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Existence of strong solutions of Pucci extremal equations with superlinear growth in Du

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Abstract. We prove existence of strong solutions of Pucci extremal equations with superlinear growth in Du and unbounded coefficients. We apply this result to establish the weak Harnack inequality for L^p-viscosity supersolutions of fully nonlinear uniformly elliptic PDEs with superlinear growth terms with respect to Du.

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1. Introduction

Pucci extremal equations [3, 8, 4] are one of the fundamental objects in the theory of fully nonlinear uniformly elliptic partial differential equations. For the ellipticity constants $0 < \lambda < \Lambda$, the Pucci extremal operators $\mathcal{P}^{\pm} : S^n \to \mathbb{R}$, where S^n is the set of $n \times n$ symmetric matrices, are defined in the following way:

$$\mathcal{P}^{+}(X) = \max\{-\mathrm{trace}(AX) \mid A \in S^n,\ \lambda I \le A \le \Lambda I\},$$

and $\mathcal{P}^{-}(X) = -\mathcal{P}^{+}(-X)$. In this paper we obtain existence of L^p-strong solutions of Dirichlet boundary value problems for Pucci extremal equations with superlinear growth in Du and unbounded coefficients of the form

$$\mathcal{P}^{\pm}(D^2u) \pm \gamma(x)|Du| \pm \mu(x)|Du|^m = f(x) \quad \text{in } \Omega, \qquad u = \phi \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^{1,1}$, $\mu \in L^q(\Omega), \gamma \in L^{q_1}(\Omega), f \in L^p(\Omega)$, and $m > 1$, under some conditions on p, q, q_1. The result is proved with the help of the standard Schauder fixed point theorem and the recent global $W^{2,p}$-estimates for solutions of fully nonlinear equations [16]. The global $W^{2,p}$-estimates of [16] are extensions of the interior $W^{2,p}$-estimates of Caffarelli [2, 3] to the boundary. Our proof is very simple and gives existence of solutions of (1.1) in $W^{2,p}(\Omega)$; however, it comes at a price of assuming that $\partial\Omega$ is of class $C^{1,1}$

and $\psi \in W^{2,p}(\Omega)$. The existence of strong solutions of (1.1) is key in establishing the weak Harnack inequality for L^p-viscosity solutions of equations with superlinear growth in Du, which in turn implies the global C^α-continuity of L^p-viscosity solutions of such equations. This opens up the possibility of other existence results for (1.1) and general equations (2.1) which we do not pursue in this paper.

There exist several results on existence of L^p-viscosity and L^p-strong solutions of (1.1) and its more general versions. The precise meaning of these notions of solutions is explained in the next section. Here we just mention that throughout the paper we will always assume that

$$p > p_0(n, \Lambda/\lambda) \in [n/2, n),$$

where p_0 is the constant which gives the range of exponents for which an Aleksandrov–Bakelman–Pucci type maximum principle holds for (1.1) with $\gamma \equiv \mu \equiv 0$ (see [5, 6]). If $\gamma, \mu \in L^\infty(\Omega)$ and $\|\mu\|_\infty \|f\|_p$ is small, existence of L^p-viscosity solutions of (1.1) can be deduced from Theorem 5.1 of [9], and if $\gamma \in L^q(\Omega)$, $q > n$, $\mu \in L^\infty(\Omega)$ and $\|\mu\|_\infty \|f\|_n$ is small, existence of L^n-viscosity solutions of (1.1) follows from Theorem 1 of [14]. For $\mu \equiv 0$, existence of L^p-strong solutions of (1.1) has been established in [4] if $\gamma \in L^\infty(\Omega)$, and in [7] if $\gamma \in L^{2n}(\Omega)$. In [11] it was shown that L^p-strong solutions of (1.1) with $\mu \equiv 0$ exist if $\gamma \in L^q(\Omega), q > n$. In this paper we consider the case $m > 1$, $p > p_0$, and $q, q_1 > n$.

As regards the Harnack inequality, the case of bounded coefficients (i.e. $\gamma, \mu \in L^\infty(\Omega)$) and $m = 2$ was handled in [15] and, by a different method, in [12] if $1 < m < 2$. If $\mu \equiv 0$ and $\gamma \in L^q(\Omega), q > n$, the weak Harnack inequality was established in [11]. Finally, global C^α-continuity estimates for L^n-viscosity solutions of (1.1) if $m = 2, \mu \in L^\infty(\Omega)$ and $\gamma \in L^q(\Omega)$, $q > n$, have been obtained in [14]. Our weak Harnack inequality, Theorem 4.2, is restricted to $1 < m < 2$.

The paper is organized as follows. In Section 2, we introduce the notation and present some preliminary results. Section 3 is devoted to the strong solvability of (1.1) via the Schauder fixed point argument. In Section 4 we show the weak Harnack inequality for (1.1) and prove Hölder continuity of L^p-viscosity solutions of general PDE.

2. Preliminaries

Throughout this paper, in order to avoid keeping track of the dependence on the size of Ω, (unless specified otherwise) we will always assume that

$$\Omega \subset B_1,$$

where $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$ for $r > 0$, and $B_r(y) = y + B_r$ for $y \in \mathbb{R}^n$. It is standard to track the dependence on Ω by scaling.

For $1 \le p \le \infty$ we will often write $\|\cdot\|_p$, respectively $\|\cdot\|_{1,p}, \|\cdot\|_{2,p}$, for $\|\cdot\|_{L^p(\Omega)}$, respectively $\|\cdot\|_{W^{1,p}(\Omega)}, \|\cdot\|_{W^{2,p}(\Omega)}$, if there is no possibility of confusion. We will denote by $L^p_+(\Omega)$ the set of nonnegative functions in $L^p(\Omega)$.

Consider the equation

$$F(x, u(x), Du(x), D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given measurable functions, $F(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. Here are the definitions of L^p-strong and L^p-viscosity solutions. Recall that we assume $n/2 \le p_0 < p < \infty$.

