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Strongly relatively nonexpansive sequences in Banach spaces and applications

Koji Aoyama, Fumiaki Kohsaka and Wataru Takahashi

Abstract. We introduce the concept of a strongly relatively nonexpansive sequence in a Banach space and investigate its properties. Then we apply our results to the problem of approximating a common fixed point of a countable family of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach space.

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1. Introduction

Let E be a smooth Banach space and let $\phi\colon E \times E \to \mathbb{R}$ be a function defined by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$, where J is the normalized duality mapping of E. Let C be a nonempty subset of E and let T be a mapping of C into E. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be relatively nonexpansive [7, 22, 23] if it satisfies the following conditions: (i) $F(T) \neq \emptyset$; (ii) $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$; (iii) $F(T) = \hat{F}(T)$, where $\hat{F}(T)$ is the set of asymptotic fixed points of T; see [8, 24].

The class of relatively nonexpansive mappings includes all resolvents of maximal monotone operators with zero points on a uniformly convex and uniformly smooth Banach space and all nonexpansive mappings with fixed points in a Hilbert space; see, for example, [23]. There are various results concerning that class: Matsushita and Takahashi [22, 23] discussed the problem of finding a fixed point of a relatively nonexpansive mapping in a Banach space. Kohsaka and Takahashi [17] proved weak convergence theorems for approximating a common asymptotic fixed point of a finite family of relatively nonexpansive mappings; see also [4, 5].

On the other hand, Reich [24] introduced the concept of strong nonexpansiveness for relatively nonexpansive mappings, which is deeply related to the strong nonexpansiveness for nonexpansive mappings due to Bruck and Reich [6].

Censor and Reich [8] also introduced a convex combination based on Bregman distance and obtained several results for approximating a common asymptotic fixed point of a family of paracontractions in a finite-dimensional space. Using the convex combination, some results on proximal-type algorithms for a maximal monotone operator in a Banach space were obtained by Kamimura, Kohsaka, and Takahashi [13] and Kohsaka and Takahashi [16].

In this paper, motivated by all the literature mentioned above, we introduce the concept of a strongly relatively nonexpansive sequence in a Banach space and investigate its properties. Then we consider an iterative sequence $\{x_n\}$ defined by $x_1 = x \in C$ and $x_{n+1} = T_n x_n$ for $n \in \mathbb{N}$, where $\{T_n\}$ is a strongly relatively nonexpansive sequence of self-mappings of C and x is an arbitrary point in C. In §4 we give a condition on $\{T_n\}$ under which $\{x_n\}$ converges weakly or strongly to a common fixed point of $\{T_n\}$. In §5 and §6, using this result, we prove weak and strong convergence theorems for a countable family of relatively nonexpansive mappings.

2. Preliminaries

Throughout the present paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the dual of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, $\mathbb{N}$ the set of positive integers, and $\mathbb{R}$ the set of real numbers. The norm of E^* is also denoted by $\|\cdot\|$ for the sake of convenience. Strong convergence of $\{x_n\}$ to $x \in E$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$, where $\{x_n\}$ is a sequence in E. The (normalized) duality mapping of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. It is known that $\|y\|^2 \geq \|x\|^2 + 2\langle y - x, j\rangle$ for all $x, y \in E$ and $j \in Jx$; see, for example, [27]. Replacing y by $(x+y)/2$ in this inequality, we see that

$$\left\| \frac{x+y}{2} \right\|^2 \geq \|x\|^2 + 2\left\langle \frac{y-x}{2}, j \right\rangle = \langle y, j\rangle \tag{2.1}$$

for all $x, y \in E$ and $j \in Jx$.

Let S_E denote the unit sphere of E, that is, $S_E = \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in S_E$. In this case E is said to be *smooth* and it is known that the duality mapping J of E is single-valued. The norm of E is said to be *uniformly Gâteaux differentiable* (resp. *Fréchet differentiable*) if for each $y \in S_E$ (resp. for each $x \in S_E$) the limit (2.2) is attained uniformly for $x \in S_E$ (resp. uniformly for

$y \in S_E$). In this case it is known that J is uniformly norm-to-weak* continuous on each bounded subset of E (resp. norm-to-norm continuous). A Banach space E is said to be *uniformly smooth* if the limit (2.2) is attained uniformly for $x, y \in S_E$. In this case it is known that J is uniformly norm-to-norm continuous on each bounded subset of E; see [27] for more details.

A Banach space E is said to be *strictly convex* if $x, y \in S_E$ and $x \neq y$ imply $\|x + y\| < 2$. A Banach space E is said to be *uniformly convex* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S_E$ and $\|x - y\| \geq \epsilon$ imply $\|x + y\|/2 \leq 1 - \delta$. It is known that the duality mapping J of E is single-valued and one-to-one if E is smooth and strictly convex; J is surjective if E is reflexive; E is reflexive and strictly convex if E is uniformly convex; E is uniformly smooth if and only if E^* is uniformly convex; see [27] for more details. It is also known that if E is uniformly convex, then the function $\|\cdot\|^2$ is uniformly convex on every bounded convex subset B of E, that is, for each $\epsilon > 0$, there is $\delta > 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\delta$$

for all $\lambda \in [0, 1]$ and $x, y \in B$ with $\|x - y\| \geq \epsilon$; see, for example, [9, 30]. This fact implies the following (see [3]).

Lemma 2.1. *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex Banach space E and $\{\lambda_n\}$ a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \lambda_n > 0$. Suppose that*

$$\lambda_n\|x_n\|^2 + (1 - \lambda_n)\|y_n\|^2 - \|\lambda_n x_n + (1 - \lambda_n)y_n\|^2 \to 0.$$

Then $(1 - \lambda_n)\|x_n - y_n\| \to 0$.

Let E be a smooth Banach space. We deal with a function $\phi\colon E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$. This function was studied in [1]. From the definition of ϕ, it is clear that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.3}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y, z \in E$. The following result is known:

Lemma 2.2 ([14]). *Let E be a uniformly convex and smooth Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a smooth Banach space E. It is obvious from the definition of ϕ that $\phi(x_n, y_n) \to 0$ whenever $\|x_n - y_n\| \to 0$. From this fact and Lemma 2.2, we deduce the following: Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex and uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \;\Leftrightarrow\; \|Jx_n - Jy_n\| \to 0 \;\Leftrightarrow\; \phi(x_n, y_n) \to 0. \tag{2.5}$$

The following lemma is used in the proof of Lemma 2.6 below.

Lemma 2.3. *Let E be a uniformly convex and smooth Banach space. Let $\{s_n\}$ be a convergent sequence in $\mathbb{R}$ and $\{x_n\}$ a sequence in E such that*

$$\phi(x_m, x_n) \leq |s_m - s_n|$$

for all $m, n \in \mathbb{N}$ with $m < n$. Then $\{x_n\}$ is a strongly convergent sequence.

Proof. It is clear from (2.3) that

$$(\|x_m\| - \|x_n\|)^2 \leq \phi(x_m, x_n) \leq |s_m - s_n|$$

for all $m, n \in \mathbb{N}$ with $m < n$. Therefore $\{\|x_n\|\}$ is a Cauchy sequence in $\mathbb{R}$ and hence $\{x_n\}$ is bounded. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that $m_i < n_i$ and $\|x_{m_i} - x_{n_i}\| \geq \epsilon$ for every $i \in \mathbb{N}$. Since E is uniformly convex, $\|\cdot\|^2$ is uniformly convex on $B = \{z \in E : \|z\| \leq \sup_{n\in\mathbb{N}}\|x_n\|\}$. Thus for ϵ there exists $\delta > 0$ such that

$$\left\|\frac{x_{m_i} + x_{n_i}}{2}\right\|^2 \leq \frac{1}{2}\|x_{m_i}\|^2 + \frac{1}{2}\|x_{n_i}\|^2 - \delta$$

for every $i \in \mathbb{N}$. Hence it follows from (2.1) that

$$\langle x_{m_i}, Jx_{n_i}\rangle \leq \frac{1}{2}\|x_{m_i}\|^2 + \frac{1}{2}\|x_{n_i}\|^2 - \delta$$

and thus

$$0 < 2\delta \leq \|x_{m_i}\|^2 - 2\langle x_{m_i}, Jx_{n_i}\rangle + \|x_{n_i}\|^2 = \phi(x_{m_i}, x_{n_i}) \leq |s_{m_i} - s_{n_i}|$$

for every $i \in \mathbb{N}$. This contradicts the assumption that $\{s_n\}$ is convergent. Therefore we conclude that $\{x_n\}$ is a Cauchy sequence. □

Let E be a strictly convex, smooth, and reflexive Banach space and C a nonempty closed convex subset of E. It is known that, for each $x \in E$, there is a unique point x_0 such that

$$\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}.$$

Such a point x_0 is denoted by $\Pi_C x$, and Π_C is said to be the *generalized projection* of E onto C; see [1] and [14]. The following lemma is known:

Lemma 2.4 ([1], [14] and [20]). *Let E be a strictly convex, smooth, and reflexive Banach space. Let C be a nonempty closed convex subset of E and Π_C the generalized projection of E onto C. Then the following inequalities hold:*

$$\langle \Pi_C x - u, Jx - J\Pi_C x\rangle \geq 0, \tag{2.6}$$

$$\phi(u, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(u, x), \tag{2.7}$$

$$\langle \Pi_C x - \Pi_C y, J\Pi_C x - J\Pi_C y\rangle \leq \langle \Pi_C x - \Pi_C y, Jx - Jy\rangle \tag{2.8}$$

for all $x, y \in E$ and $u \in C$.

