

Existence of solitons in the nonlinear beam equation

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To Dick Palais on the occasion of his 80th birthday

Abstract. This paper concerns the existence of solitons, namely stable solitary waves in the nonlinear beam equation with a suitable nonlinearity. An equation of this type has been introduced in [P. J. McKenna and W. Walter, *Arch. Ration. Mech. Anal.*, 98 (1987), 167–177] as a model of a suspension bridge. We prove both the existence of solitary waves for a large class of nonlinearities and their stability. As far as we know this is the first result about stability of solitary waves in nonlinear beam equation.

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1. Introduction

Let us consider the nonlinear beam equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + W'(u) = 0, \quad (1)$$

where $u = u(t, x)$ and $W \in C^1(\mathbb{R})$. In this paper, we prove that, under suitable assumptions, equation (1) admits soliton solutions. Roughly speaking, a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some form of stability so that it has a particle-like behavior (see, e.g., [3] or [5]). Following [3], a soliton or solitary wave is called *hylomorphic* if its stability is due to a particular ratio between *energy* E and the *hyleneic charge* C which is another integral of motion. More precisely, a soliton \mathbf{u}_0 is hylomorphic if

$$E(\mathbf{u}_0) = \min\{E(\mathbf{u}) \mid C(\mathbf{u}) = C(\mathbf{u}_0)\}.$$

The physical meaning of C depends on the problem (in this case C is the *momentum*, see Section 3.1). The main result of this paper is the proof of the existence of hylomorphic solitons for equation (1) provided that W satisfies suitable assumptions (namely, (W-i), (W-ii) and (W-iii) of Section 3.1). In particular, these assumptions are satisfied by

$$W(s) = \begin{cases} \frac{1}{2}s^2 & \text{for } s \leq 1, \\ s - \frac{1}{2} & \text{for } s \geq 1. \end{cases} \tag{2}$$

Equation (1) with $W(s)$ as in (2) has been proposed as a model for a suspension bridge (see [9], [7], [8]). In particular, in [10] and [11] the existence of travelling waves has been proved.

Observe that $u(t, x) - 1$ denotes the displacement of the beam from the unloaded state $u(x) \equiv 1$, and the bridge is seen as a vibrating beam supported by cables which are treated as springs. The force relative to the potential $W(s)$ in (2) is given by

$$F(s) = -W'(s) = \begin{cases} -s & \text{for } s \leq 1, \\ -1 & \text{for } s \geq 1; \end{cases}$$

namely, for $s \geq 1$, only the constant gravity force -1 acts; while, for $s \leq 1$, an elastic force (of intensity $1 - s$), due to the suspension cables, must be added to the constant gravity force -1 . Of course, assumptions (W-i), (W-ii) and (W-iii) are satisfied also by the potential

$$W(s) = s - 1 + e^{-s} \tag{3}$$

which has been considered in [10] and [11] as an alternative smooth model for a suspension bridge.

2. Hylomorphic solitary waves and solitons

2.1. An abstract definition of solitary waves and solitons

Solitary waves and solitons are particular *states* of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more *fields* which mathematically are represented by functions

$$\mathbf{u} : \mathbb{R}^N \rightarrow V, \tag{4}$$

where V is a vector space with norm $|\cdot|_V$ which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system (X, γ) , where X is the set of the states and $\gamma : \mathbb{R} \times X \rightarrow X$ is the time evolution map. If $\mathbf{u}_0(x) \in X$, the evolution of the system will be described by the function

$$\mathbf{u}(t, x) := \gamma_t \mathbf{u}_0(x). \tag{5}$$

We assume that the states of X have “finite energy” so that they decay at ∞ sufficiently fast.

We give a formal definition of a solitary wave.

Definition 1. A state $\mathbf{u}(x) \in X$ is called a solitary wave if there is $\xi(t)$ such that

$$\gamma_t \mathbf{u}(x) = \mathbf{u}(x - \xi(t)).$$

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well-known notions in the theory of dynamical systems.

Definition 2. A set $\Gamma \subset X$ is called *invariant* if $\forall \mathbf{u} \in \Gamma, \forall t \in \mathbb{R}, \gamma_t \mathbf{u} \in \Gamma$.

Definition 3. Let (X, d) be a metric space and let (X, γ) be a dynamical system. An invariant set $\Gamma \subset X$ is called *stable*, if $\forall \varepsilon > 0, \exists \delta > 0, \forall \mathbf{u} \in X,$

$$d(\mathbf{u}, \Gamma) \leq \delta$$

implies that

$$\forall t \geq 0, d(\gamma_t \mathbf{u}, \Gamma) \leq \varepsilon.$$

Let G be the group induced by the translations in \mathbb{R}^N , namely, for every $\tau \in \mathbb{R}^N$, the transformation $g_\tau \in G$ is defined as follows:

$$(g_\tau \mathbf{u})(x) = \mathbf{u}(x - \tau). \tag{6}$$

Definition 4. A subset $\Gamma \subset X$ is called *G-invariant* if

$$\forall \mathbf{u} \in \Gamma, \forall \tau \in \mathbb{R}^N, g_\tau \mathbf{u} \in \Gamma.$$

Definition 5. A closed *G-invariant* set $\Gamma \subset X$ is called *G-compact* if for any sequence $\mathbf{u}_n(x)$ in Γ there is a sequence $\tau_n \in \mathbb{R}^N$ such that $\mathbf{u}_n(x - \tau_n)$ has a converging subsequence.

Now we are ready to give the definition of a soliton.

Definition 6. A solitary wave $\mathbf{u}(x)$ is called *soliton* if there is an invariant set Γ such that

- (i) $\mathbf{u}(x) \in \Gamma,$
- (ii) Γ is stable,
- (iii) Γ is *G-compact*.

Usually, in the literature, the kind of stability described by the above definition is called *orbital stability*.