Definition 2.1. A function $u \in C(\Omega)$ is called an L^p-*strong subsolution* (resp., *supersolution*) of (2.1) if $u \in W^{2,p}_{\text{loc}}(\Omega)$, and

$$F(x, u(x), Du(x), D^2u(x)) \le f(x) \quad (\text{resp., } \ge f(x)) \quad \text{a.e. in } \Omega.$$

We call $u \in C(\Omega)$ an L^p-*strong solution* of (2.1) if it is both an L^p-strong sub- and supersolution of (2.1).

Definition 2.2. A function $u \in C(\Omega)$ is called an L^p-*viscosity subsolution* (resp., *supersolution*) of (2.1) if

$$\operatorname{ess}\liminf_{y\to x}(F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)) \le 0$$
$$(\text{resp., } \operatorname{ess}\limsup_{y\to x}(F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)) \ge 0)$$

whenever $u - \phi$ attains its local maximum (resp., minimum) at $x \in \Omega$ for some $\phi \in W^{2,p}_{\text{loc}}(\Omega)$.

We call $u \in C(\Omega)$ an L^p-*viscosity solution* of (2.1) if it is both an L^p-viscosity sub- and supersolution of (2.1).

Regarding the constant p_0 introduced earlier, the following result is true [10, 11]. It is not stated in the most general form but it will be sufficient for our purposes here.

Theorem 2.3. *Let $\Omega \subset B_1$, $p_0 < p \le q_1$, $q_1 > n$, $f \in L^p_+(\Omega)$, $\gamma \in L^{q_1}_+(\Omega)$. If $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution (respectively, supersolution) of*

$$\mathcal{P}^-(D^2u) - \gamma(x)|Du| = f(x) \quad \textit{in } \Omega \tag{2.2}$$
$$(\textit{resp., } \mathcal{P}^+(D^2u) + \gamma(x)|Du| = -f(x) \quad \textit{in } \Omega), \tag{2.3}$$

then

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C(n, p, q_1, \lambda, \Lambda, \|\gamma\|_{q_1})\|f\|_p \tag{2.4}$$
$$(\textit{resp., } \inf_\Omega u \ge \inf_{\partial\Omega} u - C(n, p, q_1, \lambda, \Lambda, \|\gamma\|_{q_1})\|f\|_p). \tag{2.5}$$

We point out that by [11], L^p-strong sub- and supersolutions considered in this paper are also L^p-viscosity sub- and supersolutions.

Recently, Winter [16] established global up to the boundary $W^{2,p}$-estimates for solutions of fully nonlinear uniformly elliptic equations. The theorem below uses these estimates to show strong solvability of (1.1) in $W^{2,p}(\Omega)$ when $\mu \equiv 0$.

Proposition 2.4. *Let $\partial\Omega \in C^{1,1}$. Let $p_0 < p \le q_1$, $q_1 > n$. For every $f \in L^p(\Omega)$, $\gamma \in L^{q_1}(\Omega)$ and $\psi \in W^{2,p}(\Omega)$, there exists a unique (among strong and L^p-viscosity solutions) L^p-strong solution $u \in W^{2,p}(\Omega)$ of*

$$\mathcal{P}^{\pm}(D^2u) \pm \gamma(x)|Du| = f(x) \quad \text{in } \Omega, \qquad u = \psi \quad \text{on } \partial\Omega \tag{2.6}$$

such that

$$\|u\|_\infty \le \|\psi\|_{L^\infty(\partial\Omega)} + C\|f\|_p \tag{2.7}$$

and

$$\|u\|_{W^{2,p}(\Omega)} \le \bar{C}(\|\psi\|_{W^{2,p}(\Omega)} + \|f\|_p), \tag{2.8}$$

where C is from Theorem 2.3 and $\bar{C} = \bar{C}(n, \lambda, \Lambda, p, q_1, \|\gamma\|_{q_1}, \Omega) > 0$.

Proof. We will only show the conclusion for the upper extremal equation. It follows from [11, Theorem 7.1] that there exists a unique strong solution $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of (2.6) such that (2.7) holds. If γ is bounded then $u \in W^{2,p}(\Omega)$ by Theorem 4.3 of [16]. If we can show (2.8) for each bounded γ then the result can be obtained by a typical approximation argument and maximum principle estimates of Theorem 2.3. Therefore we can assume from the beginning that $u \in W^{2,p}(\Omega)$. Estimate (2.8) will follow from a standard covering argument once we can show local $W^{2,p}$-estimates in a neighborhood of every point $x_0 \in \overline{\Omega}$.

First we notice that $v = u - \psi$ satisfies the equation

$$\mathcal{P}^+(D^2v) + \gamma(x)|Dv| = g(x) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

where

$$g(x) = f(x) + \mathcal{P}^+(D^2(u-\psi)(x)) + \gamma(x)|D(u-\psi)(x)| - \mathcal{P}^+(D^2u(x)) - \gamma(x)|Du(x)|,$$

and hence $\|g\|_p \le C(n, p, \lambda, \Lambda, \|\gamma\|_{q_1}, \Omega)\|\psi\|_{2,p} + \|f\|_p$.

We will be using the notation from [16]. Let $x_0 \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$ there exists a neighborhood $U(x_0)$ of x_0 and a $C^{1,1}$-diffeomorphism $\Psi : U(x_0) \to B_1(0)$ such that $\Psi(x_0) = 0$ and $\Psi(\Omega \cap U(x_0)) = B_1^+$, where $B_\rho^+ = \{x \in B_\rho(0) \mid x_n > 0\}$ for $\rho > 0$. It is easy to see that the function $\tilde{v} = v \circ \Psi^{-1} \in W^{2,p}(B_1^+)$ is an L^p-strong solution of the equation

$$\tilde{F}(x, D\tilde{v}, D^2\tilde{v}) = \tilde{g}(x) \quad \text{in } B_1^+,$$

where

$$\begin{aligned}\tilde{F}(x, p, X) = \mathcal{P}^+\big(&D\Psi^T \circ \Psi^{-1}(x) X D\Psi \circ \Psi^{-1}(x) + (p\partial_{i,j}\Psi \circ \Psi^{-1}(x))_{1\le i,j\le n}\big)\\ &+ \tilde{\gamma}(x)|pD\Psi \circ \Psi^{-1}(x)| = \tilde{g}(x),\end{aligned}$$

$\tilde{\gamma} = \gamma \circ \Psi^{-1}$, and $\tilde{g} = g \circ \Psi^{-1}$. Moreover $\tilde{v} = 0$ on $\{x_n = 0\}$.