The generalized projection also has the following property.

Lemma 2.5. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Then Π_C is uniformly norm-to-norm continuous on every bounded set.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a bounded subset of E such that $\|x_n - y_n\| \to 0$. Note that $\{\Pi_C x_n\}$ and $\{\Pi_C y_n\}$ are bounded because it follows from (2.3) and (2.7) that

$$(\|u\| - \|\Pi_C x_n\|)^2 \le \phi(u, \Pi_C x_n) + \phi(\Pi_C x_n, x_n) \le \phi(u, x_n) \le (\|u\| + \|x_n\|)^2$$

for all $n \in \mathbb{N}$ and $u \in C$. Since E is uniformly smooth, the duality mapping J is uniformly norm-to-norm continuous on every bounded set. It follows from (2.8) that

$$\begin{aligned} 0 &\le \frac{1}{2}(\phi(\Pi_C x_n, \Pi_C y_n) + \phi(\Pi_C y_n, \Pi_C x_n)) \\ &= \langle \Pi_C x_n - \Pi_C y_n, J\Pi_C x_n - J\Pi_C y_n \rangle \\ &\le \langle \Pi_C x_n - \Pi_C y_n, Jx_n - Jy_n \rangle \\ &\le \|\Pi_C x_n - \Pi_C y_n\| \, \|Jx_n - Jy_n\| \to 0. \end{aligned}$$

Therefore $\phi(\Pi_C x_n, \Pi_C y_n) \to 0$. Hence Lemma 2.2 implies $\|\Pi_C x_n - \Pi_C y_n\| \to 0$. This means that Π_C is uniformly norm-to-norm continuous on a bounded set. □

We next show the following lemma. Part (1) is a generalization of [29, Lemma 3.2]. We follow the idea in [22] for the proof of (2).

Lemma 2.6. *Let E be a smooth Banach space, C a nonempty closed convex subset of E, and $\{x_n\}$ a sequence in E such that $\phi(u, x_{n+1}) \le \phi(u, x_n)$ for all $u \in C$ and $n \in \mathbb{N}$. Then the following hold:*

(1) *If E is uniformly convex, then $\{\Pi_C x_n\}$ is strongly convergent.*
(2) *If the norm of E^* is Fréchet differentiable and the interior of C is nonempty, then $\{x_n\}$ is strongly convergent.*

Proof. We first show (1). By the definition of Π_C, we see that

$$\begin{aligned} \phi(\Pi_C x_{n+m}, x_{n+m}) &= \min\{\phi(y, x_{n+m}) : y \in C\} \le \phi(\Pi_C x_n, x_{n+m}) \\ &\le \phi(\Pi_C x_n, x_{n+m-1}) \le \phi(\Pi_C x_n, x_n) \end{aligned} \tag{2.9}$$

for all $m, n \in \mathbb{N}$. Therefore $\{\phi(\Pi_C x_n, x_n)\}$ is convergent. Further, it follows from (2.7) and (2.9) that

$$\begin{aligned} \phi(\Pi_C x_n, \Pi_C x_{n+m}) &\le \phi(\Pi_C x_n, x_{n+m}) - \phi(\Pi_C x_{n+m}, x_{n+m}) \\ &\le \phi(\Pi_C x_n, x_n) - \phi(\Pi_C x_{n+m}, x_{n+m}) \end{aligned}$$

for all $m, n \in \mathbb{N}$. Hence Lemma 2.3 implies that $\{\Pi_C x_n\}$ is strongly convergent.

We next show (2). By assumption, there exist $p \in C$ and $\delta > 0$ such that $\|h\| \le \delta$ implies that $p + h \in C$. Fix $h \in E$ such that $\|h\| \le \delta$. Then it follows from (2.4) that

$$\begin{aligned}\phi(p,x_{n+1}) &= \phi(p,x_n)+\phi(x_n,x_{n+1})+2\langle p-x_n, Jx_n-Jx_{n+1}\rangle \\ &= \phi(p,x_n)+\phi(x_n,x_{n+1})+2\langle p+h-x_n, Jx_n-Jx_{n+1}\rangle \\ &\quad -2\langle h, Jx_n-Jx_{n+1}\rangle \\ &= \phi(p,x_n)+\phi(p+h,x_{n+1})-\phi(p+h,x_n) \\ &\quad -2\langle h, Jx_n-Jx_{n+1}\rangle. \end{aligned} \tag{2.10}$$

By assumption, we have

$$\phi(p+h,x_{n+1}) \le \phi(p+h,x_n). \tag{2.11}$$

By (2.10) and (2.11), we obtain

$$2\langle h, Jx_n-Jx_{n+1}\rangle \le \phi(p,x_n)-\phi(p,x_{n+1}).$$

This implies that

$$2\delta\|Jx_n-Jx_{n+1}\| = 2\sup_{\|h\|\le\delta}\langle h, Jx_n-Jx_{n+1}\rangle \le \phi(p,x_n)-\phi(p,x_{n+1})$$

for all $n\in\mathbb{N}$. Thus we obtain

$$2\delta\|Jx_n-Jx_m\| \le \phi(p,x_n)-\phi(p,x_m)$$

for all $m,n\in\mathbb{N}$ with $n<m$. By the fact that $\{\phi(p,x_n)\}$ is convergent, we know that $\{Jx_n\}$ is a Cauchy sequence in E^*. Since J^{-1} is norm-to-norm continuous, $\{x_n\}$ converges strongly in E. □

Let A be a set-valued mapping of E into E^*, which is often denoted by $A\subset E\times E^*$. The effective domain of A is denoted by $\mathrm{dom}(A)$ and the range of A by $R(A)$, that is, $\mathrm{dom}(A)=\{x\in E : Ax\neq\emptyset\}$ and $R(A)=\bigcup_{x\in\mathrm{dom}(A)}Ax$. A set-valued mapping $A\subset E\times E^*$ is said to be a *monotone operator* if $\langle x-y, x^*-y^*\rangle\ge 0$ for all $(x,x^*),(y,y^*)\in A$. A monotone operator $A\subset E\times E^*$ is said to be *maximal* if $A=A'$ whenever $A'\subset E\times E^*$ is a monotone operator such that $A\subset A'$. It is known that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex, where $A^{-1}0=\{x\in E : Ax\ni 0\}$. We know that the duality mapping J of E is a monotone operator. We also know that if E is strictly convex, then J is *strictly monotone*, that is, $\langle x-y, x^*-y^*\rangle>0$ whenever $x,y\in E$, $x\neq y$, $x^*\in Jx$, and $y^*\in Jy$.

Let E be a smooth, strictly convex, and reflexive Banach space, $A\subset E\times E^*$ a maximal monotone operator, and $r>0$. Then it is known that $R(J+rA)=E^*$; see [25]. Thus the single-valued mapping $Q_r=(J+rA)^{-1}J$ of E onto $\mathrm{dom}(A)$ is well defined and is called the *resolvent* of A. It is also known that $F(Q_r)=A^{-1}0$ and

$$\phi(u,Q_rx)+\phi(Q_rx,x)\le\phi(u,x) \tag{2.12}$$

for all $x\in E$ and $u\in F(Q_r)$, where $F(Q_r)$ is the set of fixed points of Q_r; see [12, 13, 16].

Let E be a smooth Banach space, C a nonempty subset of E, and $T\colon C \to E$ a mapping. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* of T [8, 24] if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping T is said to be *of type (r)* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is said to be *of type (sr)* if T is of type (r) and $\phi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence in C such that $\phi(p, x_n) - \phi(p, Tx_n) \to 0$ for some $p \in F(T)$. We know that if C is a nonempty closed convex subset of a strictly convex and smooth Banach space E and $T\colon C \to E$ is of type (r), then $F(T)$ is closed and convex; see [23, Proposition 2.4].

A mapping $T\colon C \to E$ is said to be *relatively nonexpansive* [7, 22, 23] if it is of type (r) and $F(T) = \hat{F}(T)$. A mapping $T\colon C \to E$ is said to be *strongly relatively nonexpansive* [5, 18, 24] if it is of type (sr) and $F(T) = \hat{F}(T)$. It is easy to check that every generalized projection is strongly relatively nonexpansive. It is known that if $r > 0$, then the resolvent Q_r of a maximal monotone operator defined in a strictly convex and reflexive Banach space whose norm is uniformly Gâteaux differentiable is strongly relatively nonexpansive [20]; see also [12, 13, 22, 23].

The following results are proved in [4]; see also [5, 17, 24].

