{R_0 R_1}$.

Proof. Since K_0 is a cone and $\|\cdot\|$ is a seminorm, we can immediately see that K is a cone too. Also, from (2.1), $\phi \neq 0$ and $\phi \in K$. Hence $K \neq \{0\}$. Next, observe that inequality (2.2) guarantees $N(K) \subset K$. Furthermore, (2.5) shows that $|Nu| \leq R_1$ for every $u \in K$ with $|u| \leq R_1$. Thus, if we denote $C := \{u \in K : |u| \leq R_1\}$, we have $N(C) \subset C$. Consequently,

$$i(N, C) = 1. \tag{2.6}$$

Here, for any subset $U \subset C$ open in C, we denote by $i(N, U)$ the fixed point index of N over U with respect to C (see, e.g., [2, p. 238] and [6]). If $R_0 = 0$ or N has a fixed point $u \in C$ with $\|u\| = R_0$, we are done. Hence assume $R_0 > 0$ and

$$Nu \neq u \quad \text{for every } u \in C, \ \|u\| = R_0. \tag{2.7}$$

Consider a subset of C, namely

$$U := \{u \in C : \|u\| < R_0\}.$$

Since the seminorm $\|\cdot\|$ is continuous, we have that U is open in C and its boundary is $\partial U = \{u \in C : \|u\| = R_0\}$. Now take $h := R_1\phi$. Clearly $|h| = R_1$. We prove that the following boundary condition is satisfied:

$$u \neq \lambda h + (1 - \lambda)\, Nu \quad \text{for every } u \in \partial U \tag{2.8}$$

and $\lambda \in (0, 1]$. Assume the contrary. Then for some $\lambda \in (0, 1]$ and $u \in \partial U$,

$$\begin{aligned} u = \lambda h + (1-\lambda)\, Nu &\geq \lambda \|h\|\, \chi + (1-\lambda)\, \|N(u)\| \chi \\ &\geq \left(\lambda R_1 \|\phi\| + (1-\lambda) \inf_{\substack{u \in K_{R_0 R_1} \\ \|u\| = R_0}} \|N(u)\| \right) \chi. \end{aligned} \tag{2.9}$$

This implies that

$$R_0 \geq \left(\lambda R_1 \|\phi\| + (1-\lambda) \inf_{\substack{u \in K_{R_0 R_1} \\ \|u\| = R_0}} \|N(u)\| \right) \|\chi\|,$$

which in view of (2.3) gives

$$R_0 > \lambda R_0 + (1-\lambda) \inf_{\substack{u \in K_{R_0 R_1} \\ \|u\| = R_0}} \|N(u)\|\, \|\chi\|,$$

which yields a contradiction with inequality (2.4). Thus (2.8) holds for every $\lambda \in (0, 1]$. According to (2.7), (2.8) also holds for $\lambda = 0$. Thus the convex combination $\lambda h + (1-\lambda)\, Nu$ is an admissible homotopy on $\overline{U}$. Also, from (2.1), we see that $\|\chi\| \leq 1$ and then by (2.3),

$$\|h\| = R_1 \|\phi\| \geq R_1 \|\phi\|\, \|\chi\| > R_0.$$

Hence $h \in C \setminus \overline{U}$. Consequently,

$$i\,(N, U) = i\,(h, U) = 0. \tag{2.10}$$

Now (2.6) and (2.10) yield

$$i\left(N, C\setminus\overline{U}\right) = i\,(N,C) - i\,(N,U) = 1.$$

Thus N has a fixed point u in $C\setminus\overline{U}$; i.e., $R_0 < \|u\|$ and $|u| \le R_1$. □

Remark 2.1. If (2.4) holds with strict inequality, then condition (2.7) is satisfied. Indeed, otherwise, $u = Nu$ for some $u \in C$ with $\|u\| = R_0$. Then

$$\frac{R_0}{\|\chi\|} \ge R_0 = \|u\| = \|Nu\| \ge \inf_{\substack{u\in K_{R_0 R_1}\\ \|u\|=R_0}} \|N(u)\|,$$

which yields a contradiction with (2.4).