Let now $R > 0$ and $r \in (0, R)$. Choose $\phi \in C_0^\infty(B_\rho)$, where $\rho = (R + r)/2$, such that

$$0 \le \phi \le 1 \text{ in } B_\rho, \qquad \phi = 1 \text{ on } B_r, \qquad |D^k\phi| \le \frac{C}{(R-r)^k} \text{ in } B_\rho \ (k = 1, 2).$$

Since Ψ is a $C^{1,1}$-diffeomorphism, by [16, Theorem 4.2],

$$\|D^2\tilde{v}\|_{L^p(B_r^+)} \le \|D^2(\phi\tilde{v})\|_p \le C(n, p, \lambda, \Lambda, \Omega)(\|\tilde{F}(x, 0, D^2(\phi\tilde{v})(x))\|_p + \|\tilde{v}\|_\infty).$$

We now notice that

$$\|\tilde{F}(x,0,D^2(\phi\tilde{v})(x))\|_p \le \|\phi\tilde{F}(x,0,D^2\tilde{v}(x))\|_p + \frac{C_1}{(R-r)^2}\|\tilde{v}\|_{L^p(B_\rho^+)} + \frac{C_1}{R-r}\|D\tilde{v}\|_{L^p(B_\rho^+)}$$

and

$$\|\phi\tilde{F}(x,0,D^2\tilde{v}(x))\|_p \le \|\tilde{g}\|_p + \frac{C_1}{R-r}\|\bar{\gamma}\|_p\|\tilde{v}\|_\infty + C_1\|\bar{\gamma}D(\phi\tilde{v})\|_p$$

for some constant $C_1 = C_1(n,p,\lambda,\Lambda,\Omega)$, where $\bar{\gamma}(x) = \tilde{\gamma}(x) + 1$ We need to estimate the last term $\|\bar{\gamma}D(\phi\tilde{v})\|_p$.

We consider three cases: (i) $p > n$, (ii) $n > p > p_0$, and (iii) $n = p$.

(i) $p > n$: In this case $\|\bar{\gamma}D(\phi\tilde{v})\|_p \le \|D(\phi\tilde{v})\|_\infty\|\bar{\gamma}\|_p \le C_2\|D^2(\phi\tilde{v})\|_{p'}\|\gamma\|_p$ for any $p' \in (n,p)$. Therefore, by the Hölder inequality, we have

$$\|\gamma D(\phi\tilde{v})\|_p \le C_2 R^{\frac{n(p-p')}{pp'}}\|\bar{\gamma}\|_{q_1}\|D^2(\phi v)\|_p.$$

Thus, for small $R = R(n,p,q_1,\lambda,\Lambda,\|\gamma\|_{q_1},\Omega) > 0$, setting

$$\Phi_j = \sup_{0<r<R}(R-r)^j\|D^j v\|_{L^p(B_r^+)},$$

we obtain $\Phi_2 \le C_3(\|\tilde{g}\|_p + \|\tilde{v}\|_\infty + \Phi_0 + \Phi_1)$. By a standard interpolation argument (see [8, p. 237]) we obtain

$$\Phi_2 \le C_4(\|\tilde{g}\|_p + \|\tilde{v}\|_\infty). \tag{2.9}$$

(ii) $p_0 < p < n$: Since $\|\bar{\gamma}D(\phi v)\|_p \le \|\bar{\gamma}\|_n\|D(\phi v)\|_{p^*} \le C\|\bar{\gamma}\|_n\|D^2(\phi v)\|_{2,p}$, where $p^* = np/(n-p)$, we have $\|\bar{\gamma}D(\phi v)\|_p \le C\|\bar{\gamma}\|_{q_1}\|D^2(\phi v)\|_p R^{(q_1-n)n/q_1}$. If $R > 0$ is small we can follow the argument in (i) to obtain (2.9).

(iii) $p = n$: This case can be treated as in (ii) because $q_1 > n$ and $W^{2,n} \hookrightarrow W^{1,q'}$ for any $q' > 1$.

Therefore, using (2.7) and $\|\tilde{g}\|_p \le C\|g\|_p$, we have shown that there exists $R = R(n,p,q_1,\lambda,\Lambda,\|\gamma\|_{q_1},\Omega) > 0$ such that for every $x_0 \in \partial\Omega$,

$$\|D^2u\|_{L^p(\Psi^{-1}(B_{R/2}^+))} \le C(n,p,q_1,\lambda,\Lambda,\|\gamma\|_{q_1},\Omega)(\|f\|_p + \|\psi\|_{2,p}).$$

Similar and in fact easier arguments give us local estimates when $x_0 \in \Omega$. Therefore (2.8) follows. □

We recall that in particular if $\partial\Omega \in C^{1,1}$ then the embedding of $W^{2,p}(\Omega)$ into $W^{1,r}(\Omega)$ is compact for $r < p^* = np/(n-p)$ ($n^* = +\infty$) if $p \le n$ and for $r = +\infty$ if $p > n$, and

$$\|u\|_{1,r} \le D\|u\|_{2,p} \tag{2.10}$$

for some constant $D = D(n,p,r,\Omega)$.

3. Existence of strong solutions of PDE with superlinear growth terms

In this section, we are concerned with the existence of L^p-strong solutions of (1.1) for $m > 1$.