Lemma 2.7. *Let C and D be nonempty subsets of a smooth and strictly convex Banach space E. Let $S\colon C \to E$ and $T\colon D \to E$ be mappings of type (r) such that $T(D) \subset C$ and $F(S) \cap F(T)$ is nonempty. Suppose that S or T is of type (sr). Then the following hold:*

1. *$F(S) \cap F(T) = F(ST)$ and ST is of type (r).*
2. *If, in addition, E is uniformly convex and both S and T are of type (sr), then ST is also of type (sr).*

Lemma 2.8. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Let C and D be nonempty subsets of E. Let $S\colon C \to E$ and $T\colon D \to E$ be relatively nonexpansive mappings such that $T(D) \subset C$ and $F(S) \cap F(T) \neq \emptyset$. Suppose that S or T is strongly relatively nonexpansive. Then $\hat{F}(ST) = F(ST) = F(S) \cap F(T)$ and ST is relatively nonexpansive.*

Lemma 2.9. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. Let $\{T_1, \dots, T_n\}$ be a finite family of mappings of type (r) of C into E such that $F = \bigcap_{i=1}^n F(T_i)$ is nonempty, where n is a positive integer. Let $V : C \to E$ be a mapping defined by $V = J^{-1}(\lambda_0 J + \sum_{i=1}^n \lambda_i J T_i)$, where $\{\lambda_i\}_{i=0}^n$ is a finite sequence in $(0, 1)$ such that $\sum_{i=0}^n \lambda_i = 1$. Then $F(V) = F$ and V is of type (sr).*

From Lemmas 2.7 and 2.9, we obtain the following:

Corollary 2.10. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty subset of E, and $T\colon C \to E$ a mapping of type (r). Suppose that*

$U\colon C \to C$ is a mapping defined by $U = \Pi_C J^{-1}(\lambda J + (1-\lambda)JT)$, where $\lambda \in (0,1)$ is a constant. Then $F(U) = F(T)$ and U is of type (sr).

Proof. Put $V = J^{-1}(\lambda J + (1-\lambda)JT)$. Since $F(T)$ is nonempty, we see that $F(V) = F(T)$ and V is of type (sr) by Lemma 2.9. Since $F(\Pi_C) \cap F(V) = C \cap F(T) = F(T) \neq \emptyset$ and both Π_C and V are of type (sr), it follows from Lemma 2.7 that $F(\Pi_C V) = F(\Pi_C) \cap F(V) = F(T)$ and $\Pi_C V$ is of type (sr). □

3. Strongly relatively nonexpansive sequences

Throughout this section, we assume that E is a smooth Banach space and C is a nonempty subset of E. We introduce the concept of a strongly relatively nonexpansive sequence and investigate its properties.

Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Then $\{T_n\}$ is said to be a *strongly relatively nonexpansive sequence* if each T_n is of type (r) and $\phi(T_n x_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and $\phi(p, x_n) - \phi(p, T_n x_n) \to 0$ for some point $p \in F$.

Example 3.1. Let $T\colon C \to E$ be a mapping of type (sr). Put $T_n = T$ for $n \in \mathbb{N}$. Then it is clear that $\{T_n\}$ is a strongly relatively nonexpansive sequence.

Example 3.2. Let E be a smooth, strictly convex, and reflexive Banach space and $\{T_n\}$ a sequence of mappings of C into E with a common fixed point. Suppose that, for each $n \in \mathbb{N}$,

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x)$$

for all $x \in C$ and $u \in F(T_n)$. Then $\{T_n\}$ is a strongly relatively nonexpansive sequence. Thus it follows from (2.7) and (2.12) that

- a sequence $\{\Pi_{C_n}\}$ of generalized projections is a strongly relatively nonexpansive sequence if $\{C_n\}$ is a sequence of closed convex subsets of E with a common intersection;
- a sequence $\{Q_{r_n}\}$ of resolvents of a maximal monotone operator A if $A^{-1}0$ is nonempty and $\{r_n\}$ is a sequence of positive real numbers.

The following lemma shows that every subsequence of a strongly relatively nonexpansive sequence is also a strongly relatively nonexpansive sequence in the appropriate setting.

Lemma 3.3. *Let C be a nonempty subset of a smooth Banach space E and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that $\{T_n\}$ is a strongly relatively nonexpansive sequence. If $\{T_{n_i}\}$ is a subsequence of $\{T_n\}$ with $F = \bigcap_{i=1}^{\infty} F(T_{n_i})$, then $\{T_{n_i}\}$ is a strongly relatively nonexpansive sequence.*

Proof. Let $\{u_i\}$ be a bounded sequence in C and $p \in \bigcap_{i=1}^{\infty} F(T_{n_i})$ such that $\phi(p, u_i) - \phi(p, T_{n_i} u_i) \to 0$. Let $z \in F$ be fixed. Define a sequence $\{x_n\}$ in C: For each $n \in \mathbb{N}$, if there is $i \in \mathbb{N}$ such that $n_i = n$, then $x_n = u_i$; if $n_i \neq n$ for all $i \in \mathbb{N}$, then $x_n = z$. Then it is clear that $\phi(p, x_n) - \phi(p, T_n x_n) \to 0$ and $p \in F$. Since $\{T_n\}$ is a strongly relatively nonexpansive sequence, we have $\phi(T_n x_n, x_n) \to 0$ and hence $\phi(T_{n_i} u_i, u_i) \to 0$, which completes the proof. $\square$

The composition of two strongly relatively nonexpansive sequences is also a strongly relatively nonexpansive sequence:

Theorem 3.4. *Let C and D be two nonempty subsets of a uniformly convex and smooth Banach space E. Let $\{S_n\}$ be a sequence of mappings of C into E and $\{T_n\}$ a sequence of mappings of D into E such that $F = \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $T_n(D) \subset C$ for every $n \in \mathbb{N}$. Suppose that both $\{S_n\}$ and $\{T_n\}$ are strongly relatively nonexpansive sequences, and that S_n or T_n is of type (sr) for every $n \in \mathbb{N}$. Then $\{S_n T_n\}$ is a strongly relatively nonexpansive sequence.*

Proof. It follows from Lemma 2.7 that $F(S_n T_n) = F(S_n) \cap F(T_n)$ and $S_n T_n$ is of type (r). Thus we have

$$\bigcap_{n=1}^{\infty} F(S_n T_n) = \bigcap_{n=1}^{\infty} (F(S_n) \cap F(T_n)) = F \neq \emptyset.$$

Let $\{x_n\}$ be a bounded sequence in D and $p \in F$ such that

$$\phi(p, x_n) - \phi(p, S_n T_n x_n) \to 0.$$

Since $p \in F \subset F(S_n T_n) = F(S_n) \cap F(T_n)$ and both S_n and T_n are of type (r), we have

$$0 \leq \phi(p, x_n) - \phi(p, T_n x_n) \leq \phi(p, x_n) - \phi(p, S_n T_n x_n) \to 0.$$

Since $\{T_n\}$ is a strongly relatively nonexpansive sequence, we obtain

$$\phi(T_n x_n, x_n) \to 0. \tag{3.1}$$

Similarly, we have

$$0 \leq \phi(p, T_n x_n) - \phi(p, S_n T_n x_n) \leq \phi(p, x_n) - \phi(p, S_n T_n x_n) \to 0.$$

Since $\{T_n x_n\}$ is bounded and $\{S_n\}$ is a strongly relatively nonexpansive sequence, we obtain

$$\phi(S_n T_n x_n, T_n x_n) \to 0. \tag{3.2}$$

Therefore, from (2.4), (3.1), (3.2) and Lemma 2.2, we have

$$\begin{aligned} &\phi(S_n T_n x_n, x_n) \\ &\quad = \phi(S_n T_n x_n, T_n x_n) + \phi(T_n x_n, x_n) + 2\langle S_n T_n x_n - T_n x_n, J T_n x_n - J x_n \rangle \\ &\quad \leq \phi(S_n T_n x_n, T_n x_n) + \phi(T_n x_n, x_n) + 2\|S_n T_n x_n - T_n x_n\| \|J T_n x_n - J x_n\| \to 0. \end{aligned}$$

Consequently $\{S_n T_n\}$ is a strongly relatively nonexpansive sequence. $\square$

The following corollary is immediately derived from Example 3.2 and Theorem 3.4.

Corollary 3.5. *Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Let $\{T_n\}$ be a sequence of mappings of D into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ is a strongly relatively nonexpansive sequence, then $\{\Pi_D T_n\}$ is a strongly relatively nonexpansive sequence.*

A sequence $\{z_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping $T\colon C \to E$ if $\|z_n - Tz_n\| \to 0$; see [10, 11]. The set of all bounded approximate fixed point sequences of T is denoted by $\tilde{F}(T)$, that is,

$$\tilde{F}(T) = \Big\{\{z_n\} : z_n \in C \text{ for all } n \in \mathbb{N}, \sup_{n\in\mathbb{N}}\|z_n\| < \infty, \text{ and } \|z_n - Tz_n\| \to 0\Big\}.$$

Let $\{T_n\}$ be a sequence of mappings of C into E. A sequence $\{z_n\}$ in C is said to be an approximate fixed point sequence of $\{T_n\}$ if $\|z_n - T_n z_n\| \to 0$. The set of all bounded approximate fixed point sequences of $\{T_n\}$ is denoted by $\tilde{F}(\{T_n\})$, that is,

$$\tilde{F}(\{T_n\}) = \Big\{\{z_n\} : z_n \in C \text{ for all } n \in \mathbb{N}, \sup_{n\in\mathbb{N}}\|z_n\| < \infty, \text{ and } \|z_n - T_n z_n\| \to 0\Big\}.$$

Clearly, if $\{T_n\}$ has a common fixed point, then every bounded sequence in the common fixed point set $\bigcap_{n=1}^{\infty} F(T_n)$ is an approximate fixed point sequence of $\{T_n\}$.