For the next results we shall assume that there is a constant $c_0 > 0$ such that

$$\|u\| \le c_0\,|u| \quad \text{for all } u \in K_0. \tag{2.11}$$

The next theorem is a three-solution existence result.

Theorem 2.2. *Under the assumptions of Theorem* 2.1*, if in addition the inequality in* (2.4) *is strict and there exists a number* R_{-1} *with*

$$0 < R_{-1} < \frac{1}{c_0} R_0 \tag{2.12}$$

and

$$Nu \ne \lambda u \quad \textit{for } |u| = R_{-1},\ \lambda \ge 1, \tag{2.13}$$

then N has three fixed points u_1, u_2, u_3 with

$$R_0 < \|u_1\|\,,\ |u_1| \le R_1; \quad R_{-1} < |u_2| \le R_1,\ \|u_2\| < R_0; \quad |u_3| < R_{-1}.$$

Proof. Theorem 2.1 and (2.7) guarantee a fixed point u_1 with $R_0 < \|u_1\|$, $|u_1| \le R_1$. Also (2.13) implies $i(N, U_0) = 1$, where $U_0 := \{u \in K : |u| < R_{-1}\}$. Hence a second fixed point u_3 exists in U_0. Finally, from (2.11) and (2.12) we have $\overline{U_0} \subset U$, and so

$$i\left(N, U\setminus\overline{U_0}\right) = i\,(N,U) - i\,(N,U_0) = 0 - 1 = -1,$$

whence a third fixed point u_2 in $U\setminus\overline{U_0}$. □

Theorem 2.1 immediately yields multiple-solution results.

Theorem 2.3. (A) *Let* $\left(R_0^i\right)_{1\le i\le k}$, $\left(R_1^i\right)_{1\le i\le k}$ $(k \le \infty)$ *be increasing finite or infinite sequences with* $0 \le R_0^i < \|\phi\|\,\|\chi\|\,R_1^i$ *and* $c_0 R_1^i < R_0^{i+1}$ *for all i. If the assumptions of Theorem* 2.1 *are satisfied for each couple* $\left(R_0^i, R_1^i\right)$*, then N has k (respectively, when $k = \infty$, an infinite sequence of) distinct fixed points u_i with*

$$R_0^i \le \|u_i\|\,, \quad |u_i| \le R_1^i. \tag{2.14}$$

(B) *Let* $(R_0^i)_{i\ge1}$, $(R_1^i)_{i\ge1}$ *be decreasing infinite sequences with* $0 < R_0^i < \|\phi\|\,\|\chi\|\,R_1^i$ *and* $c_0 R_1^{i+1} < R_0^i$ *for all i. If the assumptions of Theorem* 2.1 *are satisfied for each couple* (R_0^i, R_1^i)*, then N has an infinite sequence of distinct fixed points u_i satisfying* (2.14).

Proof. It is sufficient to remark that

$$K_{R_0^i R_1^i} \cap K_{R_0^{i+1} R_1^{i+1}} = \emptyset \quad \text{for all } i.$$

To prove this, let us first assume that the sequences (R_0^i), (R_1^i) are increasing. Then $K_{R_0^i R_1^i} \subset \{u \in K : \|u\| < R_0^{i+1}\}$. Indeed, if $u \in K_{R_0^i R_1^i}$, then $\|u\| \leq c_0 |u| \leq c_0 R_1^i < R_0^{i+1}$, whence $\|u\| < R_0^{i+1}$. Similarly, if the sequences (R_0^i), (R_1^i) are decreasing, one has $K_{R_0^{i+1} R_1^{i+1}} \subset \{u \in K : \|u\| < R_0^i\}$ since any $u \in K_{R_0^{i+1} R_1^{i+1}}$ satisfies $\|u\| \leq c_0 |u| \leq c_0 R_1^{i+1} < R_0^i$. □