Theorem 3.1. *Let* $\partial\Omega \in C^{1,1}$, $p_0 < p \le q_1$, $q_1 > n$, $f \in L^p(\Omega)$, $\gamma \in L^{q_1}(\Omega)$ *and* $\psi \in W^{2,p}(\Omega)$. *Assume that one of the following conditions holds:*

$$\begin{cases} \text{(i) } q = \infty,\ p_0 < p,\ n > m(n-p), \\ \text{(ii) } n < p \le q < \infty, \\ \text{(iii) } p_0 < p \le n < q < \infty,\ mq(n-p) < n(q-p). \end{cases} \tag{3.1}$$

Let

$$\begin{cases} r = pm & \text{for (i)}, \\ r = \infty & \text{for (ii) with } p = q, \\ r = \frac{mpq}{q-p} & \text{for (ii) with } p < q \text{ or (iii)}. \end{cases} \tag{3.2}$$

Set $\varepsilon_1 = (2\bar{C}D)^{-m} > 0$, *where* $\bar{C}$ *is from Proposition* 2.4 *and* D *is from* (2.10). *If*

$$\|\mu\|_q(\|f\|_p + \|\psi\|_{2,p})^{m-1} < \varepsilon_1, \tag{3.3}$$

then there exist L^p*-strong solutions* $u \in W^{2,p}(\Omega)$ *of*

$$\begin{cases} \mathcal{P}^\pm(D^2u) \pm \gamma(x)|Du| \pm \mu(x)|Du|^m = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Moreover

$$\|u\|_{W^{2,p}(\Omega)} \le \hat{C}(\|f\|_p + \|\psi\|_{2,p}) \tag{3.5}$$

for some $\hat{C} = \hat{C}(n, \lambda, \Lambda, p, q_1, q, m, \|\gamma\|_{q_1}, \Omega) > 0$.

Remark 3.2. We note that under (i) of (3.1), the last inequality $n > m(n-p)$ is always true for $p \ge n$.

Proof. We will only show the result for the upper extremal equation. It is easy to see from (3.2) that in each of the cases of (3.1), if $g \in L^r(\Omega)$, then $\mu g^m \in L^p(\Omega)$. Moreover the embedding of $W^{2,p}(\Omega)$ into $W^{1,r}(\Omega)$ is compact. Thanks to Proposition 2.4, we can define the mapping $T : W^{1,r}(\Omega) \to W^{2,p}(\Omega)$ in the following manner: for $v \in W^{1,r}(\Omega)$, we denote by $u = Tv$ the (unique) L^p-strong solution of

$$\begin{cases} \mathcal{P}^+(D^2u) + \gamma(x)|Du| = f(x) - \mu(x)|Dv(x)|^m & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

It follows from Proposition 2.4, (3.2) and the Hölder inequality that

$$\|Tv\|_\infty \le \|\psi\|_{L^\infty(\partial\Omega)} + C(\|f\|_p + \|\mu\|_q\|Dv\|_r^m), \tag{3.6}$$

and

$$\|u\|_{W^{2,p}(\Omega)} \le \bar{C}(\|\psi\|_{2,p} + \|f\|_p + \|\mu\|_q\|Dv\|_r^m). \tag{3.7}$$

We will prove that T has a fixed point by applying the standard Schauder fixed point theorem. To do this it is enough to show that T is continuous in

$W^{1,r}(\Omega)$, and that for some $R > 0$, $T(\mathcal{B}_R)$ is a precompact subset of $\mathcal{B}_R$, where $\mathcal{B}_R := \{v \in W^{1,r}(\Omega) : \|v\|_{W^{1,r}(\Omega)} \le R\}$.

Continuity of T: Let $v_k \to v$ in $W^{1,r}(\Omega)$ as $k \to \infty$. Define $u_k = Tv_k$. By (3.7) we get $\|Tv_k\|_{2,p} \le C_1$ for $k = 1, 2, \dots$ and so $Tv_k \rightharpoonup u$ in $W^{2,p}(\Omega)$ for some $u \in W^{2,p}(\Omega)$, and thus, using compactness of Sobolev embeddings, $Tv_k \to u$ in $W^{1,r}(\Omega)$. Moreover, setting $g_k(x) = f(x) - \mu(x)|Dv_k(x)|^m$ and $g(x) = f(x) - \mu(x)|Dv(x)|^m$, we have

$$\begin{aligned}\|g_k - g\|_p &\le \|\mu(|Dv_k|^m - |Dv|^m)\|_p \\ &\le C_2\|\mu|D(v_k - v)|(|Dv_k|^{m-1} + |Dv|^{m-1})\|_p \\ &\le C_2\|\mu\|_q\|D(v_k - v_2)\|_r(\|Dv_1\|_r^{m-1} + \|Dv_2\|_r^{m-1})\end{aligned}$$

in each case, which implies by the maximum principle (Theorem 2.3) that $Tv_k \to Tv$ in $C(\overline{\Omega})$. Therefore we must have $Tv = u$.

$T : \mathcal{B}_R \to \mathcal{B}_R$ *for some* $R > 0$: Let $\|v\|_{W^{1,r}(\Omega)} \le R$. By (2.10) and (3.7) we have

$$\begin{aligned}\|Tv\|_{1,r} &\le D\|Tv\|_{2,p} \le D\bar{C}(\|f\|_p + \|\psi\|_{2,p} + \|\mu|Dv|^m\|_p) \\ &\le D\bar{C}(\|f\|_p + \|\psi\|_{2,p} + \|\mu\|_q R^m)\end{aligned}$$

Set $R = \alpha(\|f\|_p + \|\psi\|_{2,p})$, where $\alpha = 2D\bar{C} > 0$. Then

$$\|Tv\|_{1,r} \le \frac{\alpha}{2}(\|f\|_p + \|\psi\|_{2,p})(1 + \|\mu\|_q\alpha^m(\|f\|_p + \|\psi\|_{2,p})^{m-1}).$$

Thus, if (3.3) is satisfied, then $\|Tv\|_{1,r} \le R$.