Remark 7. The above definition needs some explanation. For simplicity, we assume that Γ is a manifold (actually, this is the generic case in many situations). Then (iii) implies that Γ is finite dimensional. Since Γ is invariant, $\mathbf{u}_0 \in \Gamma \Rightarrow \gamma_t \mathbf{u}_0 \in \Gamma$ for every time. Thus, since Γ is finite dimensional, the evolution of \mathbf{u}_0 is described by a finite number of parameters. Thus the dynamical system (Γ, γ) behaves as a point in a finite-dimensional phase space. By the stability of Γ , a small perturbation of \mathbf{u}_0 remains close to Γ . However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite-dimensional system with a small perturbation. Since $\dim(G) = N, \dim(\Gamma) \geq N$ and hence, the “state” of a soliton is described by N parameters which define its position and, may be, other parameters which define its “internal state.”

2.2. Integrals of motion and hylomorphic solitons

In recent papers (see, e.g., [3], [2], [4]), the notion of *hylomorphic soliton* has been introduced and analyzed. The existence and the properties of hylomorphic solitons are guaranteed by the interplay between the *energy* E and another integral of motion which, in the general case, is called *hylenic charge* and it will be denoted by C . More precisely, we have the following definition.

Definition 8. Assume that the dynamical system has two first integrals of motion $E : X \rightarrow \mathbb{R}$ and $C : X \rightarrow \mathbb{R}$. A soliton $\mathbf{u}_0 \in X$ is hylomorphic if Γ (as in Definition 6) has the following structure:

$$\Gamma = \Gamma(e_0, p_0) = \{\mathbf{u} \in X \mid E(\mathbf{u}) = e_0, C(\mathbf{u}) = p_0\},$$

where

$$e_0 = \min\{E(\mathbf{u}) \mid C(\mathbf{u}) = p_0\}$$

for some $p_0 \in \mathbb{R}$.

Clearly, for a given p_0 the minimum of E might not exist; moreover, even if the minimum exists, it is possible that Γ does not satisfy (ii) or (iii) of Definition 6.

In this section, we present an abstract theorem which guarantees the existence of hylomorphic solitons. Before stating the abstract theorem, we need some definitions.

Definition 9. A functional J on X is called *G-invariant* if

$$\forall g \in G, \forall \mathbf{u} \in X, \quad J(g\mathbf{u}) = J(\mathbf{u}).$$

Definition 10. Let G be a group of translations acting on X . A sequence \mathbf{u}_n in X is called *G-compact* if we can extract a subsequence \mathbf{u}_{n_k} such that there exists a sequence $g_k \in G$ such that $g_k\mathbf{u}_{n_k}$ is convergent. A functional J on X is called *G-compact* if any minimizing sequence of J is G -compact.

Remark 11. Clearly, a G -compact functional admits a minimizer. Moreover, if J is G -invariant and \mathbf{u}_0 is a minimizer, then $\{g\mathbf{u}_0 \mid g \in G\}$ is a set of minimizers; so, if G is not compact, the set of minimizers is not compact (unless \mathbf{u}_0 is a constant). This fact adds an extra difficulty to this kind of problems.

We make the following (abstract) assumptions on the dynamical system (X, γ) .

(EC-1) There are two first integrals $E : X \rightarrow \mathbb{R}$ and $C : X \rightarrow \mathbb{R}$.

(EC-2) $E(\mathbf{u})$ and $C(\mathbf{u})$ are G -invariant.

Theorem 12. Assume that the dynamical system (X, γ) satisfies (EC-1) and (EC-2). Moreover, we set

$$J(\mathbf{u}) = \frac{E(\mathbf{u})}{|C(\mathbf{u})|} + \delta E(\mathbf{u}), \tag{7}$$

where δ is a positive constant and assume that J is G -compact. Then $J(\mathbf{u})$ has a minimizer \mathbf{u}_0 . Moreover, if we set

$$e_0 = E(\mathbf{u}_0), \quad p_0 = C(\mathbf{u}_0), \tag{8}$$

$$\Gamma = \Gamma(e_0, p_0) = \{\mathbf{u} \in X \mid E(\mathbf{u}) = e_0, C(\mathbf{u}) = p_0\}, \tag{9}$$

then every $\mathbf{u} \in \Gamma$ is a hylomorphic soliton according to Definition 8.

Proof. The proof of this theorem is in [4]. Here we just give an idea of it. Let \mathbf{u}_n be a minimizing sequence of J . J is G -compact, then, for a suitable subsequence \mathbf{u}_{n_k} and a suitable sequence g_k , we get $g_k \mathbf{u}_{n_k} \rightarrow \mathbf{u}_0$. Clearly, \mathbf{u}_0 is a minimizer of J .

Now, let Γ be defined as in (9). It remains to show that every $\mathbf{u} \in \Gamma$ is a hylomorphic soliton according to Definition 8. First of all, notice that \mathbf{u}_0 is a minimizer of E on the set

$$\mathfrak{M}_{p_0} = \{\mathbf{u} \in X \mid C(\mathbf{u}) = p_0\}$$

and hence, according to Definition 8, every $\mathbf{u} \in \Gamma$ is a hylomorphic soliton provided that Γ satisfies (i), (ii), (iii) of Definition 6. Clearly, (i) and (iii) are satisfied. In order to prove (ii), namely that Γ is stable, we set

$$V(\mathbf{u}) = (E(\mathbf{u}) - e_0)^2 + (C(\mathbf{u}) - p_0)^2. \tag{10}$$

It can be shown that V is a Lyapunov function, since J is G -compact. Then it is sufficient to apply the classical Lyapunov theorem. \square

Remark 13. The reader may wonder why we use the functional J rather than minimizing E on the manifold \mathfrak{M}_p , $p \in \mathbb{R}$. As a matter of fact, in general E does not have a minimum on \mathfrak{M}_p ; on the contrary, if you choose p_0 given by (8), E has a minimum on \mathfrak{M}_{p_0} . In general, there is a set I of real values such that $\delta \in I$ implies that J given by (7) is G -compact; then for every $\delta \in I$, there is a $p = p(\delta)$ such that E has a minimum on $\mathfrak{M}_{p(\delta)}$. Moreover, if you perform a numerical simulation, it is more efficient to minimize the functional J rather than the functional E constrained on $\mathfrak{M}_{p(\delta)}$.

