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Solitary waves in the nonlinear wave equation and in gauge theories

Vieri Benci and Donato Fortunato

Dedicated to the memory of Jean Leray

Abstract. Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. This paper is an introduction to the study of solitary waves relative to the nonlinear wave equation and to the Abelian gauge theories. Abelian gauge theories consist of a class of field equations obtained by coupling in a suitable way the nonlinear wave equation with the Maxwell equations. They provide a model for the interaction of matter with the electromagnetic field. One of the motivations of this study lies in the fact that the nonlinear wave equation and the Abelian gauge theories are the simplest equations which satisfy the basic principles of modern physics.

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1. Introduction

By *solitary wave* we mean a solution of a field equation whose energy travels as a localized packet; by *soliton*, we mean a solitary wave which exhibits some form of stability. In this respect solitary waves and solitons have a particle-like behavior and they occur in many questions of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, plasma physics (see e.g. [21], [16]).

This paper is a short introduction to the study of solitary waves and solitons for the *basic equations* of modern physics. By *basic equations,* we do not mean equations which model some basic physical theory, but the simplest equations which satisfy the fundamental principles of modern physics, namely:

A-1. *The equations are variational.*
A-2. *The equations are invariant for the Poincaré group.*
A-3. *The equations are invariant for a gauge group.*

These three principles are shared by every fundamental theory in physics. We are not interested in any particular physical theory. Our point is the investigation of the consequences of these three principles for the existence and properties of solitary waves related to the equations satisfying **A-1**, **A-2**, **A-3**.

2. The basic principles

2.1. Variational principles

Assumption **A-1** states that the fundamental equations of physics are the Euler–Lagrange equations of a suitable functional. This fact is quite surprising. There is no logical reason for this. It is just an empirical fact: all the fundamental equations which have been discovered until now derive from a variational principle. If the creator of the universe were a mathematician, he would work in variational calculus!

For example, the equations of motion of k particles whose positions at time t are given by $x_j(t)$, $x_j \in \mathbb{R}^3$, $j = 1, \ldots, k$, are obtained as the Euler–Lagrange equations relative to the functional

$$\mathcal{S} = \int \left(\sum_j \frac{m_j}{2} |\dot{x}_j|^2 - V\left(t, x_1, \ldots, x_k\right) \right) dt \tag{1}$$

where m_j is the mass of the j-th particle and V is the potential energy of the system.

Also the dynamics of fields can be determined by variational principles. The basic fields of physics can be regarded as a modification of an entity which in the nineteenth century was called "ether" and which is now called "vacuum". From the mathematical point of view a field is a function

$$\psi : \mathbb{R}^{3+1} \to \mathbb{R}^k, \qquad \psi = (\psi_1, \ldots, \psi_k),$$

where $\mathbb{R}^{3+1}$ is the space-time continuum and $\mathbb{R}^k$ is called the internal parameter space. The space and time coordinates will be denoted by $x = (x_1, x_2, x_3)$ and t respectively. The function $\psi(t, x)$ describes the *internal* state of the ether (or vacuum) at the point x and at time t.

From the mathematical point of view, assumption **A-1** states that the field equations are obtained by the variation of the action functional defined as follows:

$$\mathcal{S} = \int \int \mathcal{L}\left(t, x, \psi, \nabla\psi, \partial_t\psi\right)\, dx\, dt. \tag{2}$$

The function $\mathcal{L}$ is called a Lagrangian density function, but in the following for simplicity we will call it just a *Lagrangian*.