Next we examine a relationship between strongly relatively nonexpansive sequences and approximate fixed point sequences.

Lemma 3.6. *Let E be a uniformly convex Banach whose norm is uniformly Gâteaux differentiable. Let C and D be two nonempty subsets of E. Let $\{S_n\}$ be a sequence of mappings of C into E and $\{T_n\}$ a sequence of mappings of D into E such that $F = \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $T_n(D) \subset C$ for every $n \in \mathbb{N}$. Suppose that both S_n and T_n are of type (r) for every $n \in \mathbb{N}$ and that $\{S_n\}$ or $\{T_n\}$ is a strongly relatively nonexpansive sequence. If $\{z_n\} \in \tilde{F}(\{S_n T_n\})$, then $\{z_n\} \in \tilde{F}(\{T_n\})$ and $\{T_n z_n\} \in \tilde{F}(\{S_n\})$.*

Proof. Let $\{z_n\} \in \tilde{F}(\{S_n T_n\})$ and $p \in F$. Since both $\{z_n\}$ and $\{S_n T_n z_n\}$ are bounded and the duality mapping J is uniformly norm-to-weak* continuous on a bounded set, we have

$$\begin{aligned}
&\phi(p, z_n) - \phi(p, S_n T_n z_n) \\
&\quad = -2\langle p, Jz_n\rangle + \|z_n\|^2 + 2\langle p, JS_n T_n z_n\rangle - \|S_n T_n z_n\|^2 \\
&\quad = (\|z_n\| + \|S_n T_n z_n\|)(\|z_n\| - \|S_n T_n z_n\|) - 2\langle p, Jz_n - JS_n T_n z_n\rangle \\
&\quad \leq (\|z_n\| + \|S_n T_n z_n\|)(\|z_n - S_n T_n z_n\|) - 2\langle p, Jz_n - JS_n T_n z_n\rangle \to 0.
\end{aligned}$$

As in the proof of Theorem 3.4, we see that

$$\phi(p, z_n) - \phi(p, T_n z_n) \to 0 \quad \text{and} \quad \phi(p, T_n z_n) - \phi(p, S_n T_n z_n) \to 0. \tag{3.3}$$

Note that $\{T_n z_n\}$ is a bounded sequence in C. Now suppose that $\{T_n\}$ is a strongly relatively nonexpansive sequence. Then we obtain $\phi(T_n z_n, z_n) \to 0$. In this case, according to Lemma 2.2, we have $\|z_n - T_n z_n\| \to 0$. By assumption, we have $\|z_n - S_n T_n z_n\| \to 0$. These facts yield

$$\|T_n z_n - S_n T_n z_n\| \le \|T_n z_n - z_n\| + \|z_n - S_n T_n z_n\| \to 0.$$

Therefore $\{z_n\} \in \tilde{F}(\{T_n\})$ and $\{T_n z_n\} \in \tilde{F}(\{S_n\})$.

On the other hand, suppose that $\{S_n\}$ is a strongly relatively nonexpansive sequence. It follows from (3.3) that $\phi(S_n T_n z_n, T_n z_n) \to 0$. So, we deduce from Lemma 2.2 that $\|S_n T_n z_n - T_n z_n\| \to 0$ and

$$\|z_n - T_n z_n\| \le \|z_n - S_n T_n z_n\| + \|S_n T_n z_n - T_n z_n\| \to 0.$$

Hence we also obtain the desired result. This completes the proof. □

As a special case of Lemma 3.6, we obtain the following:

Corollary 3.7. *Let E be a uniformly convex Banach whose norm is uniformly Gâteaux differentiable. Let C and D be two nonempty subsets of E. Let $S\colon C \to E$ be a mapping of type (r) and $\{T_n\}$ a sequence of mappings of D into E such that $F = F(S) \cap \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $T_n(D) \subset C$ for every $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is a strongly relatively nonexpansive sequence. If $\{z_n\} \in \tilde{F}(\{ST_n\})$, then $\{z_n\} \in \tilde{F}(\{T_n\})$ and $\{T_n z_n\} \in \tilde{F}(S)$.*

The following theorem shows an easy way to construct a strongly relatively nonexpansive sequence from an arbitrary sequence of mappings of type (r).

Theorem 3.8. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^\infty F(T_n)$ is nonempty. Let $V_n\colon C \to E$ be a mapping defined by $V_n = J^{-1}(\lambda_n J + (1-\lambda_n) J T_n)$ for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $(0,1)$. Suppose that T_n is of type (r) for every $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} \lambda_n > 0$. Then $\{V_n\}$ is a strongly relatively nonexpansive sequence. Moreover, if $\sup_{n\in\mathbb{N}} \lambda_n < 1$, then $\tilde{F}(\{T_n\}) = \tilde{F}(\{V_n\})$.*

Proof. Lemma 2.9 implies that $F(V_n) = F(T_n)$ and V_n is of type (r). Thus $F = \bigcap_{n=1}^\infty F(V_n)$. Let $\{x_n\}$ be a bounded sequence in C such that $\phi(p, x_n) - \phi(p, V_n x_n) \to 0$ for some $p \in F$. Then we have

$$\begin{aligned}
\lambda_n \|Jx_n\|^2 &+ (1-\lambda_n)\|JT_n x_n\|^2 - \|JV_n x_n\|^2 \\
&= \lambda_n \phi(p, x_n) + (1-\lambda_n)\phi(p, T_n x_n) - \phi(p, V_n x_n) \\
&\le \lambda_n \phi(p, x_n) + (1-\lambda_n)\phi(p, x_n) - \phi(p, V_n x_n) \\
&= \phi(p, x_n) - \phi(p, V_n x_n) \to 0.
\end{aligned}$$

Since E^* is uniformly convex and both $\{Jx_n\}$ and $\{JT_n x_n\}$ are bounded, it follows from Lemma 2.1 that

$$\|Jx_n - JV_n x_n\| = (1-\lambda_n)\|Jx_n - JT_n x_n\| \to 0. \tag{3.4}$$

Therefore, from (2.5) we conclude that $\phi(V_n x_n, x_n) \to 0$ and hence $\{V_n\}$ is a strongly relatively nonexpansive sequence.

Suppose that $\sup_{n\in\mathbb{N}} \lambda_n < 1$. From (2.5) and (3.4), it is easy to check that $\tilde{F}(\{T_n\}) = \tilde{F}(\{V_n\})$. □

We can also obtain the following corollary.

Corollary 3.9. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $U_n : C \to C$ be a mapping defined by $U_n = \Pi_C J^{-1}(\lambda_n J + (1-\lambda_n)JT_n)$ for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $(0,1)$. Suppose that T_n is of type (r) for every $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} \lambda_n > 0$. Then $\{U_n\}$ is a strongly relatively nonexpansive sequence. Moreover, if $\sup_{n\in\mathbb{N}} \lambda_n < 1$, then $\tilde{F}(\{U_n\}) = \tilde{F}(\{T_n\})$.*

Proof. Let V_n be a mapping of C into E defined by $V_n = J^{-1}(\lambda_n J + (1-\lambda_n)JT_n)$ for $n \in \mathbb{N}$. By Theorem 3.8, $\{V_n\}$ is a strongly relatively nonexpansive sequence, and moreover, $\tilde{F}(\{V_n\}) = \tilde{F}(\{T_n\})$ if $\sup_{n\in\mathbb{N}} \lambda_n < 1$.

On the other hand, Corollary 3.7 implies that $\tilde{F}(\{U_n\}) \subset \tilde{F}(\{V_n\})$. It is easy to see that the converse inclusion also holds. In fact, if $\{x_n\} \in \tilde{F}(\{V_n\})$, then it follows from $x_n \in C$ and the definition of Π_C that

$$\phi(U_n x_n, V_n x_n) = \phi(\Pi_C V_n x_n, V_n x_n) \le \phi(x_n, V_n x_n) \to 0.$$

This implies that $\|U_n x_n - V_n x_n\| \to 0$. Thus $\{x_n\} \in \tilde{F}(\{U_n\})$. □

4. Convergence theorem for strongly relatively nonexpansive sequences

In this section, we discuss a convergence theorem for a strongly relatively nonexpansive sequence.

Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. We say that $\{T_n\}$ satisfies the *condition* (Z) if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $\|T_n x_n - x_n\| \to 0$.