Remark 2.2. Taking into account Remark 2.1, if the inequality in (2.4) is strict for each R_0^i, then in Theorem 2.3(A) there exists an additional sequence $(v_i)_{1 \leq i \leq k}$ of fixed points of N with

$$\|v_i\| < R_0^{i+1}, \quad |v_i| > R_1^i, \quad 1 \leq i \leq k-1.$$

Similarly, in Theorem 2.3(B), there exists an additional sequence $(v_i)_{i \geq 1}$ of fixed points of N with

$$\|v_i\| < R_0^i, \quad |v_i| > R_1^{i+1}, \quad i \geq 1.$$

2.1. Case of Hammerstein operator equations

In applications, the operator N appears as the composition SF of two mappings (a Hammerstein-type operator) and it is useful that the abstract weak Harnack inequality (2.2) is fulfilled provided that S satisfies such kind of condition. More precisely, if $S, F : K_0 \to K_0$ (hence are both positive operators) and

$$Sv \geq \|Sv\| \chi \quad \text{for every } v \in K_0, \tag{2.15}$$

then inequality (2.2) holds for $N = SF$. Indeed, if $u \in K_0$, then $Fu \in K_0$ and applied to $v := Fu$, (2.15) yields (2.2).

2.2. Case of isotone operators

A more workable form of conditions (2.4) and (2.5) from Theorem 2.1 is possible when N is isotone; that is,

$$0 \leq u \leq v \quad \text{implies} \quad Nu \leq Nv,$$

and there exists an element $\psi \in K_0$ such that

$$u \leq |u| \psi \quad \text{for all } u \in K_0. \tag{2.16}$$

Then if $u \in K$ and $|u| \leq R_1$, one has $u \leq |u| \psi \leq R_1 \psi$, whence $Nu \leq N(R_1 \psi)$. Consequently, $|Nu| \leq |N(R_1 \psi)|$, and condition (2.5) holds if

$$|N(R_1 \psi)| \leq R_1. \tag{2.17}$$

Also, if $u \in K_{R_0 R_1}$ and $\|u\| = R_0$, then $R_0 \chi = \|u\| \chi \leq u$, whence $N(R_0 \chi) \leq Nu$ and so $\|N(R_0 \chi)\| \leq \|Nu\|$. Hence (2.4) holds if

$$\|N(R_0 \chi)\| \geq \frac{R_0}{\|\chi\|}, \tag{2.18}$$

and the inequality in (2.4) is strict provided that (2.18) holds with strict inequality. Now Theorems 2.1, 2.2 and 2.3 yield the following results.

Corollary 2.4. *Assume that* (2.11) *and* (2.16) *hold and that* N *is isotone.*

(i) *If*

$$\limsup_{r\to 0} \frac{\|N(r\chi)\|}{r} > \frac{1}{\|\chi\|} \quad \textit{and} \quad \liminf_{r\to\infty} \frac{|N(r\psi)|}{r} < 1, \tag{2.19}$$

then N *has at least one fixed point* $u \in K_0 \setminus \{0\}$.

(ii) *If there exists* $R_0 > 0$ *satisfying* (2.18) *with strict inequality,*

$$\liminf_{r\to 0} \frac{|N(r\psi)|}{r} < 1 \quad \textit{and} \quad \liminf_{r\to\infty} \frac{|N(r\psi)|}{r} < 1, \tag{2.20}$$

then N *has at least two fixed points* $u, v \in K_0 \setminus \{0\}$ *with* $\|u\| < R_0$, $\|v\| > R_0$. *A third solution* $w \in K_0 \setminus \{0\}$ *is guaranteed in case that* $N(0) \neq 0$.

(iii) *If*

$$\limsup_{r\to\infty} \frac{\|N(r\chi)\|}{r} > \frac{1}{\|\chi\|} \quad \textit{and} \quad \liminf_{r\to\infty} \frac{|N(r\psi)|}{r} < 1, \tag{2.21}$$

then N *has a sequence of fixed points* $u_k \in K_0$ *with* $\|u_k\| \to \infty$ *as* $k \to \infty$.

(iv) *If*

$$\limsup_{r\to 0} \frac{\|N(r\chi)\|}{r} > \frac{1}{\|\chi\|} \quad \textit{and} \quad \liminf_{r\to 0} \frac{|N(r\psi)|}{r} < 1, \tag{2.22}$$

then N *has a sequence of fixed points* $u_k \in K_0 \setminus \{0\}$ *with* $|u_k| \to 0$ *as* $k \to \infty$.