3. The existence result

3.1. Statement of the main results

Equation (1) has a variational structure, namely it is the Euler–Lagrange equation with respect to the functional

$$S = \frac{1}{2} \int \int (u_t^2 - u_{xx}^2) dx dt - \int \int W(u) dx dt. \tag{11}$$

The Lagrangian relative to the action (11) is

$$\mathcal{L} = \frac{1}{2} (u_t^2 - u_{xx}^2) - W(u). \tag{12}$$

This Lagrangian does not depend on t nor x . Then, by Noether’s theorem (see, e.g., [6], [5]), the energy E and the momentum C defined by

$$E = \int \left(\frac{\partial \mathcal{L}}{\partial u_t} u_t - \mathcal{L} \right) dx = \frac{1}{2} \int (u_t^2 + u_{xx}^2) dx + \int W(u) dx,$$

$$C = - \int \left(\frac{\partial \mathcal{L}}{\partial u_x} u_x \right) dx = - \int u_t u_x dx$$

are constant along the solutions of (1).

Equation (1) can be rewritten as a Hamiltonian system as follows:

$$\begin{cases} \partial_t u = v, \\ \partial_t v = -\partial_x^4 u - W'(u). \end{cases} \tag{13}$$

The phase space is given by

$$X = H^2(\mathbb{R}) \times L^2(\mathbb{R})$$

and the generic point in X will be denoted by

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Here $H^2(\mathbb{R})$ denotes the usual Sobolev space.

The norm of X is given by

$$\|\mathbf{u}\| = \left(\int (v^2 + u_{xx}^2 + u^2) dx \right)^{\frac{1}{2}}.$$

We shall assume that the initial value problem for (13) is globally well-posed for any initial data $\mathbf{u} \in X$.

The energy and the momentum, as functionals defined on X , take the following form:

$$E(\mathbf{u}) = \frac{1}{2} \int (v^2 + u_{xx}^2) dx + \int W(u) dx,$$

$$C(\mathbf{u}) = - \int v u_x dx.$$

Next, we will apply the abstract theory of Section 2 where the momentum $C(\mathbf{u})$ plays the role of the hylenic charge.

We make the following assumptions.

(W-i) **(Positivity)** $\exists \eta > 0$ such that $W(s) \geq \eta s^2$ for $|s| \leq 1$ and $W(s) \geq \eta$ for $|s| \geq 1$.

(W-ii) **(Nondegeneracy at 0)** $W'(0) = 0$ and $W''(0) = 1$.

(W-iii) **(Hylomorphy)** $\exists M > 0, \exists \alpha \in [0, 2) \forall s \geq 0,$

$$W(s) \leq M|s|^\alpha.$$

Here there are some comments on assumptions (W-ii) and (W-iii).

(W-ii) The assumption $W''(0) = 1$ can be weakened just assuming the existence of $W''(0)$. In fact, by (W-i) we have $W''(0) > 0$ and we can reduce

to the case $W''(0) = 1$ by rescaling space and time. By this assumption we can write

$$W(s) = \frac{1}{2}s^2 + N(s), \quad N(s) = o(s^2). \tag{14}$$

(W-iii) This is the crucial assumption which characterizes the potentials which might produce hylomorphic solitons; notice that this assumption concerns W only for the positive values of s .

We have the following results.

Theorem 14. *Assume that (W-i), (W-ii) and (W-iii) hold, then there exists an open interval $I \subset \mathbb{R}$ such that, for every $\delta \in I$, there is a hylomorphic soliton \mathbf{u}_δ for the dynamical system (13).*

Theorem 15. *Let $\mathbf{u}_\delta = (u_\delta, v_\delta)$ be a soliton as in Theorem 14. Then the solution of equation (1) with initial data (u_δ, v_δ) has the following form:*

$$u(t, x) = u_\delta(x - ct),$$

where u_δ is a solution of the following equation:

$$\frac{\partial^4 u_\delta}{\partial x^4} + c^2 \frac{\partial^2 u_\delta}{\partial x^2} + W'(u_\delta) = 0, \tag{15}$$

and c is a constant which depends on u_δ .

Remark 16. So we get the existence of solutions of (15) by a different proof from that in [10] and [11]. We point out that (15) could have solutions which are not minimizers. In this case, these solutions give rise to solitary waves which are not solitons.

The proofs of Theorems 14 and 15 will be given in the next section.

3.2. Proof of the main results

By (W-ii), we have that, for $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \in X = H^2(\mathbb{R}) \times L^2(\mathbb{R})$,

$$E(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 + \int N(u) dx. \tag{16}$$

Lemma 17. *Let $M > 0$. Then there exists a constant $C > 0$ such that*

$$(E(\mathbf{u}) \leq M) \Rightarrow (\|\mathbf{u}\| \leq C).$$

Proof. Assume that

$$E(\mathbf{u}) = \frac{1}{2} \int (v^2 + u_{xx}^2) dx + \int W(u) dx \leq M. \tag{17}$$

Then, since $W(u) \geq 0$, we have that

$$\int (v^2 + u_{xx}^2) dx \leq M. \tag{18}$$

It remains to prove that also

$$\int u^2 dx \text{ is bounded.} \tag{19}$$

We now set

$$\Omega_u^+ = \{x \mid u(x) > 1\}; \quad \Omega_u^- = \{x \mid u(x) < -1\}.$$

Then, if (17) holds, by (W-i) we have

$$M \geq \int W(u) dx^+ \geq \int_{\Omega_u^+ \cup \Omega_u^-} W(u) dx \geq \eta |\Omega_u^+| + \eta |\Omega_u^-|, \tag{20}$$

where $|\Omega|$ denotes the measure of Ω . Now we show that

$$\int_{\Omega_u^+} u^2 dx \text{ is bounded.} \tag{21}$$

Set $v = u - 1$, then, since $v = 0$ on $\partial\Omega_u^+$, by the Poincarè inequality, there is a constant $c > 0$ such that

$$\int_{\Omega_u^+} v^2 dx \leq c \int_{\Omega_u^+} v_x^2 dx. \tag{22}$$

Since we are in dimension one, it is easy to check that $c \leq |\Omega_u^+|^2$.