Theorem 4.1. *Let E be a uniformly convex and smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of self-mappings of C such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that $\{T_n\}$ is a strongly relatively nonexpansive sequence and satisfies the condition (Z). Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and $x_{n+1} = T_n x_n$ for $n \in \mathbb{N}$. Then the following hold:*

1. *If C is compact or the interior of F is nonempty, then $\{x_n\}$ converges strongly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*
2. *If the duality mapping J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*

Proof. Let $p \in F$ be fixed. Since each T_n is of type (r), $F(T_n)$ is closed and convex and

$$\phi(p, x_{n+1}) = \phi(p, T_n x_n) \le \phi(p, x_n) \tag{4.1}$$

for every $n \in \mathbb{N}$. Therefore F is a closed convex subset of E and it follows from (1) in Lemma 2.6 that $\{\Pi_F x_n\}$ converges strongly to some point in F. Put $v = \lim_{n\to\infty} \Pi_F x_n$. The inequality (4.1) also shows that $\{\phi(p, x_n)\}$ is bounded and convergent. Thus $\{x_n\}$ is also bounded and $\phi(p, x_n) - \phi(p, T_n x_n) \to 0$. Since $\{T_n\}$ is a strongly relatively nonexpansive sequence, we conclude that $\phi(T_n x_n, x_n) \to 0$. Hence Lemma 2.2 implies that $\|T_n x_n - x_n\| \to 0$.

We first show (1). Suppose that C is compact. Then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to u$. We claim that $u = v$. From the condition (Z), we see that $u \in F$. Thus (2.6) shows that

$$\langle \Pi_F x_n - u, Jx_n - J\Pi_F x_n \rangle \geq 0 \tag{4.2}$$

for every $n \in \mathbb{N}$. Since J is norm-to-weak continuous, $\{Jx_{n_i} - J\Pi_F x_{n_i}\}$ converges weakly to $Ju - Jv$. Therefore $\langle v - u, Ju - Jv\rangle \geq 0$ and hence $u = v$. Consequently, we conclude that $\{x_n\}$ converges strongly to v.

If the interior of F is nonempty, then (2) in Lemma 2.6 implies that $\{x_n\}$ converges strongly to u. By the condition (Z), we have $u \in F$. Taking the limit in (4.2), we obtain

$$\langle v - u, Ju - Jv\rangle \geq 0$$

and hence $u = v$. Thus we conclude that $\{x_n\}$ converges strongly to v.

We finally show (2). Since E is reflexive, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$. From the condition (Z), we see that $u \in F$. Since the duality mapping J is weakly sequentially continuous, $\{Jx_{n_i} - J\Pi_F x_{n_i}\}$ converges weakly to $Ju - Jv$. Therefore $\langle v - u, Ju - Jv\rangle \geq 0$ and hence $u = v$. Consequently, we conclude that $\{x_n\}$ converges weakly to v. □

Remark 4.2. In §5, we deal with an example of a sequence of mappings which satisfies the condition (Z).

In the remainder of this section, we consider an equivalent condition and a sufficient condition for the condition (Z).

Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Then

- we say that $\{T_n\}$ satisfies the *condition* (A) if $p \in F$ whenever $\{T_{n_i}\}$ is a subsequence of $\{T_n\}$ and $\{z_i\}$ is a sequence in C such that
$$\|T_{n_i} z_i - z_i\| \to 0 \quad \text{and} \quad z_i \rightharpoonup p;$$
- we say that $\{T_n\}$ satisfies the *condition* (B) if, for any nonempty bounded subset B of C and for any increasing sequence $\{n_i\}$ in $\mathbb{N}$, there exist a mapping $T\colon C \to E$ and a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ such that
$$\lim_{j\to\infty} \sup_{y\in B} \|Ty - T_{n_{i_j}} y\| = 0 \quad \text{and} \quad \hat{F}(T) = F.$$

Lemma 4.3. *Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Then the following hold:*

(1) $\{T_n\}$ *satisfies the condition* (Z) *if and only if* $\{T_n\}$ *satisfies the condition* (A).
(2) *If* $\{T_n\}$ *satisfies the condition* (B), *then it satisfies the condition* (Z).

Proof. We first show (1). It is obvious that the condition (A) implies the condition (Z). Thus we prove the inverse implication. Let $\{T_{n_i}\}$ be a subsequence of $\{T_n\}$ and $\{z_i\}$ a sequence in C such that $\|T_{n_i} z_i - z_i\| \to 0$ and $z_i \rightharpoonup p$. Let $u \in F$ be fixed. Define a sequence $\{x_n\}$ in C as follows:

$$x_n = \begin{cases} z_i & \text{if there exists } i \in \mathbb{N} \text{ such that } n = n_i, \\ u & \text{if } n \neq n_i \text{ for all } i \in \mathbb{N}, \end{cases}$$

for $n \in \mathbb{N}$. Then it is clear that $\{x_n\}$ is bounded, $T_n x_n - x_n \to 0$, and p is a weak cluster point of $\{x_n\}$. Thus the condition (Z) implies that $p \in F$.

We next show that the condition (B) implies the condition (Z). Let $\{x_n\}$ be a bounded sequence in C such that $\|T_n x_n - x_n\| \to 0$ and p a weak cluster point of $\{x_n\}$. Then there is an increasing sequence $\{n_i\}$ in $\mathbb{N}$ such that $x_{n_i} \rightharpoonup p$. Since $\{x_{n_i}\}$ is bounded, there is a bounded subset B of C such that $\{x_{n_i} : i \in \mathbb{N}\} \subset B$. By the condition (B), for $\{n_i\}$ and B, there exist a mapping $T\colon C \to E$ and a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ such that

$$\lim_{j\to\infty} \sup_{y\in B} \|Ty - T_{n_{i_j}} y\| = 0 \quad \text{and} \quad \hat{F}(T) = F.$$

Therefore we have

$$\begin{aligned} \|x_{n_{i_j}} - Tx_{n_{i_j}}\| &\leq \|x_{n_{i_j}} - T_{n_{i_j}} x_{n_{i_j}}\| + \|T_{n_{i_j}} x_{n_{i_j}} - Tx_{n_{i_j}}\| \\ &\leq \|x_{n_{i_j}} - T_{n_{i_j}} x_{n_{i_j}}\| + \sup_{y\in B} \|T_{n_{i_j}} y - Ty\| \end{aligned}$$

and hence $\|x_{n_{i_j}} - Tx_{n_{i_j}}\| \to 0$ as $j \to \infty$. Thus $p \in \hat{F}(T) = F$. This means that $\{T_n\}$ satisfies the condition (Z). This completes our proof. □

Remark 4.4. In §6, we deal with an example of a sequence of mappings which satisfies the condition (B).

5. W-mappings

In this section, we apply results in §3 and §4 to the problem of approximating a common fixed point of a countable family of relatively nonexpansive mappings. Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of mappings of C into E and $\{\alpha_k^n\}$ a sequence in $(0, 1)$ with $n \in \mathbb{N}$ and $k = 1, \ldots, n$. For each $n \in \mathbb{N}$ we define a finite family $\{U_k^n : k = 1, \dots, n+1\}$ of mappings as follows:

$$\begin{aligned} U_{n+1}^n &= I, \\ U_n^n &= \Pi_C J^{-1}(\alpha_n^n J + (1-\alpha_n^n) J T_n), \\ U_{n-1}^n &= \Pi_C J^{-1}(\alpha_{n-1}^n J + (1-\alpha_{n-1}^n) J T_{n-1} U_n^n), \\ &\vdots \\ U_k^n &= \Pi_C J^{-1}(\alpha_k^n J + (1-\alpha_k^n) J T_k U_{k+1}^n), \\ &\vdots \\ U_1^n &= \Pi_C J^{-1}(\alpha_1^n J + (1-\alpha_1^n) J T_1 U_2^n), \end{aligned} \tag{5.1}$$

for $n \in \mathbb{N}$, where I is the identity mapping on C. In this case a mapping U_1^n is denoted by W_n and we say that $W_n \colon C \to C$ is the *W-mapping* generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n^n, \alpha_{n-1}^n, \dots, \alpha_1^n$ for $n \in \mathbb{N}$. Takahashi [26] discussed the problem of approximating a common fixed point of a family of nonexpansive mappings by using W-mappings; see also [28], [15] and [21].

We begin with a property of the fixed point set of U_k^n above.

Lemma 5.1. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of mappings of type (r) of C into E such that $F = \bigcap_{n=1}^\infty F(T_n)$ is nonempty. Let $\{\alpha_k^n\}$ be a sequence in $(0,1)$ with $n \in \mathbb{N}$ and $k = 1, \dots, n$. Let $U_k^n \colon C \to C$ be defined by* (5.1) *for $n \in \mathbb{N}$ and $k = 1, \dots, n+1$. Then*

$$F(U_k^n) = F(T_k U_{k+1}^n) = F(T_k) \cap F(U_{k+1}^n) = \bigcap_{i=k}^n F(T_i), \tag{5.2}$$

$T_k U_{k+1}^n$ are of type (r) and U_k^n are of type (sr) for all $n \in \mathbb{N}$ and $k = 1, \dots, n$.