Finally, (3.7) ensures that $T(\mathcal{B}_R)$ is precompact in $W^{1,r}(\Omega)$ and this implies that $T : \mathcal{B}_R \to \mathcal{B}_R$ has a fixed point which is an L^p-strong solution of (3.4). Estimates (3.6) and (3.7), together with (3.3), yield (3.5) for some constant $\hat{C}$. □

Using Theorem 3.1 we can obtain another proof of the Aleksandrov–Bakelman–Pucci maximum principle for Pucci extremal equations with superlinear growth in Du.

Theorem 3.3. *Let* $p_0 < p \le q_1$, $q_1 > n$, *and let one of* (3.1) *hold. Let* ε_1 *be from Theorem* 3.4 *for* $\Omega = B_1$ *and* r *as in* (3.2). *Then there exists* $C = C(n, \lambda, \Lambda, p, q_1, q, m, \|\gamma\|_{q_1})$ *such that if* $\gamma \in L_+^{q_1}(\Omega)$, $\mu \in L_+^q(\Omega)$ *and* $f \in L_+^p(\Omega)$ *satisfy*

$$\|\mu\|_q(2\|f\|_p)^{m-1} < \varepsilon_1, \tag{3.8}$$

and if $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution (resp., supersolution) of*

$$\mathcal{P}^-(D^2u) - \gamma(x)|Du| - \mu(x)|Du|^m \le f(x) \quad \text{in } \Omega^+ = \{x \in \Omega : \ u(x) > \max_{\partial\Omega} u\}$$

$$(\text{resp., } \mathcal{P}^+(D^2u) + \gamma(x)|Du| + \mu(x)|Du|^m \ge -f(x) \quad \text{in } \Omega^- = \{x \in \Omega : \ -u(x) > \max_{\partial\Omega}(-u)\}),$$

then

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C\|f\|_{L^p(\Omega^+)} \tag{3.9}$$

$$(\textit{resp.},\ \max_{\overline{\Omega}}(-u) \le \max_{\partial\Omega}(-u) + C\|f\|_{L^p(\Omega^-)}). \tag{3.10}$$

Remark 3.4. We point out that condition (3.8) and estimate (3.9) are in fact equivalent to those in Theorem 2.6 of [11] and Theorems 3.4 and 3.5 of [13].

Proof. In view of Theorem 3.1, if $\delta > 0$ is such that $2^{m-1}\|\mu\|_q(\|f\|_p + \delta)^{m-1} < \varepsilon_1$, we can find an L^p-strong solution $v_\delta \in W^{2,p}(B_1)$ of

$$\begin{cases} \mathcal{P}^+(D^2 v_\delta) + \gamma(x)|Dv_\delta| + 2^{m-1}\mu(x)\chi_{\Omega^+}(x)|Dv_\delta|^m = -f(x)\chi_{\Omega^+}(x) - \delta & \text{in } B_1, \\ v_\delta = 0 & \text{on } \partial B_1. \end{cases}$$

Estimate (3.5) in particular implies that

$$\|v_\delta\|_\infty \le C(\|f\|_{L^p(\Omega^+)} + \delta) \tag{3.11}$$

for some $C = C(n, \lambda, \Lambda, p, q_1, q, m, \|\gamma\|_{q_1}) > 0$. Setting $w := u + v_\delta$, we easily verify that w is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w) - \gamma(x)|Dw| - 2^{m-1}\mu(x)|Dw|^m \le -\delta \quad \text{in } \Omega^+.$$

Hence, from the definition of viscosity solution, we have

$$\max_{\overline{\Omega}^+} w = \max_{\partial\Omega^+} w,$$

which, together with (3.11), concludes the proof after we send $\delta \to 0$. □

4. Weak Harnack inequality

In this section, as an application of the existence of L^p-strong solutions of extremal PDE, we obtain the weak Harnack inequality for L^p-viscosity supersolutions of (1.1). We denote by Q_R the closed cube with side-length $R > 0$ and center at 0. We set $Q_R(x) = x + Q_R$ for $x \in \mathbb{R}^n$.

We will first construct an auxiliary function, which was explicitly given by Caffarelli [2] if $\mu \equiv 0$, and by Koike–Takahashi [12] if $\mu \in L^\infty(\Omega)$. However, since we need to solve Pucci extremal equations with unbounded μ, we follow the argument of [11].

We choose and fix two domains $U, V \subset \mathbb{R}^n$ with smooth boundaries such that

$$Q_{1/2} \subset U \subset Q_{3/4}, \qquad Q_3 \subset V \subset Q_4.$$

The constants in Lemma 4.1 and Theorems 4.2 and 4.3 may depend on the choice of U and V, but we omit this dependence since the sets are fixed.

Lemma 4.1. *Assume that one of* (3.1) *holds. Set* $\tilde{p} = n + 1$ *in case* (i)*, and* $\tilde{p} = q$ *in cases* (ii) *and* (iii) *of* (3.1). *There exists* $\varepsilon_2 = \varepsilon_2(n, p, q, \lambda, \Lambda, m) > 0$ *such that if* $\mu \in L^q_+(V)$ *satisfies*

$$\|\mu\|_{L^q(V)} < \varepsilon_2,$$

then there exist $\phi \in W^{2,\tilde{p}}(V)$ *and* $\xi \in L^{\tilde{p}}(V)$ *such that*

$$\begin{aligned} &\text{(i) } \mathcal{P}^-(D^2\phi) - \mu(x)|D\phi|^m = \xi(x) && \textit{in } V,\\ &\text{(ii) } \phi = 0 && \textit{on } \partial V,\\ &\text{(iii) } \phi \le -2 && \textit{in } Q_3,\\ &\text{(iv) } \xi = 0 && \textit{in } V \setminus U,\\ &\text{(v) } \|\xi\|_{\tilde{p}} \le \tilde{C}(n,p,q,\lambda,\Lambda,m). \end{aligned}$$