2.2. The Poincaré invariance

An equation

$$F(u) = 0, \qquad u \in V,$$

is called *invariant* for a representation T_g of a Lie group if, given any solution u, $T_g u$ is also a solution. If this equation is variational, i.e. $F(u) = dJ(u)$, then it is invariant for T_g if J is invariant, i.e. $J(T_g u) = J(u)$. In particular, if J has the form (2), it is sufficient to have $\mathcal{L}$ invariant:

$$\mathcal{L}\left(t, x, T_g\psi, \nabla T_g\psi, \partial_t T_g\psi\right) = \mathcal{L}\left(t, x, \psi, \nabla\psi, \partial_t\psi\right).$$

Assumption **A-2** states that the fundamental equations of physics are invariant for the Poincaré group: it is the *only* principle on which the special theory of relativity is based, in other words, its full content.

The Poincaré group $\mathfrak{P}$ is a generalization of the isometry group $\mathfrak{E}$. The isometry group $\mathfrak{E}$ in $\mathbb{R}^3$ is the group of transformations which preserve the quadratic form

$$|x|^2 = \sum_{i=1}^{3} x_i^2$$

and hence the Euclidean distance

$$d_E(x, y) = \sqrt{\sum_{i=1}^{3} |x_i - y_i|^2},$$

that is, if $g \in \mathfrak{E}$, then

$$d_E(gx, gy) = d_E(x, y).$$

If we identify the physical space with $\mathbb{R}^3$, the isometry group is also called the congruence group. Roughly speaking, the Euclidean geometry is the study of the properties of geometric objects which are preserved by the congruence group. This group is generated by translations and rotations. For this reason it is also called the group of *rototranslations.* In fact an element of this group can be represented as

$$gx = Ox + v$$

where $O \in O(3)$ is an orthogonal matrix (rotation) and $v \in \mathbb{R}^3$ is a vector (translation). Thus $\mathfrak{E}$ is a Lie group with six generators.

The Poincaré group $\mathfrak{P}$ is the transformation group in $\mathbb{R}^4$ which preserves the quadratic form

$$|x|_M^2 = -x_0^2 + \sum_{i=1}^{3} x_i^2$$

which is induced by the Minkowski bilinear form

$$\langle x, y\rangle_M = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i.$$

The 4-vectors $v = (v_0, \dots, v_3) \equiv (v_0, \mathbf{v})$ are classified according to their *causal* nature as follows:

- a 4-vector is called *space-like* if $|v|_M^2 > 0$,
- a 4-vector is called *light-like* if $|v|_M^2 = 0$,
- a 4-vector is called *time-like* if $|v|_M^2 < 0$.

The Poincaré group is a 10-parameter Lie group generated by the following one-parameter transformations:

- **Space translations.** This invariance guarantees that space is homogeneous, i.e. the laws of physics are independent of space: if an experiment is performed here or there, it gives the same results.
- **Space rotations.** This invariance guarantees that space is isotropic: the laws of physics are independent of orientation.
- **Time translations.** This invariance guarantees that time is isotropic, i.e. the laws of physics are independent of time: if an experiment is performed earlier or later, it gives the same results.
- **Lorentz transformations.** This invariance guarantees the principle of relativity which states that an experiment performed in an inertial frame gives the same results as an experiment performed in a non-moving frame. The Lorentz transformations form the Lorentz group which is a three-parameter Lie subgroup of $\mathfrak{P}$. The generators of the Lorentz group are the following:

$$\begin{array}{lll} x' = \gamma(x - v_1 t), & x' = x, & x' = x, \\ y' = y, & y' = \gamma(y - v_2 t), & y' = y, \\ z' = z, & z' = z, & z' = \gamma(z - v_3 t), \\ t' = \gamma(t - v_1 x); & t' = \gamma(t - v_2 y); & t' = \gamma(t - v_3 z), \end{array}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \tag{3}$$

with $v = v_i,\ i = 1, 2, 3$.

The causal nature of a vector is not changed by a Lorentz transformation, and hence the Lorentz group is not a transitive group: space and time are mixed, but… not so much.

The Lorentz group is not the only group which guarantees the principle of relativity. For example the Galileo group also does. Thus the Lorentz invariance is an empirical fact and, as will be shown below (see Theorem 6), it implies the remarkable facts of the Einstein special theory of relativity, such as the celebrated formula $E = mc^2$.