On the other hand,

$$\int_{\Omega_u^+} v_x^2 dx = - \int_{\Omega_u^+} v v_{xx} dx \leq \|v\|_{L^2(\Omega_u^+)} \|v_{xx}\|_{L^2(\Omega_u^+)}. \tag{23}$$

Then, since $v = u - 1$, by (22) and (23),

$$\|u - 1\|_{L^2(\Omega_u^+)}^2 \leq c \|u - 1\|_{L^2(\Omega_u^+)} \|u_{xx}\|_{L^2(\Omega_u^+)},$$

we easily get

$$\|u\|_{L^2(\Omega_u^+)}^2 - 2|\Omega_u^+|^{\frac{1}{2}} \|u\|_{L^2(\Omega_u^+)} + |\Omega_u^+| \leq c \left(\|u\|_{L^2(\Omega_u^+)} + |\Omega_u^+| \right) \|u_{xx}\|_{L^2(\Omega_u^+)}. \tag{24}$$

By (18) and (20) we have

$$\|u_{xx}\|_{L^2(\Omega_u^+)} \leq \sqrt{M}, \quad |\Omega_u^+| \leq \frac{M}{\eta}. \tag{25}$$

By (24) and (25) we get

$$\|u\|_{L^2(\Omega_u^+)}^2 - 2 \left(\frac{M}{\eta} \right)^{\frac{1}{2}} \|u\|_{L^2(\Omega_u^+)} \leq c \sqrt{M} \left(\|u\|_{L^2(\Omega_u^+)} + \frac{M}{\eta} \right).$$

From which we easily deduce (21). Analogously, we get also that

$$\int_{\Omega_u^-} u^2 dx \text{ is bounded.} \tag{26}$$

By (W-i),

$$\begin{aligned} M &\geq \int W(u) dx \\ &= \int_{|u(x)| \leq 1} W(u(x)) dx + \int_{\Omega_u^+ \cup \Omega_u^-} W(u(x)) dx \\ &\geq \eta \int_{|u(x)| \leq 1} u^2 dx. \end{aligned}$$

So, by (21), (26) and the above inequality, there is a constant R such that

$$\int u^2 dx = \int_{|u(x)| \leq 1} u^2 dx + \int_{\Omega_u^+ \cup \Omega_u^-} u^2 dx \leq \frac{M}{\eta} + R.$$

We conclude that $\int u^2 dx$ is bounded. □

Lemma 18. *Let \mathbf{u}_n be a sequence in X such that*

$$E(\mathbf{u}_n) \rightarrow 0. \tag{27}$$

Then, up to a subsequence, we have $\|\mathbf{u}_n\|_X \rightarrow 0$.

Proof. Let $\mathbf{u}_n = (u_n, v_n)$, with $u_n \in H^2(\mathbb{R})$ and $v_n \in L^2(\mathbb{R})$, be a sequence such that $E(\mathbf{u}_n) \rightarrow 0$. Then clearly $\|v_n\|_{L^2} \rightarrow 0$. By Lemma 17, u_n is bounded in $H^2(\mathbb{R})$ and hence, by the Sobolev embedding theorem, u_n is bounded in $L^\infty(\mathbb{R})$; moreover, for all n we have $u_n(x) \rightarrow 0$ for $|x| \rightarrow \infty$.

For each n let τ_n be a maximum point of $|u_n|$ and set

$$u'_n(x) = u_n(\tau_n + x), \quad v'_n(x) = v_n(\tau_n + x),$$

so that

$$|u'_n(0)| = \max |u'_n|. \tag{28}$$

Clearly, u'_n is bounded in $H^2(\mathbb{R})$, then, up to a subsequence, we get

$$u'_n \rightharpoonup u \text{ weakly in } H^2(\mathbb{R}) \tag{29}$$

and consequently

$$\frac{d^2 u'_n}{dx^2} \rightharpoonup \frac{d^2 u}{dx^2} \text{ weakly in } L^2(\mathbb{R}). \tag{30}$$

On the other end, since $E(\mathbf{u}_n) \rightarrow 0$, we have $\frac{d^2 u_n}{dx^2} \rightarrow 0$ in $L^2(\mathbb{R})$. Then also

$$\frac{d^2 u'_n}{dx^2} \rightarrow 0 \text{ in } L^2(\mathbb{R}). \tag{31}$$

From (30) and (31) we get

$$\frac{d^2 u}{dx^2} = 0.$$

So $u \in H^2(\mathbb{R})$ is linear and consequently

$$u = 0. \tag{32}$$

Now set

$$B_R = \{x \in \mathbb{R} : |x| < R\}, \quad R > 0,$$

then, by the compact embedding $H^2(B_R) \Subset L^\infty(B_R)$, by (29) and (32), we get

$$u'_n \rightarrow 0 \text{ in } L^\infty(B_R). \tag{33}$$

By (28) and (33) we get

$$\|u'_n\|_{L^\infty(\mathbb{R})} = |u'_n(0)| \rightarrow 0.$$

So, if n is sufficiently large, we have $|u'_n(x)| \leq 1$ for all x .