Proof. Let ε_1 be from Theorem 3.4 for $\Omega = V\setminus\overline{U}$, $\gamma = 0$ and r as in (3.2). We choose $\zeta \in C^2(\overline{V} \setminus U)$ such that $\zeta = 0$ on ∂V and $\zeta = -1$ on ∂U. Set $C_1 = \|\zeta\|_{W^{2,\tilde{p}}(V\setminus\overline{U})}$. Then if $\tilde{\mu} \in L^q_+(V)$ with $\|\tilde{\mu}\|_q < C_1^{1-m}\varepsilon_1$, by Theorem 3.1 we can find an $L^{\tilde{p}}$-strong solution $\phi_0 \in W^{2,\tilde{p}}(V \setminus \overline{U})$ of

$$\begin{cases} \mathcal{P}^-(D^2\phi_0) - \tilde{\mu}(x)|D\phi_0|^m = 0 & \text{in } V \setminus \overline{U},\\ \phi_0 = \zeta & \text{on } \partial V \cup \partial U. \end{cases} \tag{4.1}$$

We claim that there exists $\sigma > 0$ such that if $\|\tilde{\mu}\|_q$ is sufficiently small then $\phi \le -\sigma$ on $Q_3 \setminus U$. If not then there exist $\tilde{\mu}_k \in L^q(V)$ with $\|\tilde{\mu}_k\|_q \to 0$ and $\sup_{Q_3} \phi_k \to 0$, where ϕ_k are solutions of (4.1) with $\tilde{\mu}$ replaced by $\tilde{\mu}_k$. Since ϕ_k are also $L^{\tilde{p}}$-viscosity solutions [11] and $\|\phi_k\|_{2,\tilde{p}} \le C$ we find (see the appendix of [11]) that $\phi_k \to \bar{\phi}$ in $C(\overline{V} \setminus U)$, where ϕ is an $L^{\tilde{p}}$-viscosity solution of (4.1) with $\tilde{\mu} = 0$ and $\sup_{Q_3\setminus U} \bar{\phi} = 0$. This is however impossible by the strong maximum principle. Therefore there must exist $\tilde{\varepsilon}_2 \le C_1^{1-m}\varepsilon_1$ small enough so that the claim holds for some $\sigma > 0$ if $\|\tilde{\mu}\|_q \le \tilde{\varepsilon}_2$.

We now extend ϕ_0 onto the whole V as a $W^{2,\tilde{p}}$ function ϕ so that (iii) is preserved. Since by (3.5), $\|\phi_0\|_{W^{2,\tilde{p}}(V\setminus U)} \le \hat{C}\|\zeta\|_{2,\tilde{p}}$, (i) and (v) will be satisfied for ϕ_0 with some ξ_1 and constant $\tilde{C}$.

We now set $\phi = 2\phi_0/\sigma$. This function satisfies the equation

$$\mathcal{P}^-(D^2\phi) - \frac{\sigma^{m-1}}{2^{m-1}}\tilde{\mu}(x)|D\phi|^m = \frac{2}{\sigma}\xi_1(x) \quad \text{in } V$$

and therefore it is enough to set $\tilde{\mu}(x) = 2^{m-1}\mu(x)/\sigma^{m-1}$, $\xi(x) = 2\xi_1(x)/\sigma$ and take $\varepsilon_2 = \sigma^{m-1}\tilde{\varepsilon}_2/2^{m-1}$. □

We can now establish the weak Harnack inequality for (1.1).

Theorem 4.2. *Assume one of* (3.1), *and*

$$2 - n/q > m > 1. \tag{4.2}$$

Let $M \ge 0$, $f \in L^p_+(V)$ *and* $\mu \in L^q_+(V)$. *Then there exists* $\varepsilon_3 = \varepsilon_3(n,p,\lambda,\Lambda,m,M) > 0$, $C = C(n,\lambda,\Lambda,p,q,m) > 0$ *and* $r = r(n,\lambda,\Lambda,p,q,m) > 0$ *such that if*

$$\|\mu\|_q(1 + \|f\|_p^{m-1}) < \varepsilon_3, \tag{4.3}$$

and $u \in C(\overline{V})$ *is an* L^p*-viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \ge -f(x) \quad \textit{in } V$$

satisfying $0 \le u \le M$ *in* V*, then*

$$\left(\int_{Q_1} u^r\,dx\right)^{1/r} \le C(\inf_{Q_1} u + \|f\|_{L^p(V)}).$$

Proof. We first reduce the proof to the case $f \equiv 0$. Let $\varepsilon_1 = \varepsilon_1(n, p, \lambda, \Lambda, m)$ be from Theorem 3.1 for V. If $\|\mu\|_q (4\|f\|_p)^{m-1} < \varepsilon_1$, due to Theorem 3.1, we can find $v \in W^{2,p}(V)$ such that

$$\mathcal{P}^-(D^2v) - 2^{m-1}\mu(x)|Dv|^m = f(x) \quad \text{ in } V,$$

and $v = 0$ on ∂V. By Theorem 3.3, we have

$$0 \le v \le C\|f\|_p \quad \text{ in } V. \tag{4.4}$$

We notice that $w := u + v$ is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2w) + 2^{m-1}\mu(x)|Dw|^m \ge 0 \quad \text{ in } V.$$

If the assertion holds for $f \equiv 0$, then we have

$$\left(\int_{Q_1} w^r\,dx\right)^{1/r} \le C \inf_{Q_1} w.$$

Hence, due to (4.4), we obtain the result.

Therefore we can assume that $f \equiv 0$. Let ε_2 be from Lemma 4.1. Set $v = u/(m_0 + \varepsilon)$, where $m_0 = \inf_{Q_1} u$ and $\varepsilon > 0$. We need to show that if $\inf_{Q_1} v \le 1$, then $(\int_{Q_1} v^r dx)^{1/r} \le C$. The function v is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2v) + \mu(x)(m_0 + \varepsilon)^{m-1}|Dv|^m \ge 0 \quad \text{ in } V.$$

Suppose that $\|\mu\|_q M^{m-1} < \varepsilon_2$. Let ϕ be the function from Lemma 4.1 applied with $\mu := (m_0+\varepsilon)^{m-1}\mu$ for ε sufficiently small. Then $w := v+\phi$ is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2w) + 2^{m-1}(m_0 + \varepsilon)^{m-1}\mu(x)|Dw|^m \ge \xi(x) \quad \text{ in } V,$$

where ξ is from Lemma 4.1. Define $C_n = \|\xi\|_n$.