Proof. Clearly (2.19) implies the existence of two positive numbers R_0, R_1 satisfying (2.18), (2.17) and (2.3). Hence Theorem 2.1 applies. Next, from (2.20) there are positive numbers R_{-1} (small enough) and R_1 (large enough) such that all the assumptions of Theorem 2.2 are satisfied. Finally (iii) and (iv) follow from Theorem 2.3. □

3. Multiple positive solutions of p-Laplace equations

In this section, the abstract results from Section 2 are applied to the boundary value problem (1.1). First we recall some basic results concerning the inverse of the operator $-\Delta_p$ under the Dirichlet boundary condition. The first one gathers together earlier contributions of several authors.

Lemma 3.1 (See [1]**).** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain of class* $C^{1,\beta}$ *for some* $\beta \in (0,1)$ *and* $v \in L^\infty(\Omega)$. *Then the problem*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx = \int_\Omega vw \, dx \quad \textit{for all } w \in C_0^\infty(\Omega), \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{3.1}$$

has a unique solution $u \in C_0^1(\overline{\Omega})$. *Moreover, if we define the operator* $S : L^\infty(\Omega) \to C_0^1(\overline{\Omega})$, $v \mapsto u$ *where* u *is the unique solution of* (3.1), *then* S *is completely continuous, isotone and* $Sv > 0$ *in* Ω *for every* $v \geq 0$, $v \neq 0$.

Thus, if $f:\overline{\Omega}\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous, then problem (1.1) is equivalent to the fixed point problem

$$u=SFu,\quad u\in K_0,$$

where $X=L^\infty(\Omega)$ with sup norm $|\cdot|=|\cdot|_\infty$, $K_0:=\{u\in L^\infty(\Omega):u\geq 0\}$ and $F:K_0\to K_0$ is the superposition operator given by $(Fu)(x)=f(x,u(x))$, $x\in\Omega$. If in addition $f(x,u)$ is nondecreasing in u for every x, then both F and S are isotone. Also the complete continuity of S guarantees the same property for the Hammerstein-type operator $N=SF$, and (2.16) is true with $\psi\equiv 1$. In what follows, for simplicity, we shall assume that $f(x,u)$ is nondecreasing in u for every x.

Also recall the local weak Harnack inequality for nonnegative p-superharmonic functions, due to Trudinger [26, Theorem 1.2] (see also Lindqvist [13, Corollary 3.18]) and anticipated by Moser [15].

Lemma 3.2 (Trudinger [26]**).** *For each $q\in\left[1,\frac{n(p-1)}{n-p}\right)$ and $\rho>0$ such that $B_{3\rho}\subset\Omega$, there exists a constant $C=C(n,p,q)>0$ such that*

$$\inf_{B_\rho} u\geq C\rho^{-\frac{n}{q}}\left(\int_{B_{2\rho}}u^q dx\right)^{\frac{1}{q}}$$

for every nonnegative p-superharmonic function u, i.e., nonnegative weak solution of the inequality $-\Delta_p u\geq 0$ in Ω.

Following a standard reasoning based on finite cover by balls of any compact set (see the proof of Theorem 1.3 in [21]), we immediately obtain the following variant.

Lemma 3.3. *For each $q\in\left[1,\frac{n(p-1)}{n-p}\right)$ and each compact $D\subset\Omega$, there is a constant $M=M(n,p,q,D,\Omega)>0$ such that for every nonnegative p-superharmonic function u in Ω, the following inequality holds:*

$$\inf_D u\geq M\left(\int_D u^q dx\right)^{\frac{1}{q}}.$$

Since for each $v\geq 0$, the function Sv is nonnegative and p-superharmonic in Ω, according to Lemma 3.3, inequality (2.15) is satisfied, with the seminorm $\|\cdot\|$ defined by

$$\|u\|=M\left(\int_D|u(x)|^q dx\right)^{\frac{1}{q}},\quad u\in L^\infty(\Omega),$$

D and q being fixed, and with $\chi=\chi_D$, where χ_D is the characteristic function of D, i.e., $\chi_D(x)=1$ for $x\in D$ and $\chi_D(x)=0$ on $\Omega\setminus D$. Notice that (2.11) holds with $c_0=M(\operatorname{mes}D)^{1/q}$. Also note that for (2.1), one can choose $\phi:=\frac{S1}{|S1|_\infty}$.