Concluding, the Poincaré group is a 10-parameter Lie group generated by the above transformations (plus the time inversion, $t \mapsto -t$, and the parity inversion $(x, y, z) \mapsto (-x, -y, -z)$).

The Poincaré group acts on a scalar field ψ by the following representation:

$$(T_g \psi)(t, x, y, z) = \psi(t', x', y', z'), \qquad (t', x', y', z') = g(t, x, y, z).$$

The simplest equation invariant for the Poincaré group is the d'Alembert equation

$$\Box\psi = 0 \tag{4}$$

where

$$\Box\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi \quad \text{and} \quad \Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}.$$

Actually, the d'Alembert equation is the simplest *variational* field equation invariant for the Poincaré group. In fact it is obtained from the variation of the action

$$\mathcal{S}_0 = \int \mathcal{L}_0 \, dx \, dt, \qquad \mathcal{L}_0 = -\frac{1}{2}\langle d\psi, d\psi\rangle_M = \frac{1}{2}|\partial_t\psi|^2 - \frac{1}{2}|\nabla\psi|^2. \tag{5}$$

We end this section by recalling the notion of 4-vector field. A 4-tuple of quantities

$$F = (F^0, F^1, F^2, F^3)$$

is called a 4-*vector field* if, under a Lorentz transformation, it changes as do the quantities (t, x, y, z), namely

$$\begin{aligned} F'_0 &= \gamma(F_0 - vF_1),\\ F'_1 &= \gamma(F_1 - vF_0),\\ F'_2 &= F_2,\\ F'_3 &= F_3. \end{aligned}$$

For example, if ψ is scalar field, then the Minkowski gradient $\nabla_M\psi := (-\partial_t\psi, \nabla\psi)$ is a 4-vector field. We recall that $\nabla_M\psi$ is defined by the duality map provided by the Minkowski quadratic form $\langle\cdot,\cdot\rangle_M$:

$$\langle\nabla_M\psi, w\rangle_M = d\psi[w] \quad \text{for every } w \in \mathbb{R}^4.$$

The components of a cotangent vector, i.e. the components of a differential 1-form, transform in a different way; in fact, given a 1-form $\sum_{i=0}^3 A_i dx^i$ we have

$$\begin{aligned} A'_0 &= \gamma(A_0 + vA_1),\\ A'_1 &= \gamma(A_1 + vA_0),\\ A'_2 &= A_2,\\ A'_3 &= A_3. \end{aligned}$$

2.3. The gauge invariance

Take a function $\psi : \mathbb{R}^4 \to V$ and assume that the representation T_g of some group $(G, \circ)$ acts on V. This action induces two possible actions on ψ:

- a global action: $\psi(x) \mapsto T_g\psi\,(x)$ where $g \in G$;
- a local action: $\psi(x) \mapsto T_{g(x)}\psi(x)$ where $g(x)$ is a smooth function with values in G.

In the second case, we have a representation of the infinite-dimensional group

$$\mathfrak{G} = \mathcal{C}^{\infty}(\mathbb{R}^4, G)$$

equipped with the group operation

$$(g \circ h)(x) = g(x) \circ h(x).$$

If a Lagrangian $\mathcal{L}$ satisfies the condition

$$\mathcal{L}(t, x, \psi, \nabla\psi, \partial_t\psi) = \mathcal{L}(t, x, T_g\psi, \nabla(T_g\psi), \partial_t(T_g\psi)), \quad g \in G,$$

we say that $\mathcal{L}$ is invariant for a global action of the group G, or for a trivial gauge action of the group $\mathfrak{G}$; if

$$\mathcal{L}(t, x, \psi, \nabla\psi, \partial_t\psi) = \mathcal{L}(t, x, T_{g(x)}\psi, \nabla(T_{g(x)}\psi), \partial_t(T_{g(x)}\psi)), \quad g(x) \in \mathfrak{G},$$

we say that $\mathcal{L}$ is invariant for a local action of G, or for a gauge action of $\mathfrak{G}$.