Then, setting $\mathbf{u}'_n = (u'_n, v'_n)$, by (W-i), we have that

$$\begin{aligned} E(\mathbf{u}'_n) &= \int \left(\frac{1}{2} \left(v'^2_n + (\partial_{xx} u'_n)^2 \right) + W(u'_n) \right) dx \\ &\geq \int \left(\frac{1}{2} \left(v'^2_n + (\partial_{xx} u'_n)^2 \right) + \eta u'^2_n \right) dx \\ &\geq c \|\mathbf{u}'_n\|^2, \end{aligned} \tag{34}$$

where c is a positive constant.

Since

$$E(\mathbf{u}'_n) = E(\mathbf{u}_n), \quad \|\mathbf{u}'_n\| = \|\mathbf{u}_n\|,$$

by (34), (27) we have

$$\|\mathbf{u}_n\|_X \rightarrow 0. \quad \square$$

We set

$$\begin{aligned} \Lambda_0 &= \inf_{\mathbf{u} \in X} \frac{\frac{1}{2} \|\mathbf{u}\|^2}{|C(\mathbf{u})|}, \\ \Lambda_* &= \inf_{\mathbf{u} \in X} \frac{E(\mathbf{u})}{|C(\mathbf{u})|} = \inf_{\mathbf{u} \in X} \frac{\frac{1}{2} \|\mathbf{u}\|^2 + \int N(u) dx}{|C(\mathbf{u})|}. \end{aligned}$$

Lemma 19. *The following inequality holds:*

$$\Lambda_0 \geq 1.$$

Proof. For $\mathbf{u} = (v, u)$ we have

$$\begin{aligned} |C(\mathbf{u})| &\leq \int |v \partial_x u| dx \leq \left(\int v^2 dx \right)^{1/2} \cdot \left(\int |\partial_x u|^2 dx \right)^{1/2} \\ &\leq \frac{1}{2} \int v^2 dx + \frac{1}{2} \int |\partial_x u|^2 dx \\ &= \frac{1}{2} \int v^2 dx - \frac{1}{2} \int uu_{xx} dx \\ &\leq \frac{1}{2} \int v^2 dx + \frac{1}{2} \int \frac{1}{2} [u^2 + u_{xx}^2] dx \\ &\leq \frac{1}{2} \int [v^2 + u_{xx}^2 + u^2] dx = \frac{1}{2} \|\mathbf{u}\|^2. \end{aligned}$$

Then, for every \mathbf{u} ,

$$\Lambda_0 \geq \frac{\frac{1}{2} \|\mathbf{u}\|^2}{|C(\mathbf{u})|} \geq 1. \quad \square$$

The next lemma provides a crucial estimate for the existence of solitons.

Lemma 20. *We have*

$$\Lambda_* < 1.$$

Proof. Let $U \in C^2$ be a positive function with compact support such that

$$\frac{\int (U_{xx})^2}{\int (U_x)^2} < \frac{1}{2}. \tag{35}$$

Such a function exists; in fact if U_0 is any positive function with compact support, $U(x) = U_0(\frac{x}{\lambda})$ satisfies (35) for λ sufficiently large. Take

$$\mathbf{u}_R = (u_R, v) = (RU, RU_x).$$

By the definition of X , $\mathbf{u}_R \in X$. Now we can estimate Λ_* :

$$\begin{aligned} \Lambda_* &= \inf_{\mathbf{u} \in X} \frac{\frac{1}{2} \|\mathbf{u}\|^2 + \int N(u) dx}{|C(\mathbf{u})|} \leq \frac{\frac{1}{2} \|\mathbf{u}_R\|^2 + \int N(u_R) dx}{|C(\mathbf{u}_R)|} \\ &= \frac{\frac{1}{2} \int [(RU_x)^2 + (RU_{xx})^2 + (RU)^2] dx + \int N(RU) dx}{\int (RU_x)^2 dx} \\ &= \frac{\frac{1}{2} \int [(RU_x)^2 + (RU_{xx})^2] dx}{\int (RU_x)^2 dx} + \frac{\int W(RU) dx}{\int (RU_x)^2 dx} \\ &= \frac{1}{2} + \frac{1}{2} \frac{\int (U_{xx})^2 dx}{\int (U_x)^2 dx} + \frac{\int W(RU) dx}{\int (RU_x)^2 dx} \quad (\text{by (W-iii)}) \\ &\leq \frac{1}{2} + \frac{1}{2} \frac{\int (U_{xx})^2 dx}{\int (U_x)^2 dx} + \frac{\int M |RU|^\alpha dx}{\int (RU_x)^2 dx} \quad (\text{by (35)}) \\ &< \frac{1}{2} + \frac{1}{4} + \frac{M}{R^{2-\alpha}} \cdot \frac{\int |U|^\alpha dx}{\int U_x^2 dx}. \end{aligned}$$

Then, for R sufficiently large, we get the conclusion. □

Lemma 21. *Consider any sequence*

$$\mathbf{u}_n = \mathbf{u} + \mathbf{w}_n \in X,$$

where \mathbf{w}_n converges weakly to 0. Then

$$E(\mathbf{u}_n) = E(\mathbf{u}) + E(\mathbf{w}_n) + o(1) \tag{36}$$

and

$$C(\mathbf{u}_n) = C(\mathbf{u}) + C(\mathbf{w}_n) + o(1). \tag{37}$$

Proof. First of all, we introduce the following notation:

$$K(u) = \int N(u) dx \quad \text{and} \quad K_\Omega(u) = \int_\Omega N(u) dx, \quad \Omega \text{ open subset in } \mathbb{R}.$$

As usual u, w_n will denote the first components, respectively, of $\mathbf{u}, \mathbf{w}_n \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$.