Denote

$$l_0 := \liminf_{r\to 0} \frac{\max_{x\in\overline{\Omega}} f(x,r)}{r^{p-1}}, \quad l_\infty := \liminf_{r\to\infty} \frac{\max_{x\in\overline{\Omega}} f(x,r)}{r^{p-1}},$$

$$L_0 := \limsup_{r\to 0} \frac{\min_{x\in D} f(x,r)}{r^{p-1}}, \quad L_\infty := \limsup_{r\to\infty} \frac{\min_{x\in D} f(x,r)}{r^{p-1}},$$

and notice that, in particular if $f(x,u) = f(u)$, then

$$l_0 = \liminf_{r\to 0} \frac{f(r)}{r^{p-1}}, \quad l_\infty = \liminf_{r\to\infty} \frac{f(r)}{r^{p-1}},$$

$$L_0 = \limsup_{r\to 0} \frac{f(r)}{r^{p-1}}, \quad L_\infty = \limsup_{r\to\infty} \frac{f(r)}{r^{p-1}}.$$

Also denote

$$A := \frac{1}{|S1|^{p-1}} \quad \text{and} \quad B := \frac{1}{(\|\chi\| \, \|S1\|)^{p-1}}.$$

Theorem 3.4. (a) *If*

$$l_\infty < A \quad \textit{and} \quad L_0 > B,$$

then (1.1) *has at least one solution.*

(b) *If*

$$l_\infty < A \quad \textit{and} \quad L_\infty > B,$$

then (1.1) *has a sequence of solutions* (u_k) *with* $|u_k|_\infty \to \infty$ *as* $k \to \infty$.

(c) *If*

$$l_0 < A \quad \textit{and} \quad L_0 > B,$$

then (1.1) *has a sequence of solutions* (u_k) *with* $u_k \to 0$ *as* $k \to \infty$.

Proof. The proof key is the following simple formula:

$$Sr = r^{\frac{1}{p-1}} S1, \quad r > 0$$

(here 1 and r stand for constant functions). This ensures that for any $r > 0$,

$$\begin{aligned} \|N(r\chi)\| &= M\left(\int_D SF(r)^q \, dx\right)^{\frac{1}{q}} \\ &\geq \left(\min_D f(x,r)\right)^{\frac{1}{p-1}} \|S1\| \end{aligned} \tag{3.2}$$

and

$$|N(r\psi)| = |N(r)| \leq \left(\max_{\overline{\Omega}} f(x,r)\right)^{\frac{1}{p-1}} |S1|. \tag{3.3}$$

On this base, the assumption of (a) guarantees (2.19), the assumption of (b) implies (2.21) and that of (c) guarantees (2.22). Thus the conclusions follow from Corollary 2.4. □

We emphasize that the conclusion (a) from Theorem 3.4, when $l_\infty = 0$ and $L_0 = \infty$, also follows from a result by Hai and Wang [8]. Also note the quasisimilarity of results (b) and (c), for $l_\infty = 0$, $L_\infty = \infty$ and $l_0 = 0$, $L_0 = \infty$, respectively, to those obtained by Omari and Zanolin [16, 17] using a totally different approach.

Finally, we shall discuss the parametrized problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\lambda > 0$.

Theorem 3.5. (i) *If*

$$L_0 = \infty,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (0, \lambda^*)$, (3.4) *has at least one solution.*

(ii) *If*

$$l_\infty = 0,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$, (3.4) *has at least one solution.*

(iii) *If*

$$l_0 = l_\infty = 0,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$, (3.4) *has at least two solutions. A third solution is guaranteed in case that* $f(\cdot, 0) \neq 0$.

(iv) *If*

$$l_\infty = 0 \quad \textit{and} \quad L_\infty > 0,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$, (3.4) *has a sequence* (u_k) *of solutions with* $|u_k| \to \infty$ *as* $k \to \infty$.