Let us consider two simple examples: the functional

$$\int \mathcal{L}(\nabla u)\, dx, \quad u \in \mathbb{R},$$

is invariant for a global action of the group $(\mathbb{R}, +)$. In fact, if we set $T_r u = u + r$, $r \in \mathbb{R}$, we have

$$\mathcal{L}(\nabla u) = \mathcal{L}(\nabla(T_r u)).$$

Next, consider the functional

$$\int \mathcal{L}(d\alpha)\, dx$$

where α is a 1-form and d is the exterior derivative. In this case, $\mathcal{L}(d\alpha)$ is not only invariant for a trivial action of $(\mathbb{R}, +)$, but also for the local action

$$T_{g(x)}\alpha = \alpha + dg(x), \quad g(x) \in \mathfrak{G} := \mathcal{C}^{\infty}(\mathbb{R}^4, \mathbb{R});$$

in fact

$$\mathcal{L}(d(\alpha + dg(x))) = \mathcal{L}(d\alpha).$$

2.4. The simplest nonlinear model

The d'Alembert equation is the simplest equation invariant for the Poincaré group, moreover it is invariant for the gauge transformation

$$\psi \mapsto \psi + c.$$

Also, if ψ is complex-valued, it is invariant for the following representation of the group $S^1 = \left\{e^{i\theta} : \theta \in \mathbb{R}\right\}$:

$$\psi \mapsto e^{i\theta}\psi. \tag{6}$$

It satisfies assumptions **A-1**, **A-2** and **A-3**, but it is linear and it produces a "model of world" rather boring. There exist only nondispersive waves. Let us add to the Lagrangian (5) a nonlinear term:

$$\mathcal{L} = |\partial_t\psi|^2 - |\nabla\psi|^2 - W(\psi) \tag{7}$$

where $W : \mathbb{C} \to \mathbb{R}$ satisfies

$$W(e^{i\theta}\psi) = W(\psi),$$

i.e. $W(\psi) = F(|\psi|)$ for some function $F : \mathbb{R} \to \mathbb{R}$. This is the simplest Lagrangian which gives rise to nonlinear Euler equations and which is invariant for the Poincaré group and the trivial gauge action (6).

The equation of motion relative to the Lagrangian (7) is the semilinear wave equation

$$\Box\psi + W'(\psi) = 0 \tag{8}$$

where

$$W'(\psi) = \frac{\partial W}{\partial \psi_1} + i\frac{\partial W}{\partial \psi_2},$$

that is,

$$W'(\psi) = F'(|\psi|)\frac{\psi}{|\psi|}.$$

In the following sections we will see that equation (8), together with some mild assumption on W, produces a very rich model in which there are solitary waves which behave as relativistic particles.

If $W'(\psi)$ is linear, $W'(\psi) = \Omega^2\psi$ with $\Omega^2 > 0$, then (8) reduces to the Klein–Gordon equation. Among the solutions of the Klein–Gordon equations there are *wave packets* which behave as solitary waves but disperse in space as time goes on. On the contrary, if W has a suitable nonlinear component, the wave packets do not disperse, actually they are solitary waves.

Sometimes, it will be useful to write ψ in polar form,

$$\psi(t, x) = u(t, x)e^{iS(t,x)}. \tag{9}$$

In this case the action $\int \mathcal{L}\, dx\, dt$ takes the fom

$$\mathcal{S}(u, S) = \frac{1}{2}\int \{(\partial_t u)^2 - |\nabla u|^2 + [(\partial_t S)^2 - |\nabla S|^2]u^2\}\, dx\, dt - \int W(u)\, dx\, dt = 0 \tag{10}$$

and equation (8) becomes

$$\Box u + [-(\partial_t S)^2 + |\nabla S|^2]u + W'(u) = 0, \tag{11}$$

$$\partial_t(u^2\partial_t S) - \nabla \cdot (u^2\nabla S) = 0. \tag{12}$$

2.5. Conservation laws

Noether's theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion, i.e. a quantity which is preserved in time by the solutions (see e.g. [12]). Thus (8) has 10 integrals.