We have to show that $\lim_{n \rightarrow \infty} |E(\mathbf{u} + \mathbf{w}_n) - E(\mathbf{u}) - E(\mathbf{w}_n)| = 0$. By (16) we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} |E(\mathbf{u} + \mathbf{w}_n) - E(\mathbf{u}) - E(\mathbf{w}_n)| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{2} \left| \|\mathbf{u} + \mathbf{w}_n\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{w}_n\|^2 \right| \\ & \quad + \lim_{n \rightarrow \infty} \left| \int (N(u + w_n) - N(u) - N(w_n)) dx \right|. \end{aligned} \tag{38}$$

If (\cdot, \cdot) denotes the inner product induced by the norm $\|\cdot\|$, we have

$$\lim_{n \rightarrow \infty} \left| \|\mathbf{u} + \mathbf{w}_n\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{w}_n\|^2 \right| = \lim_{n \rightarrow \infty} |2(\mathbf{u}, \mathbf{w}_n)| = 0. \tag{39}$$

Then by (38) and (39) we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} |E(\mathbf{u} + \mathbf{w}_n) - E(\mathbf{u}) - E(\mathbf{w}_n)| \\ & \leq \lim_{n \rightarrow \infty} \left| \int (N(u + w_n) - N(u) - N(w_n)) dx \right|. \end{aligned} \tag{40}$$

Choose $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ such that

$$\left| \int_{B_R^c} N(u) \right| < \varepsilon, \quad \int_{B_R^c} |u|^2 < \varepsilon, \tag{41}$$

where

$$B_R^c = \mathbb{R} - B_R \quad \text{and} \quad B_R = \{x \in \mathbb{R} : |x| < R\}.$$

Since $w_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R})$, by usual compactness arguments, we have that

$$K_{B_R}(w_n) \rightarrow 0 \quad \text{and} \quad K_{B_R}(u + w_n) \rightarrow K_{B_R}(u). \tag{42}$$

Then we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \int [N(u + w_n) - N(u) - N(w_n)] \right| \\ & = \lim_{n \rightarrow \infty} \left| K_{B_R^c}(u + w_n) + K_{B_R}(u + w_n) \right. \\ & \quad \left. - K_{B_R^c}(u) - K_{B_R}(u) - K_{B_R^c}(w_n) - K_{B_R}(w_n) \right|. \end{aligned} \tag{43}$$

The above equality with (42) and (41) gives

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \int [N(u + w_n) - N(u) - N(w_n)] \right| \\ & = \lim_{n \rightarrow \infty} \left| K_{B_R^c}(u + w_n) - K_{B_R^c}(u) - K_{B_R^c}(w_n) \right| \\ & \leq \lim_{n \rightarrow \infty} \left| K_{B_R^c}(u + w_n) - K_{B_R^c}(w_n) \right| + \varepsilon. \end{aligned} \tag{44}$$

By the intermediate value theorem there are ζ_n in $(0, 1)$ such that

$$\left| K_{B_R^c}(u + w_n) - K_{B_R^c}(w_n) \right| = \left| \int_{B_R^c} N'(\zeta_n u + w_n) u dx \right|. \tag{45}$$

Since w_n is bounded in $H^2(\mathbb{R})$, $\zeta_n u + w_n$ is bounded in L^∞ , so that there exists a positive constant M such that

$$\|\zeta_n u + w_n\|_{L^\infty} \leq M. \tag{46}$$

Now by (14) there exists a constant $C_M > 0$ depending on M such that

$$|N'(s)| \leq C_M |s| \quad \text{for } |s| \leq M. \tag{47}$$

Then by (45), (46) and (47) we get

$$\begin{aligned} |K_{B_R^c}(u + w_n) - K_{B_R^c}(w_n)| &\leq \int_{B_R^c} |N'(\zeta_n u + w_n)u| dx \\ &\leq C_M \int_{B_R^c} |(\zeta_n u + w_n)u| dx \\ &\leq C_M \left(\|u\|_{L^2(B_R^c)}^2 + \|w_n\|_{L^2(B_R^c)} \|u\|_{L^2(B_R^c)} \right) \\ &\leq C_M (\varepsilon + c_1 \varepsilon) = c_2 \varepsilon \quad (\text{by (41)}), \end{aligned}$$

where $c_1 = \sup \|w_n\|_{L^2(B_R^c)}$ and $c_2 = C_M(1 + c_1)$.

Then we get

$$|K_{B_R^c}(u + w_n) - K_{B_R^c}(w_n)| \leq c_2 \varepsilon. \tag{48}$$

So by (44) and (48),

$$\lim_{n \rightarrow \infty} \left| \int [N(u + w_n) - N(u) - N(w_n)] dx \right| \leq c_2 \varepsilon + \varepsilon. \tag{49}$$

Finally, by (40) and (49) and since ε is arbitrary, we get

$$\lim_{n \rightarrow \infty} |E(\mathbf{u} + \mathbf{w}_n) - E(\mathbf{u}) - E(\mathbf{w}_n)| = 0,$$

and so (36) is proved. The proof of (37) is immediate. □

By Lemmas 19 and 20, we have that

$$\Lambda_* < \Lambda_0.$$

So there exist $\mathbf{u}_0 \in X$ and $b > 0$ such that

$$\frac{E(\mathbf{u}_0)}{|C(\mathbf{u}_0)|} \leq \Lambda_0 - b.$$

Then we can choose $\delta > 0$ such that

$$\frac{E(\mathbf{u}_0)}{|C(\mathbf{u}_0)|} + \delta E(\mathbf{u}_0) \leq \Lambda_0 - \frac{b}{2} \tag{50}$$

and we define

$$J(\mathbf{u}) = \frac{E(\mathbf{u})}{|C(\mathbf{u})|} + \delta E(\mathbf{u}). \tag{51}$$

Then we have that

$$J_* := \inf J(\mathbf{u}) \leq J(\mathbf{u}_0) \leq \Lambda_0 - \frac{b}{2}. \tag{52}$$