We impose the last restriction that $\|\mu\|_q (4MC_n)^{m-1} < \varepsilon_1(n, n, \lambda, \Lambda, m)$, where ε_1 is from Theorem 3.1 for V and $p = n$. Then, for small $\varepsilon > 0$, we deduce from Theorem 3.3 that

$$\sup_V(-w) \le C\|\xi\|_{L^n(Q_1^+)},$$

where $Q_1^+ = \{x \in Q_1 \mid w(x) \le 0\}$. (The three restrictions we imposed on $\|f\|_p$ and $\|\mu\|_q$ provide us with condition (4.3) for some $\varepsilon_3 > 0$.) On the other hand, since $1 \le 2 - \inf_{Q_1} v \le 2 - \inf_{Q_3} v \le \sup_{Q_3}(-w) \le \sup_V(-w)$, we have

$$1 \le C|Q_1^+|^{\tilde{p}/(\tilde{p}-n)}\|\xi\|_{\tilde{p}},$$

where $\tilde{p}$ is from Lemma 4.1. This implies that there exists $\theta = \theta(n, p, q, \lambda, \Lambda, m) \in (0, 1)$ such that

$$|\{x \in Q_1 \mid v(x) > K\}| \le \theta, \tag{4.5}$$

where $K = \max_V(-\phi) > 1$.

To prove the claim it is enough to show that

$$|\{x \in Q_1 \mid v(x) > K^j\}| \leq \theta^j \quad (j = 1, 2, \ldots).$$

To this end, assuming $|B| \leq \theta^{j-1}$ for $j \geq 2$, where $B := \{x \in Q_1 \mid v(x) > K^{j-1}\}$, we will show that

$$|A| \leq \theta |B|, \tag{4.6}$$

where $A = \{x \in Q_1 \mid v(x) > K^j\}$. Following the argument in [12] (cf. Lemma 4.2 in [3]), by (4.2), we divide Q_1 into cubes with length $2^{-\ell}$, where $\ell \in \mathbb{N}$ satisfies

$$2^{\ell(2-m-n/q)} \geq K^{(j-1)(m-1)}, \tag{4.7}$$

such that $Q_1 = \bigcup_{i=1}^{2^{\ell n}} Q_{2^{-\ell}}(x_i)$ with $x_i \in Q_1$. To get (4.6), it is enough to show

$$|A \cap Q_{2^{-\ell}}(x_i)| \leq \theta |B \cap Q_{2^{-\ell}}(x_i)| \quad \text{for } i = 1, 2, \ldots, 2^{\ell n}.$$

We fix $i \in \{1, 2, \ldots, 2^{\ell n}\}$. By translation, we may suppose that $x_i = 0$. Thus, by Lemma 4.2 in [3], it is sufficient to prove that for any dyadic cube $Q := Q_{2^{-\ell-i}}(\hat{x}) \subset Q_{2^{-\ell}}$ for $i \in \mathbb{N}$ and $\hat{x} \in Q_{2^{-\ell}}$, if

$$|A \cap Q| > \theta |Q| = \frac{\theta}{2^{n(\ell+i)}}, \tag{4.8}$$

then $\tilde{Q} \subset B$, where $\tilde{Q}$ is the predecessor of Q.

Again, we will assume that $\hat{x} = 0$ for simplicity of notation. Suppose that there exists $\tilde{x} \in \tilde{Q}$ such that $v(\tilde{x}) \leq K^{j-1}$. We set $\zeta(x) = K^{1-j} v(2^{-\ell-i}x)$. It is easy to see that ζ is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2\zeta) + \hat{\mu}(x)|D\zeta|^m \geq 0,$$

where

$$\hat{\mu}(x) = \frac{K^{(j-1)(m-1)} \mu(2^{-\ell-i}x)}{2^{(\ell+i)(2-m)}}.$$

We notice that (4.7) yields

$$\|\hat{\mu}\|_{L^q(V)} \leq \|\mu\|_q.$$

Remembering that we only needed $\inf_{Q_3} v \leq 1$ to get (4.5), we obtain

$$|\{x \in Q_1 \mid \zeta(x) > K\}| \leq \theta,$$

which implies

$$|\{x \in Q \mid v(x) > K^j\}| \leq \frac{\theta}{2^{n(\ell+i)}}.$$

This contradicts (4.8) and therefore the proof is complete. □

It is standard to extend the weak Harnack inequality to the boundary $\partial\Omega$. Therefore we omit the proof of it referring to [11] for the details.

Theorem 4.3. *Let* $0 \in \partial\Omega$. *Assume one of* (3.1), *and* (4.2). *Let* ε_3 *be from Theorem* 4.2. *Let* $M \geq 0$. *There exist* $C = C(n, \lambda, \Lambda, p, q, m) > 0$ *and* $r = r(n, \lambda, \Lambda, p, q, m) > 0$ *such that if* $f \in L^p_+(\Omega)$ *and* $\mu \in L^q_+(\Omega)$ *satisfy* (4.3), *and* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -f(x) \quad \text{in } \Omega$$

such that $0 \le u \le M$ *in* Ω*, then*

$$\left(\int_{Q_1} u_m^r dx\right)^{1/r} \le C(\inf_{Q_1} u_m + \|f\|_{L^p(V\cap\Omega)}),$$

where $m = \inf_{V\cap\partial\Omega} u$*, and*

$$u_m(x) = \begin{cases} \min\{u(x), m\} & \text{for } x \in V \cap \Omega, \\ m & \text{for } x \in V \setminus \Omega. \end{cases}$$

Remark 4.4. By using a different version of Lemma 4.1 or by scaling and applying a covering argument of Cabré [1], we may replace V in Theorems 4.2 and 4.3 by Q_R for any $R > 1$.