- **Energy.** Energy, by definition, is a quantity which is preserved due to the time invariance of the Lagrangian; it has the following form (see e.g. [12]):

  $$\mathcal{E} = \mathrm{Re}\int \left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\overline{\partial_t\psi} - \mathcal{L}\right) dx$$

  where $\overline{z}$ denotes the complex conjugate of z.

In particular, if we take the Lagrangian (7), we get

$$\mathcal{E} = \int \left[\frac{1}{2} \left| \partial_t \psi \right|^2 + \frac{1}{2} \left| \nabla \psi \right|^2 + W(\psi) \right] dx. \tag{13}$$

Using (9) we get

$$\mathcal{E} = \int \left[\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} ((\partial_t S)^2 + |\nabla S|^2) u^2 + W(u) \right] dx. \tag{14}$$

- **Momentum.** Momentum, by definition, is a quantity which is preserved due to the space invariance of the Lagrangian; invariance for translations in the x_i direction gives the invariant (see e.g. [12])

$$P_i = -\operatorname{Re} \int \frac{\partial \mathcal{L}}{\partial (\partial_t \psi)} \overline{\partial_i \psi} \, dx.$$

In particular, if we take the Lagrangian (7), we get

$$P_i = -\operatorname{Re} \int \partial_t \psi \overline{\partial \psi_i} \, dx$$

and if $\mathbf{P} = (P_1, P_2, P_3)$, we can write

$$\mathbf{P} = -\operatorname{Re} \int \partial_t \psi \overline{\nabla \psi} \, dx. \tag{15}$$

Using (9) we get

$$\mathbf{P} = -\int \left(\partial_t u \, \nabla u + \partial_t S \, \nabla S \; u^2 \right) \, dx. \tag{16}$$

- **Angular momentum.** The angular momentum, by definition, is a quantity which is preserved due to the invariance of the Lagrangian under space rotations about the origin (see e.g. [12]).

  In particular, if we take the Lagrangian (7), we get

$$\mathbf{M} = \operatorname{Re} \int \mathbf{x} \times \nabla \psi \overline{\partial_t \psi} \, dx. \tag{17}$$

Using (9) we get

$$\mathbf{M} = \int \mathbf{x} \times (\nabla u \, \partial_t u + \nabla S \, \partial_t S \, u^2) \, dx. \tag{18}$$

- **Ergocenter velocity.** If we take the Lagrangian (7), the following quantity is preserved under the Lorentz transformations (by standard computations, see e.g. [12]):

$$\mathbf{K} = \int \mathbf{x} \left[\frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx - t\mathbf{P}. \tag{19}$$

Let us interpret it in a more meaningful way. If we define the *ergocenter* as follows:

$$\mathbf{Q} := \frac{\int \mathbf{x}[\frac{1}{2}|\partial_t\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + W(\psi)]\,dx}{\int[\frac{1}{2}|\partial_t\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + W(\psi)]\,dx} = \frac{\int \mathbf{x}[\frac{1}{2}|\partial_t\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + W(\psi)]\,dx}{\mathcal{E}}, \tag{20}$$

by the conservation of $\mathcal{E}$, $\mathbf{P}$ and $\mathbf{K}$ we get

$$\dot{\mathbf{Q}} = \frac{\mathbf{P}}{\mathcal{E}}. \tag{21}$$

Thus, the three components of $\dot{\mathbf{Q}}$ are another three integrals of motion.

Finally, we have another integral given by the action (6).