(v) *If*

$$l_\infty < \infty \quad \textit{and} \quad L_\infty = \infty,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (0, \lambda^*)$, (3.4) *has a sequence* (u_k) *of solutions with* $|u_k| \to \infty$ *as* $k \to \infty$.

(vi) *If*

$$l_0 = 0 \quad \textit{and} \quad L_0 > 0,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$, (3.4) *has a sequence* (u_k) *of solutions with* $u_k \to 0$ *as* $k \to \infty$.

(vii) *If*

$$l_0 < \infty \quad \textit{and} \quad L_0 = \infty,$$

then there exists a $\lambda^* > 0$ *such that for each* $\lambda \in (0, \lambda^*)$, (3.4) *has a sequence* (u_k) *of solutions with* $u_k \to 0$ *as* $k \to \infty$.

Proof. We shall take into account formulas (3.2) and (3.3).

(i) The hypothesis guarantees the existence of a number $R_0 > 0$ (sufficiently small) with property (2.4). Next, fix any $R_1 > R_0 / \|\phi\| \|\chi\|$. Clearly condition (2.5) is fulfilled if λ is sufficiently small. Now apply Theorem 2.1.

(ii) The hypothesis guarantees the existence of a number $R_1 > 0$ (sufficiently large) with property (2.5). Next, fix any $R_0 < \|\phi\| \|\chi\| R_1$. Clearly condition (2.4) is fulfilled if λ is sufficiently large. Now Theorem 2.1 yields the result.

(iii) From the assumption it follows that there are positive numbers R_{-1} (small enough) and R_1 (large enough) such that condition (2.5) is satisfied for both R_{-1} and R_1. Next, we fix a number R_0 such that (2.3) and (2.12) hold. Finally, condition (2.4) with strict inequality holds for every sufficiently large λ and thus Theorem 2.2 applies.

(iv) Since $L_\infty > 0$, we can find a sufficiently large number $\lambda^* > 0$ and an increasing sequence $\left(R_0^i\right)_i$ tending to infinity, such that condition (2.4) is satisfied for every R_0^i and every $\lambda > \lambda^*$. Next, in view of $l_\infty = 0$, for each $\lambda > \lambda^*$, we can find an increasing sequence $\left(R_1^i\right)_i$ satisfying (2.5) and $R_0^i < \|\phi\| \, \|\chi\| \, R_1^i$. Passing eventually to subsequences, we may suppose that $c_0 R_1^i < R_0^{i+1}$ for every i. Thus Theorem 2.3(A) applies.

(v) Since $l_\infty < \infty$, we can find a sufficiently small number $\lambda^* > 0$ and an increasing sequence $\left(R_1^i\right)_i$ tending to infinity, such that condition (2.5) is satisfied for every R_1^i and every $\lambda \in (0, \lambda^*)$. Next, in view of $L_\infty = \infty$, for each $\lambda \in (0, \lambda^*)$, we can find a sequence $\left(R_0^i\right)_i$ satisfying (2.4) and $c_0 R_1^i < R_0^{i+1}$. Passing eventually to subsequences, we may suppose that $R_0^i < \|\phi\| \, \|\chi\| \, R_1^i$ for every i. Thus Theorem 2.3(A) applies.

The proofs of (vi) and (vii) are omitted being somewhat similar to those of (iv) and (v). □

We note that the result in (i) can be deduced from the paper by Hai and Wang [8]. Also the existence of two positive solutions under the assumption of (iii) was previously established by Hai [7].