Lemma 22. *The functional defined by (51) is G -compact (where G is defined by (6)).*

Proof. Let $\mathbf{u}_n = (u_n, v_n)$ be a minimizing sequence for J . Since the G -compactness depends on subsequences, we can take a subsequence in which all the $C(\mathbf{u}_n)$ have the same sign. So, to fix the ideas, we can assume that

$$C(\mathbf{u}_n) > 0; \tag{53}$$

thus we have that

$$J(\mathbf{u}_n) = \frac{E(\mathbf{u}_n)}{C(\mathbf{u}_n)} + \delta E(\mathbf{u}_n).$$

It is immediate to see that $E(\mathbf{u}_n) = \frac{1}{2} \|\mathbf{u}_n\|^2 + \int N(u_n) dx$ is bounded. Then, by Lemma 17, $\|\mathbf{u}_n\|$ is bounded and hence, passing eventually to a suitable subsequence, we have $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in X . Now, starting from \mathbf{u}_n , we construct a minimizing sequence \mathbf{u}'_n which weakly converges to

$$\bar{\mathbf{u}} \neq 0. \tag{54}$$

To this end we first show that

$$\|u_n\|_{L^\infty} \text{ does not converge to } 0. \tag{55}$$

Arguing by contradiction, assume that

$$\|u_n\|_{L^\infty} \rightarrow 0.$$

Then, since $N(s) = o(s^2)$, there is a sequence of positive real numbers ε_n with $\varepsilon_n \rightarrow 0$ such that

$$\begin{aligned} \Lambda(\mathbf{u}_n) &= \frac{E(\mathbf{u}_n)}{C(\mathbf{u}_n)} \geq \frac{\frac{1}{2} \left(\left\| \frac{d^2 u_n}{dx^2} \right\|_{L^2}^2 + \|u_n\|_{L^2}^2 \right) - \int |N(u_n)| dx}{C(\mathbf{u}_n)} \\ &= \frac{\frac{1}{2} \left(\left\| \frac{d^2 u_n}{dx^2} \right\|_{L^2}^2 + \|u_n\|_{L^2}^2 \right) - \frac{\varepsilon_n}{2} \|u_n\|_{L^2}^2}{C(\mathbf{u}_n)} \\ &\geq \frac{\frac{1}{2} \left(\left\| \frac{d^2 u_n}{dx^2} \right\|_{L^2}^2 + \|u_n\|_{L^2}^2 \right) - \frac{\varepsilon_n}{2} \left(\left\| \frac{d^2 u_n}{dx^2} \right\|_{L^2}^2 + \|u_n\|_{L^2}^2 \right)}{C(\mathbf{u}_n)} \\ &= \frac{\frac{1}{2} \left(\left\| \frac{d^2 u_n}{dx^2} \right\|_{L^2}^2 + \|u_n\|_{L^2}^2 \right)}{C(\mathbf{u}_n)} (1 - \varepsilon_n) \\ &\geq \Lambda_0 (1 - \varepsilon_n) \quad (\text{by definition of } \Lambda_0). \end{aligned}$$

And hence

$$J(\mathbf{u}_n) \geq \Lambda_0 (1 - \varepsilon_n). \tag{56}$$

On the other hand, by (52),

$$\lim J(\mathbf{u}_n) \leq \Lambda_0 - \frac{b}{2}. \tag{57}$$

Clearly (57) contradicts (56).

So (55) holds and consequently there exist $b > 0$ and a sequence x_n such that, up to a subsequence,

$$|u_n(x_n)| \geq b. \tag{58}$$

Now we set

$$\mathbf{u}'_n(x) = \mathbf{u}_n(x + x_n), \quad u'_n(x) = u_n(x + x_n).$$

Clearly also $\mathbf{u}'_n(x)$ is a minimizing sequence; moreover, by (58),

$$|u'_n(0)| \geq b. \tag{59}$$

Since, up to a subsequence, $\mathbf{u}'_n \rightharpoonup \bar{\mathbf{u}} \in X$ weakly in X , we have, by a compact embedding result, that

$$u'_n \rightarrow \bar{u} \text{ in } L^\infty(-1, 1),$$

where \bar{u} denotes the first component of $\bar{\mathbf{u}}$. Then by (59) we have $\bar{u} \neq 0$ and then $\bar{\mathbf{u}} \neq 0$. So (54) is proved.

Now set

$$\mathbf{u}'_n = \bar{\mathbf{u}} + \mathbf{w}_n$$

with $\mathbf{w}_n \rightharpoonup 0$ weakly in X .

We finally show that there is no splitting, namely that $\mathbf{w}_n \rightarrow \mathbf{0}$ strongly in X . To this end, first we show that

$$C(\bar{\mathbf{u}} + \mathbf{w}_n) \text{ does not converge to } 0. \tag{60}$$

Arguing by contradiction assume that $C(\bar{\mathbf{u}} + \mathbf{w}_n)$ converges to 0. Then, since $\bar{\mathbf{u}} + \mathbf{w}_n$ is a minimizing sequence for J , also $E(\bar{\mathbf{u}} + \mathbf{w}_n)$ converges to 0 and then, by Lemma 18, we get

$$\bar{\mathbf{u}} + \mathbf{w}_n \rightarrow 0 \text{ in } X. \tag{61}$$

From (61) and since $\mathbf{w}_n \rightharpoonup 0$ weakly in X , we have that $\bar{\mathbf{u}} = 0$, contradicting (54). So (60) holds and, passing eventually to a subsequence, we can assume

$$C(\bar{\mathbf{u}} + \mathbf{w}_n) \geq \delta > 0. \tag{62}$$

By Lemma 21, we have

$$E(\mathbf{u}'_n) = E(\bar{\mathbf{u}} + \mathbf{w}_n) = E(\bar{\mathbf{u}}) + E(\mathbf{w}_n) + o(1)$$

and

$$\begin{aligned} C(\mathbf{u}'_n) &= C(\bar{\mathbf{u}} + \mathbf{w}_n) \\ &= C(\bar{\mathbf{u}}) + C(\mathbf{w}_n) + o(1) \\ &\geq \delta \quad (\text{by (62)}) \\ &> 0. \end{aligned} \tag{63}$$