- **Charge.** The charge, by definition, is a quantity which is preserved by the trivial gauge action (6). The charge has the following expression (see e.g. [3]):

$$C = \operatorname{Im} \int \frac{\partial \mathcal{L}}{\partial(\partial_t\psi)}\overline{\psi}\,dx.$$

In particular, if we take the Lagrangian (7), we get

$$C = \operatorname{Im} \int \partial_t\psi\overline{\psi}\,dx.$$

Using (9) we get

$$C = \int \partial_t S\, u^2\,dx. \tag{22}$$

3. Solitary waves and solitons

By *solitary wave* we mean a solution of a field equation whose energy travels as a localized packet; by *soliton*, we mean a solitary wave which exhibits some form of stability. This is a rather weak definition of soliton but probably the most commonly used, and it will be adopted in this paper. In order to prove the existence of solitons, first it is necessary to prove the existence of solitary waves and then to prove their stability.

3.1. Existence of solitary waves and solitons

The easiest way to produce solitary waves of (8) consists in solving the static equation

$$-\Delta u + W'(u) = 0 \tag{23}$$

and setting

$$\psi_v(t,x) = \psi_v(t,x_1,x_2,x_3) = u\left(\frac{x_1 - vt}{\sqrt{1-v^2}}, x_2, x_3\right); \tag{24}$$

$\psi_v(t,x)$ is a solution of (8) which represents a bump which travels in the x_1-direction with speed v.

In [15] and [19], it has been proved that (23) has nontrivial solutions provided that W has the following form:

$$W(u) = \frac{1}{2}\Omega^2 u^2 - \frac{1}{p}u^p, \qquad \Omega > 0,\ 2 < p < 6, \tag{25}$$

Moreover Shatah [18] found a necessary and sufficient condition which guarantees the "orbital stability" of the solitary waves of (8); if W is given by (25), this condition becomes $2 < p < 10/3$ (see e.g. [2] or [3]).

However, it would be interesting to assume

$$W \geq 0; \tag{26}$$

in fact the energy of a solution ψ of equation (8) is given by

$$E(\psi) = \int \left[\frac{1}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{1}{2}|\nabla \psi|^2 + W(\psi)\right] dx.$$

In this case, the positivity of the energy is not only an important requirement for the related physical models, but it provides good *a priori* estimates for the solutions of the relevant Cauchy problem. These estimates allow one to prove existence and well-posedness results under very general assumptions on W.

Unfortunately Derrick [11], in a well known paper, has proved that (26) implies that equation (23) has only the trivial solution. His proof is based on an equality which in a different form was also found by Pohozaev (for details see also [3]). The Derrick–Pohozaev identity states that for any finite energy solution u of (23),

$$\frac{1}{6}\int |\nabla u|^2\, dx + \int W(u)\, dx = 0. \tag{27}$$

Clearly the above equality and (26) imply that $u \equiv 0$.

However, we can try to prove the existence of solitons for (8) (under assumption (26)) exploiting the possible existence of *standing waves*, since this fact is not prevented by (27). A *standing wave* is a finite energy solution of (8) having the following form:

$$\psi_0(t,x) = u(x)e^{-i\omega_0 t}, \qquad u \geq 0. \tag{28}$$

Substituting (28) in (8), we get

$$-\Delta u + W'(u) = \omega_0^2 u. \tag{29}$$

Since the Lagrangian (7) is invariant for the Lorentz group, we can obtain other solutions $\psi_1(t,x)$ just making a Lorentz transformation. Namely, if we take the velocity $\mathbf{v} = (v,0,0)$, $|v| < 1$, and set

$$t' = \gamma(t - vx_1),\ x_1' = \gamma(x_1 - vt),\ x_2' = x_2,\ x_3' = x_3 \qquad \text{with } \gamma = \frac{1}{\sqrt{1-v^2}}$$

it turns out that

$$\psi_1(t,x) = \psi_0(t',x')$$

is a solution of (8).