Proof. Let $n \in \mathbb{N}$ be fixed. Since T_n is of type (r), we see that (5.2) holds for $k = n$ and U_n^n is of type (sr) by Corollary 2.10. Assume that (5.2) holds and U_k^n is of type (sr) for some $k \in \{2, \dots, n\}$. Since T_{k-1} is of type (r), U_k^n is of type (sr), and

$$F(T_{k-1}) \cap F(U_k^n) = F(T_{k-1}) \cap \bigcap_{i=k}^n F(T_i) = \bigcap_{i=k-1}^n F(T_i) \supset F \neq \emptyset,$$

it follows from Lemma 2.7 that

$$F(T_{k-1} U_k^n) = F(T_{k-1}) \cap F(U_k^n) = \bigcap_{i=k-1}^n F(T_i)$$

and $T_{k-1} U_k^n$ is of type (r). Thus Corollary 2.10 implies that (5.2) holds for $k-1$ and U_{k-1}^n is of type (sr). By induction on k, we obtain the desired result. □

Using Lemma 5.1, we can prove the following:

Lemma 5.2. *Let E, C, $\{T_n\}$, F, $\{\alpha_k^n\}$, and $\{U_k^n\}$ be as in Lemma* 5.1. *Define $V_k^n \colon C \to E$ by*

$$V_k^n = J^{-1}(\alpha_k^n J + (1-\alpha_k^n) J T_k U_{k+1}^n) \tag{5.3}$$

for $n \in \mathbb{N}$ *and* $k = 1, \dots, n$. *Suppose that* $0 < \inf\{\alpha_k^n : n \in \mathbb{N}, n \geq k\}$ *for every* $k \in \mathbb{N}$. *Then the following hold:*

(1) $\{U_k^n\}_{n=k}^\infty$ *and* $\{V_k^n\}_{n=k}^\infty$ *are strongly relatively nonexpansive sequences for every* $k \in \mathbb{N}$.

(2) *If* T_n *is relatively nonexpansive for every* $n \in \mathbb{N}$ *and* $\sup\{\alpha_k^n : n \in \mathbb{N}, n \geq k\} < 1$ *for every* $k \in \mathbb{N}$, *then* $\{U_1^n\}$ *and* $\{V_1^n\}$ *satisfy the condition* (Z).

Proof. We first prove (1). Let $k \in \mathbb{N}$ be fixed. From Lemma 5.1, we see that

$$\bigcap_{n=k}^{\infty} F(U_k^n) = \bigcap_{n=k}^{\infty} F(T_k U_{k+1}^n) = \bigcap_{n=k}^{\infty} \bigcap_{i=k}^{n} F(T_i) = \bigcap_{n=k}^{\infty} F(T_n) \supset F \neq \emptyset \tag{5.4}$$

and $T_k U_{k+1}^n$ is of type (r) for every $n = k, k+1, \dots$. Thus Theorem 3.8 implies that $\{V_k^n\}_{n=k}^\infty$ is a strongly relatively nonexpansive sequence and hence $\{U_k^n\}_{n=k}^\infty$ is also a strongly relatively nonexpansive sequence by Corollary 3.5.

We next prove (2). Let $\{x_n\}$ be a bounded sequence in C with $\|x_n - U_1^n x_n\| \to 0$ and $x_{n_i} \rightharpoonup p$. Then (5.4) and the relative nonexpansiveness of each T_n imply that

$$\bigcap_{n=1}^{\infty} F(U_1^n) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} \hat{F}(T_n).$$

Thus, to prove that $\{U_1^n\}$ satisfies the condition (Z), it is sufficient to show that $p \in \bigcap_{n=1}^\infty \hat{F}(T_n)$. The proof is given by induction on n. By assumption, we have $\{x_n\} \in \tilde{F}(\{U_1^n\})$. Note that (5.4) ensures that

$$\bigcap_{n=1}^{\infty} F(T_1 U_2^n) = \bigcap_{n=1}^{\infty} F(T_n) = F \neq \emptyset.$$

Then Corollary 3.9 implies that $\tilde{F}(\{U_1^n\}) = \tilde{F}(\{T_1 U_2^n\})$. Since $\{U_2^n\}$ is a strongly relatively nonexpansive sequence and

$$F(T_1) \cap \bigcap_{n=2}^{\infty} F(U_2^n) = F(T_1) \cap \bigcap_{n=2}^{\infty} F(T_n) = F \neq \emptyset$$

by (5.4), Corollary 3.7 implies that $\tilde{F}(\{T_1 U_2^n\}) \subset \tilde{F}(\{U_2^n\})$. Thus we obtain

$$\tilde{F}(\{U_1^n\}) = \tilde{F}(\{T_1 U_2^n\}) \subset \tilde{F}(\{U_2^n\}). \tag{5.5}$$

Hence we have

$$\|U_2^n x_n - T_1 U_2^n x_n\| \leq \|U_2^n x_n - x_n\| + \|x_n - T_1 U_2^n x_n\| \to 0$$

and

$$U_2^{n_i} x_{n_i} = (U_2^{n_i} x_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p.$$

Therefore $p \in \hat{F}(T_1)$. We also know that $\{x_n\}_{n=2}^\infty \in \tilde{F}(\{U_2^n\}_{n=2}^\infty)$.

Suppose that $p \in \hat{F}(T_{k-1})$ and $\{x_n\}_{n=k}^\infty \in \tilde{F}(\{U_k^n\}_{n=k}^\infty)$ for some $k = 2, 3, \dots$. By (5.4), we have

$$\bigcap_{n=k}^{\infty} F(T_k U_{k+1}^n) \supset F \neq \emptyset$$

and

$$F(T_k) \cap \bigcap_{n=k+1}^{\infty} F(U_{k+1}^n) = \bigcap_{n=k}^{\infty} F(T_n) \supset F \neq \emptyset.$$

As in the proof of (5.5), Corollaries 3.7 and 3.9 ensure that

$$\tilde{F}(\{U_k^n\}_{n=k}^{\infty}) = \tilde{F}\left(\{T_k U_{k+1}^n\}_{n=k}^{\infty}\right) \subset \tilde{F}\left(\{U_{k+1}^n\}_{n=k}^{\infty}\right).$$

This gives us that

$$\|U_{k+1}^n x_n - T_k U_{k+1}^n x_n\| \leq \|U_{k+1}^n x_n - x_n\| + \|x_n - T_k U_{k+1}^n x_n\| \to 0$$

and

$$U_{k+1}^{n_i} x_{n_i} = \left(U_{k+1}^{n_i} x_{n_i} - x_{n_i}\right) + x_{n_i} \rightharpoonup p$$

and hence we obtain $p \in \hat{F}(T_k)$. We also know that

$$\{x_n\}_{n=k+1}^{\infty} \in \tilde{F}\left(\{U_{k+1}^n\}_{n=k+1}^{\infty}\right).$$

Therefore we conclude that $p \in \bigcap_{n=1}^{\infty} \hat{F}(T_n)$.

We can similarly prove that $\{V_1^n\}$ satisfies the condition (Z). This completes the proof. □

Using Theorem 4.1 and Lemmas 5.1 and 5.2, we immediately get the following:

Theorem 5.3. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_k^n\}$ be a sequence in $(0,1)$ with $n \in \mathbb{N}$ and $k = 1, \dots, n$ such that*

$$0 < \inf\{\alpha_k^n : n \in \mathbb{N},\, n \geq k\} \leq \sup\{\alpha_k^n : n \in \mathbb{N},\, n \geq k\} < 1$$

for every $k \in \mathbb{N}$. Let $W_n \colon C \to C$ be the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n^n, \alpha_{n-1}^n, \dots, \alpha_1^n$ for $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and $x_{n+1} = W_n x_n$ for $n \in \mathbb{N}$.

1. *If C is compact or the interior of F is nonempty, then $\{x_n\}$ converges strongly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*
2. *If the duality mapping J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*

Proof. It is clear from Lemma 5.1 that $F = \bigcap_{n=1}^{\infty} F(W_n)$. Lemma 5.2 shows that $\{W_n\}$ is a strongly relatively nonexpansive sequence and satisfies the condition (Z). Thus Theorem 4.1 implies the conclusion. □

The following two results are known:

Theorem 5.4 ([4, Theorem 4.2]). *Let E be a uniformly convex and smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of mappings of type (r) of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $\{T_n\}$ satisfies the condition* (A). *Let $x \in E$ and let $\{x_n\}$ be the sequence in C defined by $x_1 = \Pi_C x$ and*

$$\begin{cases} H_n = \{z \in C : \phi(z, T_n x_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F(x)$.

Theorem 5.5 ([4, Theorem 4.4]). *Let E be a uniformly convex and smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of mappings of type (r) of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $\{T_n\}$ satisfies the condition* (A). *Let $x \in E$ and let $\{x_n\}$ be the sequence in C defined by $x_1 = \Pi_C x \in C = C_0$ and*

$$\begin{cases} C_n = \{z \in C_{n-1} : \phi(z, T_n x_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_n}(x) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F(x)$.