Then

$$\begin{aligned}
 J_* &:= \lim J(\mathbf{u}'_n) = \lim \frac{E(\mathbf{u}'_n)}{C(\mathbf{u}'_n)} + \delta E(\mathbf{u}'_n) \\
 &= \lim \left[\frac{E(\bar{\mathbf{u}}) + E(\mathbf{w}_n) + o(1)}{C(\bar{\mathbf{u}}) + C(\mathbf{w}_n) + o(1)} + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) + o(1) \right] \\
 &= \lim \left[\frac{E(\bar{\mathbf{u}}) + E(\mathbf{w}_n)}{C(\bar{\mathbf{u}}) + C(\mathbf{w}_n)} + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) \right] \\
 &\geq \lim \left[\frac{E(\bar{\mathbf{u}}) + E(\mathbf{w}_n)}{|C(\bar{\mathbf{u}})| + |C(\mathbf{w}_n)|} + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) \right] \\
 &\geq \lim \left[\min \left(\frac{E(\bar{\mathbf{u}})}{|C(\bar{\mathbf{u}})|}, \frac{E(\mathbf{w}_n)}{|C(\mathbf{w}_n)|} \right) + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) \right].
 \end{aligned}$$

Now we consider two cases. First case:

$$\frac{E(\bar{\mathbf{u}})}{|C(\bar{\mathbf{u}})|} \geq \frac{E(\mathbf{w}_n)}{|C(\mathbf{w}_n)|};$$

then

$$\begin{aligned}
 J_* &\geq \lim \left[\frac{E(\mathbf{w}_n)}{|C(\mathbf{w}_n)|} + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) \right] \\
 &= \lim [J(\mathbf{w}_n) + \delta E(\bar{\mathbf{u}})] \geq J_* + \delta E(\bar{\mathbf{u}}).
 \end{aligned}$$

This case cannot occur since it implies $\delta E(\bar{\mathbf{u}}) \leq 0$ and this contradicts (54).

Then we have the second case:

$$\frac{E(\bar{\mathbf{u}})}{|C(\bar{\mathbf{u}})|} < \frac{E(\mathbf{w}_n)}{|C(\mathbf{w}_n)|}.$$

In this case,

$$\begin{aligned}
 J_* &\geq \lim \left[\frac{E(\bar{\mathbf{u}})}{|C(\bar{\mathbf{u}})|} + \delta E(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n) \right] \\
 &= \lim [J(\bar{\mathbf{u}}) + \delta E(\mathbf{w}_n)] \geq J_* + \delta \lim E(\mathbf{w}_n).
 \end{aligned}$$

Then

$$\delta \lim E(\mathbf{w}_n) \leq 0. \tag{64}$$

Then by Lemma 18 and by (64) we have $\mathbf{w}_n \rightarrow 0$ strongly in X . \square

Proof of Theorem 14. We shall use Theorem 12. Obviously assumptions (EC-1) and (EC-2) are satisfied with G given by (6). Then by Lemma 22 and Theorem 12, we have the existence of soliton solutions. \square

Proof of Theorem 15. Since $\mathbf{u}_\delta = (u_\delta, v_\delta) \in X = H^2(\mathbb{R}) \times L^2(\mathbb{R})$ is a minimizer, we have $J'(\mathbf{u}_\delta) = 0$. Then

$$\frac{E'(\mathbf{u}_\delta)}{C(\mathbf{u}_\delta)} - \frac{E(\mathbf{u}_\delta)}{C(\mathbf{u}_\delta)^2} C'(\mathbf{u}_\delta) + \delta E'(\mathbf{u}_\delta) = 0,$$

namely

$$(C(\mathbf{u}_\delta) + \delta C(\mathbf{u}_\delta)^2) E'(\mathbf{u}_\delta) = E(\mathbf{u}_\delta) C'(\mathbf{u}_\delta).$$

Since, by (53), $C(\mathbf{u}_\delta) > 0$, then $C(\mathbf{u}_\delta) + \delta C(\mathbf{u}_\delta)^2 > 0$, and hence we can divide both sides by $C(\mathbf{u}_\delta) + \delta C(\mathbf{u}_\delta)^2$ and we get

$$E'(\mathbf{u}_\delta) = cC'(\mathbf{u}_\delta), \quad (65)$$

where

$$c = \frac{E(\mathbf{u}_\delta)}{C(\mathbf{u}_\delta) + \delta C(\mathbf{u}_\delta)^2}.$$

If we write (65) explicitly, we get, for all $\varphi \in H^2(\mathbb{R})$ and all $\psi \in L^2(\mathbb{R})$,

$$\begin{aligned} \int \partial_x^2 u_\delta \partial_x^2 \varphi + W'(u_\delta) \varphi &= c \int v_\delta \partial_x \varphi, \\ \int v_\delta \psi &= c \int \psi \partial_x u_\delta, \end{aligned}$$

namely

$$\begin{aligned} \partial_x^4 u_\delta + W'(u_\delta) &= -c \partial_x v_\delta, \\ v_\delta &= c \partial_x u_\delta, \end{aligned}$$

and so we get

$$\partial_x^4 u_\delta + c^2 \partial_x^2 u_\delta + W'(u_\delta) = 0.$$

Now, we can check directly that

$$u(t, x) = u_\delta(x - ct)$$

solves equation (1) with initial conditions $(u_\delta(x), -c \partial_x u_\delta(x))$. \square

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