Example 3.1. We present an example of a nondecreasing function $f(x)$ with $0 \le l_\infty < A$ and $B < L_\infty \le \infty$, which illustrates condition (b) in Theorem 3.4 and conditions (iv) and (v) in Theorem 3.5. Let $(a_k)_{k\ge1}$, $(b_k)_{k\ge1}$ be two increasing sequences of positive numbers tending to infinity as $k \to \infty$, $(c_k)_{k\ge1}$ a nondecreasing sequence of positive numbers tending to some $\beta \le \infty$, and let $(d_k)_{k\ge0}$ be a nonincreasing sequence of positive numbers tending to some α with $0 \le \alpha < \beta$. Assume that $c_1 \ge d_0$ and

$$a_{k+1} > b_k > a_k \frac{c_k}{d_k} \quad \text{for every } k \ge 1. \tag{3.5}$$

For instance, $a_k = (k+1)!$, $b_k = (k+1)!\,(k+1)$, $c_k = \sqrt{k}$, $d_k = \frac{1}{\sqrt{k}}$ $(d_0 = 1)$ are such sequences. Note that from $c_1 \ge d_0$, since (c_k) is nondecreasing and (d_k) is nonincreasing, we have

$$c_k \ge c_1 \ge d_0 \ge d_{k-1} \ge d_k, \quad k \ge 1. \tag{3.6}$$

Hence $\frac{c_k}{d_k} \ge 1$ and by (3.5), $b_k > a_k$ for all $k \ge 1$. Let $b_0 = 0$ and consider the function $g : \mathbb{R}_+ \to (0, \infty)$,

$$g(x) = \begin{cases} \dfrac{b_k d_k - a_k c_k}{b_k - a_k} + \dfrac{a_k b_k (c_k - d_k)}{x (b_k - a_k)} & \text{in } [a_k, b_k], \\[2ex] \dfrac{c_k - d_{k-1}}{a_k - b_{k-1}} (x - b_{k-1}) + d_{k-1} & \text{in } [b_{k-1}, a_k], \end{cases} \qquad k \ge 1.$$

Define

$$f(x) = x^{p-1} g(x) \quad \text{for a given } p \geq 2.$$

Clearly f is continuous on $\mathbb{R}_+$. To show that f is nondecresing we have to check the inequality

$$(p-1)\, g(x) + x g'(x) \geq 0 \tag{3.7}$$

on every interval (b_{k-1}, a_k) and (a_k, b_k) for every $k \geq 1$. For $x \in (b_{k-1}, a_k)$, this is clear since from (3.5) and (3.6), $g'(x) = \frac{c_k - d_{k-1}}{a_k - b_{k-1}} \geq 0$. For $x \in (a_k, b_k)$, $g'(x) = -\frac{a_k b_k (c_k - d_k)}{x^2 (b_k - a_k)} \leq 0$ and (3.7) reduces to

$$(p-1)(b_k d_k - a_k c_k) + (p-2) \frac{a_k b_k (c_k - d_k)}{x} \geq 0,$$

which is true in view of the last inequality in (3.5). Thus f is nondecreasing. On the other hand,

$$\frac{f(a_k)}{a_k^{p-1}} = g(a_k) = c_k \to \beta \qquad \text{and} \qquad \frac{f(b_k)}{b_k^{p-1}} = g(b_k) = d_k \to \alpha,$$

which show that

$$\beta \leq L_\infty \leq \infty \quad \text{and} \quad 0 \leq l_\infty \leq \alpha.$$

Example 3.2. If f is the function from Example 3.1, then the nondecreasing function $\widetilde{f}$ given by

$$\widetilde{f}(x) = \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{1}{f\left(\frac{1}{x}\right)} & \text{if } x > 0 \end{cases}$$

is like in condition (c) from Theorem 3.4 and in conditions (vi) and (vii) from Theorem 3.5. Indeed, one has

$$\frac{\widetilde{f}\left(\frac{1}{a_k}\right)}{\left(\frac{1}{a_k}\right)^{p-1}} = \frac{a_k^{p-1}}{f(a_k)} = \frac{1}{c_k} \to \frac{1}{\beta} \qquad \text{and} \qquad \frac{\widetilde{f}\left(\frac{1}{b_k}\right)}{\left(\frac{1}{b_k}\right)^{p-1}} = \frac{b_k^{p-1}}{f(b_k)} = \frac{1}{d_k} \to \frac{1}{\alpha}.$$

Since both sequences $\left(\frac{1}{a_k}\right)$, $\left(\frac{1}{b_k}\right)$ tend to zero, these show that

$$0 \leq l_0 \leq \frac{1}{\beta} \quad \text{and} \quad \frac{1}{\alpha} \leq L_0 \leq \infty.$$

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