Using Theorems 5.4 and 5.5, we also obtain the following strong convergence theorem:

Theorem 5.6. *Let E, C, $\{T_n\}$, F, and $\{\alpha_k^n\}$ be as in Theorem* 5.3. *Let U_k^n be a mapping defined by* (5.1) *for $n \in \mathbb{N}$ and $k = 1, \dots, n+1$. Let $V_1^n \colon C \to E$ be defined by* (5.3) *for $n \in \mathbb{N}$. Let $x \in E$ and let $\{x_n\}$ be the sequence in E defined by $x_1 = \Pi_C x$ and*

$$\begin{cases} H_n = \{z \in C : \phi(z, V_1^n x_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for $n \in \mathbb{N}$. Let $y \in E$ and let $\{y_n\}$ be the sequence in E defined by $y_1 = \Pi_C y \in C = C_0$ and

$$\begin{cases} C_n = \{z \in C_{n-1} : \phi(z, V_1^n y_n) \leq \phi(z, y_n)\}, \\ y_{n+1} = \Pi_{C_n}(y) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $\Pi_F(x)$ and $\Pi_F(y)$, respectively.

Proof. Lemma 5.1 shows that $F(T_1 U_2^n) = \bigcap_{i=1}^{n} F(T_i)$ and $T_1 U_2^n$ is of type (r). Thus, by Lemma 2.9, we see that $F(V_1^n) = F(T_1 U_2^n)$ and V_1^n is of type (r) for every $n \in \mathbb{N}$, and hence

$$\bigcap_{n=1}^{\infty} F(V_1^n) = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} F(T_i) = F \neq \emptyset.$$

It follows from Lemmas 4.3 and 5.2 that $\{V_1^n\}$ satisfies the condition (A). Therefore, from Theorems 5.4 and 5.5, we get the conclusion. □

6. Convex combinations of relatively nonexpansive mappings

Finally, we discuss another method of approximating a common fixed point of a countable family of relatively nonexpansive mappings.

We begin with the following lemma:

Lemma 6.1. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. Let $\{T_n\}$ be a sequence of mappings of type (r) of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_k^n\}$ be a sequence in $(0,1)$ with $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$ such that $\sum_{k=0}^{n} \lambda_k^n = 1$ for every $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} \lambda_0^n > 0$. For each n, let $V_n \colon C \to E$ be defined by*

$$V_n = J^{-1} \sum_{k=0}^{n} \lambda_k^n J T_k, \tag{6.1}$$

where T_0 is the identity mapping on C. Then $\{V_n\}$ is a strongly relatively nonexpansive sequence.

Proof. By the definition of V_n, we have

$$\begin{aligned} V_n &= J^{-1}\left(\frac{\lambda_0^n}{2} J + \frac{\lambda_0^n}{2} J + \sum_{k=1}^{n} \lambda_k^n J T_k\right) \\ &= J^{-1}\left(\frac{\lambda_0^n}{2} J + \left(1 - \frac{\lambda_0^n}{2}\right) J J^{-1}\left(\frac{\lambda_0^n}{2-\lambda_0^n} J + \sum_{k=1}^{n} \frac{2\lambda_k^n}{2-\lambda_0^n} J T_k\right)\right). \end{aligned}$$

For each $n \in \mathbb{N}$, define $V_n' \colon C \to E$ by

$$V_n' = J^{-1}\left(\frac{\lambda_0^n}{2-\lambda_0^n} J + \sum_{k=1}^{n} \frac{2\lambda_k^n}{2-\lambda_0^n} J T_k\right).$$

Then we see that $F(V_n') = \bigcap_{k=1}^{n} F(T_k)$ and V_n' is of type (r) by Lemma 2.9. Thus

$$\bigcap_{n=1}^{\infty} F(V_n') = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{n} F(T_k) = F \neq \emptyset.$$

Therefore Theorem 3.8 implies that $\{V_n\}$ is a strongly relatively nonexpansive sequence. □

For the following result, see [19, Theorem 3.3].

Lemma 6.2 ([19]). *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, $\{t_n\}$ a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} t_n = 1$, and $\{\alpha_n\}$ a sequence in $(0,1)$. Then the mapping $U\colon C \to E$ defined by*

$$Ux = J^{-1} \sum_{n=1}^{\infty} t_n(\alpha_n Jx + (1-\alpha_n)JT_n x)$$

for $x \in C$ is a relatively nonexpansive mapping and $F(U) = F$.

Using Lemma 6.2, we obtain the following:

Lemma 6.3. *Let E, C, $\{T_n\}$, and F be as in Lemma 6.2. Let T_0 be the identity mapping on C and $\{\lambda_n\}_{n=0}^{\infty}$ a sequence in $(0,1)$ such that $\sum_{n=0}^{\infty} \lambda_n = 1$. Let $V\colon C \to E$ be defined by*

$$Vx = J^{-1} \sum_{n=0}^{\infty} \lambda_n JT_n x \tag{6.2}$$

for $x \in C$. Then V is relatively nonexpansive and $F(V) = F$.

Proof. Let $x \in C$ be fixed. Since each T_n is of type (r), it follows from (2.3) that $(\|u\| - \|T_n x\|)^2 \le \phi(u, T_n x) \le \phi(u,x)$ for all $u \in F$ and $n \in \mathbb{N}$. Therefore $\sup_{n\in\mathbb{N}}\|JT_n x\|$ is finite and hence Vx is well-defined. Taking into account $1-\lambda_0 = \sum_{n=1}^{\infty} \lambda_n$, we have

$$\begin{aligned} J^{-1} \sum_{n=1}^{\infty} \frac{\lambda_n}{1-\lambda_0}(\lambda_0 Jx + (1-\lambda_0)JT_n x) &= J^{-1}\left(\frac{\lambda_0}{1-\lambda_0} Jx \sum_{n=1}^{\infty} \lambda_n + \sum_{n=1}^{\infty} \lambda_n JT_n x\right) \\ &= J^{-1} \sum_{n=0}^{\infty} \lambda_n JT_n x = Vx. \end{aligned}$$

It is clear that $\sum_{n=1}^{\infty} \lambda_n/(1-\lambda_0) = 1$ and $0 < \lambda_n/(1-\lambda_0) < 1$ for every $n \in \mathbb{N}$. Lemma 6.2 implies the conclusion. □

By using the results above together with Theorem 4.1 and Lemma 4.3, the following theorem is proved.

Theorem 6.4. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_k^n\}$ be a sequence in $(0,1)$ with $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$ such that $\sum_{k=0}^{n} \lambda_k^n = 1$ for every $n \in \mathbb{N}$, $\inf_{n\in\mathbb{N}} \lambda_0^n > 0$, and $\sum_{k=0}^{n} |\lambda_k - \lambda_k^n| \to 0$ as $n \to \infty$ for some sequence $\{\lambda_k\}_{k=0}^{\infty}$ in $(0,1)$. Let T_0 be the identity mapping on C. For each n, let $V_n\colon C \to E$ be defined by (6.1). Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and $x_{n+1} = \Pi_C V_n x_n$ for $n \in \mathbb{N}$.*

1. *If C is compact or the interior of F is nonempty, then $\{x_n\}$ converges strongly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*

(2) *If the duality mapping J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*

Proof. Lemmas 2.7 and 2.9 show that $F(\Pi_C V_n) = F(\Pi_C) \cap F(V_n) = \bigcap_{k=1}^{n} F(T_k)$ and hence

$$\bigcap_{n=1}^{\infty} F(\Pi_C V_n) = \bigcap_{n=1}^{\infty} F(V_n) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{n} F(T_k) = F \neq \emptyset.$$

Thus Lemma 6.1 and Corollary 3.5 imply that $\{\Pi_C V_n\}$ is a strongly relatively nonexpansive sequence. Let us show that $\{\Pi_C V_n\}$ satisfies the condition (B). Note that $\sum_{k=0}^{\infty} \lambda_k = 1$. Indeed, clearly we have

$$\lambda_k^n - |\lambda_k - \lambda_k^n| \le \lambda_k \le \lambda_k^n + |\lambda_k - \lambda_k^n|,$$

so that

$$\sum_{k=0}^{n} \lambda_k^n - \sum_{k=0}^{n} |\lambda_k - \lambda_k^n| \le \sum_{k=0}^{n} \lambda_k \le \sum_{k=0}^{n} \lambda_k^n + \sum_{k=0}^{n} |\lambda_k - \lambda_k^n|$$

for every $n \in \mathbb{N}$. Taking the limit $n \to \infty$, we obtain $\sum_{k=0}^{\infty} \lambda_k = 1$. Let B be a bounded subset of C. Since F is nonempty and each T_n is of type (r), it follows from (2.3) that

$$(\|u\| - \|T_n y\|)^2 \le \phi(u, T_n y) \le \phi(u, y) \le (\|u\| + \|y\|)^2$$

for all $y \in B$, $u \in F$, and $n \in \mathbb{N}$. Therefore $M = \sup\{\|T_n y\| : n \in \mathbb{N},\, y \in B\}$ is finite. Define a mapping $V\colon C \to E$ by (6.2). Then we have