A consequence of Theorems 4.2 and 4.3 is the Hölder continuity of L^p-viscosity solutions of general fully nonlinear PDE

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{4.9}$$

We will just state the result and make a few comments about its proof referring for the details to [11].

We assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is measurable, $F(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$,

$$f \in L^p(\Omega), \tag{4.10}$$

$$\mathcal{P}^-(X - Y) \le F(x, r, \xi, X) - F(x, r, \xi, Y) \le \mathcal{P}^+(X - Y) \tag{4.11}$$

for $x \in \Omega$, $\xi \in \mathbb{R}^n$, $X, Y \in S^n$,

$$F(x, 0, 0, O) = 0 \quad \text{for } x \in \Omega, \tag{4.12}$$

$$|F(x, r, \xi, O)| \le f_0(x)|r| + \mu(x)|\xi|^m \quad \text{for } x \in \Omega,\ r \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{4.13}$$

where $f_0 \in L^p_+(\Omega)$ and $\mu \in L^q_+(\Omega)$ and $m > 1$. We also assume that there exist $\Theta > 0$ and $t_0 > 0$, such that

$$|Q_t(x) \setminus \Omega| \ge \Theta t^n \quad \text{for } x \in \partial\Omega \text{ and } 0 < t \le t_0. \tag{4.14}$$

Theorem 4.5. *Assume that* (3.1)*,* (4.2)*,* (4.10)*,* (4.11)*,* (4.12)*,* (4.13)*,* (4.14) *hold. For* $M \ge 1$*,* $\sigma \in (0, 1)$ *and* $g \in C^\sigma(\partial\Omega)$*, there exist* $\alpha = \alpha(n, \lambda, \Lambda, p, q, m, \sigma, \Theta) \in (0, 1)$ *and* $\tilde{C} = \tilde{C}(n, \lambda, \Lambda, p, q, m, M, \|f_0\|_p, \|f\|_p, \|\mu\|_q, \mathrm{diam}(\Omega), \Theta, t_0) > 0$ *such that if* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity solution of* (4.9) *such that* $|u(x)| \le M$ *for* $x \in \overline{\Omega}$*, and* $u = g$ *on* $\partial\Omega$*, then*

$$|u(x) - u(y)| \le \tilde{C}|x - y|^\alpha \quad \text{for } x, y \in \overline{\Omega}.$$

Remark 4.6. When $f_0 \equiv 0$ (i.e. when F does not depend on u), a bound on $\|u\|_\infty$ for L^p-viscosity solutions of (4.9) is provided by Theorem 2.11 if $\|\mu\|_q\|f\|_p^{m-1}$ is small enough.

Proof. We first observe that $\pm u$ are L^p-viscosity supersolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -|f(x)| - f_0(x)|u| \quad \text{in } \Omega.$$

Suppose that $Q_{4r} \subset \Omega$. Setting $v(x) = u(rx)$ for $x \in Q_4$, we observe that v is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2v) + \hat{\mu}(x)|Dv|^m \geq -g(x) \quad \text{in } Q_4,$$

where $\hat{\mu}(x) = r^{2-m}\mu(rx)$ and $g(x) = r^2(|f(rx)| + f_0(rx)M)$. Using (4.2) and $p > p_0 > n/2$ it is easy to verify that $\|\hat{\mu}\|_{L^q(Q_4)}$ and $\|g\|_{L^p(Q_4)}$ go to 0 as $r \to 0$ and so we can find $r_0 > 0$ such that (4.3) is satisfied for $r < r_0$. Hence we may repeat the arguments of the proof of Theorem 6.2 of [11], together with Theorems 4.2 and 4.3, to conclude the proof. □

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References

[1] X. Cabré, *On the Alexandroff–Bakelman–Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations.* Comm. Pure Appl. Math. **48** (1995), 539–570.

[2] L. A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations* Ann. of Math. **130** (1989), 189–213.

[3] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations.* Amer. Math. Soc., Providence, 1995.

[4] L. A. Caffarelli, M. G. Crandall, M. Kocan, and A. Święch, *On viscosity solutions of fully nonlinear equations with measurable ingredients.* Comm. Pure Appl. Math. **49** (1996), 365–397.

[5] L. Escauriaza, $W^{2,n}$ *a priori estimates for solutions to fully non-linear equations.* Indiana Univ. Math. J. **42** (1993), 413–423.

[6] E. B. Fabes and D. W. Stroock, *The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations.* Duke Math. J. **51** (1984), 997–1016.

[7] P. Fok, *Some maximum principles and continuity estimates for fully nonlinear elliptic equations of second order.* Ph.D. Thesis, UCSB, 1996.

[8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Reprint of the 1998 edition, Classics Math., Springer, Berlin, 2001.

[9] S. Koike and A. Święch, *Maximum principle and existence of L^p-viscosity solutions for fully nonlinear uniformly elliptic equations with measurable and quadratic terms.* Nonlinear Differential Equations Appl. **11** (2004), 491–509.

[10] S. Koike and A. Święch, *Maximum principle for fully nonlinear equations via the iterated comparison function method.* Math. Ann. **339** (2007), 461–484.

[11] S. Koike and A. Święch, *Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients.* J. Math. Soc. Japan, to appear.

[12] S. Koike and T. Takahashi, *Remarks on regularity of viscosity solutions for fully nonlinear uniformly elliptic PDEs with measurable ingredients.* Adv. Differential Equations **7** (2002), 493–512.

[13] K. Nakagawa, *Maximum principle for L^p-viscosity solutions of fully nonlinear equations with unbounded ingredients and superlinear growth terms.* Adv. Math. Sci. Appl., to appear.

[14] B. Sirakov, *Solvability of uniformly elliptic fully nonlinear PDE.* Arch. Ration. Mech. Anal., to appear.

[15] N. S. Trudinger, *Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations.* Rev. Mat. Iberoamer. **4** (1988), 453–468.

[16] N. Winter, *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations.* Z. Anal. Anwend. **28** (2009), 129–164.

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