$$\begin{aligned}
\|JVy - JV_n y\| &= \Big\| \sum_{k=0}^{n} (\lambda_k - \lambda_k^n) JT_k y + \sum_{k=n+1}^{\infty} \lambda_k JT_k y \Big\| \\
&\le \sum_{k=0}^{n} |\lambda_k - \lambda_k^n| \|JT_k y\| + \sum_{k=n+1}^{\infty} \lambda_k \|JT_k y\| \\
&\le \Big(\sum_{k=0}^{n} |\lambda_k - \lambda_k^n| + \sum_{k=n+1}^{\infty} \lambda_k \Big) M.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \sup_{y\in B} \|JVy - JV_n y\| = 0$. Since E^* is uniformly smooth, we have

$$\lim_{n\to\infty} \sup_{y\in B} \|J^{-1}JVy - J^{-1}JV_n y\| = 0.$$

By Lemma 2.5, we see that Π_C is uniformly norm-to-norm continuous on every bounded set. Thus we obtain

$$\lim_{n\to\infty} \sup_{y\in B} \|\Pi_C V y - \Pi_C V_n y\| = 0.$$

On the other hand, Lemma 6.3 shows that V is relatively nonexpansive and $F(V) = F$. Since $F(\Pi_C) \cap F(V) = F \neq \emptyset$ and Π_C is strongly relatively nonexpansive, it follows from Lemma 2.8 that $F(\Pi_C V) = \hat{F}(\Pi_C V) = F(\Pi_C) \cap F(V) = F$.

This means that $\{\Pi_C V_n\}$ satisfies the condition (B). Therefore, by Theorem 4.1 and Lemma 4.3, we obtain the conclusion. □

A direct consequence of Theorem 6.4 is as follows; see [2].

Corollary 6.5. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{V_n\}$ be the sequence of mappings of C into E defined by*

$$\begin{aligned} V_1 &= J^{-1}\left(\frac{1}{2}J + \frac{1}{2}JT_1\right), \\ V_2 &= J^{-1}\left(\frac{1}{2}J + \frac{1}{4}JT_1 + \frac{1}{4}JT_2\right), \\ &\vdots \\ V_n &= J^{-1}\left(\frac{1}{2}J + \frac{1}{4}JT_1 + \cdots + \frac{1}{2^n}JT_{n-1} + \frac{1}{2^n}JT_n\right), \\ &\vdots \end{aligned}$$

and let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and $x_{n+1} = \Pi_C V_n x_n$ for $n \in \mathbb{N}$.

(1) *If C is compact or the interior of F is nonempty, then $\{x_n\}$ converges strongly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*
(2) *If the duality mapping J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $v \in F$, where $v = \lim_{n\to\infty} \Pi_F x_n$.*

Proof. Let $\{\lambda_k^n\}$ be a sequence with $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$ defined by

$$\lambda_k^n = \begin{cases} 1/2^{k+1}, & k = 0, 1, \ldots, n-1, \\ 1/2^n, & k = n. \end{cases}$$

Then it is clear that $\inf_{n\in\mathbb{N}} \lambda_0^n = 1/2 > 0$, $\sum_{k=0}^n \lambda_k^n = 1$ for every $n \in \mathbb{N}$ and

$$\sum_{k=0}^{n} \left| \frac{1}{2^{k+1}} - \lambda_k^n \right| = \left| \frac{1}{2^{n+1}} - \frac{1}{2^n} \right| = \frac{1}{2^{n+1}} \to 0.$$

Thus Theorem 6.4 implies the conclusion. □

It is easy to verify that $\{V_n\}$ in Theorem 6.4 also satisfies the condition (B). Consequently, we obtain the following strong convergence theorem by using Theorems 5.4 and 5.5 together with Lemmas 2.9 and 4.3.

Theorem 6.6. *Let E, C, $\{T_n\}$, F, $\{\lambda_k^n\}$, T_0, and $\{V_n\}$ be as in Theorem 6.4. Let $x \in E$ and let $\{x_n\}$ be the sequence in E defined by $x_1 = \Pi_C x$ and*

$$\begin{cases} H_n = \{z \in C : \phi(z, V_n x_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x) \end{cases}$$

for $n \in \mathbb{N}$. *Let* $y \in E$ *and let* $\{y_n\}$ *be the sequence in* E *defined by* $y_1 = \Pi_C y \in C = C_0$ *and*

$$\begin{cases} C_n = \{z \in C_{n-1} : \phi(z, V_n y_n) \le \phi(z, y_n)\}, \\ y_{n+1} = \Pi_{C_n}(y) \end{cases}$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $\Pi_F(x)$ *and* $\Pi_F(y)$, *respectively.*

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References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications.* In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure Appl. Math. 178, Dekker, New York, 1996, 15–50.

[2] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space.* Nonlinear Anal. **67** (2007), 2350–2360.

[3] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *Strongly nonexpansive sequences and their applications in Banach spaces.* In: Fixed Point Theory and its Applications, Yokohama Publ., Yokohama, 2008, 1–18.

[4] K. Aoyama, F. Kohsaka, and W. Takahashi, *Strong convergence theorems by shrinking and hybrid projection methods for relatively nonexpansive mappings in Banach spaces.* In: Nonlinear Analysis and Convex Analysis, Yokohama Publ., Yokohama, 2009, 7–26.

[5] K. Aoyama and W. Takahashi, *Strong convergence theorems for a family of relatively nonexpansive mappings in Banach spaces.* Fixed Point Theory **8** (2007), 143–160.

[6] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces.* Houston J. Math. **3** (1977), 459–470.

[7] D. Butnariu, S. Reich, and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces.* J. Appl. Anal. **7** (2001), 151–174.

[8] Y. Censor and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization.* Optimization **37** (1996), 323–339.

[9] J. Diestel, *Geometry of Banach Spaces—Selected Topics.* Lecture Notes in Math. 485, Springer, Berlin, 1975.

[10] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory.* Cambridge Stud. Adv. Math. 28, Cambridge Univ. Press, Cambridge, 1990.

[11] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings.* Monogr. Textbooks Pure Appl. Math. 83, Dekker, New York, 1984.

[12] S. Kamimura, *The proximal point algorithm in a Banach space.* In: Nonlinear Analysis and Convex Analysis, Yokohama Publ., Yokohama, 2004, 143–148.

[13] S. Kamimura, F. Kohsaka, and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*. Set-Valued Anal. **12** (2004), 417–429.

[14] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*. SIAM J. Optim. **13** (2002), 938–945.

[15] Y. Kimura and W. Takahashi, *Weak convergence to common fixed points of countable nonexpansive mappings and its applications*. J. Korean Math. Soc. **38** (2001), 1275–1284.

[16] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*. Abstr. Appl. Anal. **2004**, 239–249.

[17] F. Kohsaka and W. Takahashi, *Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces*. Fixed Point Theory Appl. **2007**, art. ID 21972, 18 pp.

[18] F. Kohsaka and W. Takahashi, *Approximating common fixed points of countable families of strongly nonexpansive mappings*. Nonlinear Stud. **14** (2007), 219–234.

[19] F. Kohsaka and W. Takahashi, *The set of common fixed points of an infinite family of relatively nonexpansive mappings*. In: Banach and Function Spaces II, Yokohama Publ., Yokohama, 2008, 361–373.

[20] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*. SIAM J. Optim. **19** (2008), 824–835.

[21] P. K. F. Kuhfittig, *Common fixed points of nonexpansive mappings by iteration*. Pacific J. Math. **97** (1981), 137–139.

[22] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*. Fixed Point Theory Appl. **2004**, 37–47.

[23] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*. J. Approx. Theory **134** (2005), 257–266.

[24] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure Appl. Math. 178, Dekker, New York, 1996, 313–318.

[25] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*. Trans. Amer. Math. Soc. **149** (1970), 75–88.

[26] W. Takahashi, *Weak and strong convergence theorems for families of nonexpansive mappings and their applications*. In: Proceedings of Workshop on Fixed Point Theory (Kazimierz Dolny, 1997), Ann. Univ. Mariae Curie-Skłodowska Sect. A **51** (1997), 277–292.

[27] W. Takahashi, *Nonlinear Functional Analysis*. Yokohama Publ., Yokohama, 2000.

[28] W. Takahashi and K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*. Math. Comput. Modelling **32** (2000), 1463–1471.

[29] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*. J. Optim. Theory Appl. **118** (2003), 417–428.

[30] C. Zălinescu, *On uniformly convex functions*. J. Math. Anal. Appl. **95** (1983), 344–374.

Koji Aoyama
Department of Economics
Chiba University
Yayoi-cho, Inage-ku, Chiba-shi
Chiba 263-8522, Japan
e-mail: aoyama@le.chiba-u.ac.jp

Fumiaki Kohsaka
Department of Computer Science and Intelligent Systems
Oita University
Dannoharu, Oita-shi
Oita 870-1192, Japan
e-mail: f-kohsaka@csis.oita-u.ac.jp

Wataru Takahashi
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Ookayama, Meguro-ku
Tokyo 152-8552, Japan
e-mail: wataru@is.titech.ac.jp

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