In particular given a standing wave $\psi(t,x) = u(x)e^{-i\omega_0 t}$, the function $\psi_{\mathbf{v}}(t,x)$ $:= \psi(t',x')$ is a solitary wave which travels with velocity $\mathbf{v}$. Thus, if $u(x) = u(x_1,x_2,x_3)$ is any solution of (29), then

$$\psi_{\mathbf{v}}(t,x_1,x_2,x_3) = u(\gamma(x_1 - vt), x_2, x_3)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{30}$$

is a solution of (8) provided that

$$\omega = \gamma\omega_0 \quad \text{and} \quad \mathbf{k} = \gamma\omega_0\mathbf{v}. \tag{31}$$

Notice that (24) is a particular case of (30) when $\omega_0 = 0$.

We have the following result:

Theorem 1. *Assume that*

(i) $W(u) \geq 0$,
(ii) $W(0) = W'(0) = 0$ *and* $W''(0) = \Omega^2 > 0$,
(iii) *there exists* $u_0 \in \mathbb{R}^+$ *such that* $W(u_0) < \frac{1}{2}\Omega^2 u_0^2$.

Then (8) *has finite energy solitary waves of the form* $\psi_0(t,x) = u(x)e^{-i\omega_0 t}$ *for every frequency* $\omega_0 \in (\Omega_0, \Omega)$ *where*

$$\Omega_0 = \inf\left\{\Omega \in \mathbb{R} : \exists u \in \mathbb{R}^+,\ W(u) - \frac{1}{2}\Omega^2 u^2 < 0\right\}.$$

Notice that by (iii), $\Omega_0 < \Omega$, so the interval (Ω_0, Ω) is not empty.

Proof. By the previous discussion, it is sufficient to show that equation (29) has a solution u with finite energy. The solutions of finite energy of (29) are the critical points in the Sobolev space $H^1(\mathbb{R}^3)$ of the *reduced action* functional:

$$J(u) = \frac{1}{2}\int |\nabla u|^2 dx + \int G(u)\,dx, \qquad G(u) = W(u) - \frac{1}{2}\omega_0^2 u^2. \tag{32}$$

By a theorem of Berestycki and Lions [6], the existence of nontrivial critical points of J is guaranteed under the following assumptions on G:

- $G(0) = G'(0) = 0$,
- $G''(0) > 0$,
- $\limsup_{s\to\infty} G'(u)/s^5 \geq 0$,
- $\exists u_0 \in \mathbb{R}^+ : G(u_0) < 0$.

It is easy to check that for every frequency $\omega_0 \in (\Omega_0, \Omega)$, the above assumptions are satisfied. □

3.2. The mass of solitary waves

In classical mechanics the mass is a symmetric tensor m_{ij} which relates the momentum $\mathbf{P} = (P_1, P_2, P_3)$ to the velocity $\mathbf{v} = (v_1, v_2, v_3)$ by the formula

$$P_i = m_{ij}v_j.$$

Since the momentum of a solitary wave is defined by (15), it is possible to define the mass of a solitary wave by the above formula and to compute it. As we will see it turns out that the mass is a scalar, i.e. $m_{ij} = m\delta_{ij}$.

Theorem 2. *Let $\psi_{\mathbf{v}}$ be defined by* (30)*; then its momentum is given by*

$$\mathbf{P}(\psi_{\mathbf{v}}) = \mathbf{v}\gamma \int ((\partial_{\mathbf{n}} u)^2 + \omega_0^2 u^2)\, dx_1\, dx_2\, dx_3$$

were $\partial_{\mathbf{n}}$ denotes the directional derivative in the direction $\mathbf{n} = \mathbf{v}/|\mathbf{v}|$.

Proof. It is not restrictive to choose $\mathbf{v} = (v, 0, 0)$; then by (16), we have

$$P_1 = -\int \left(\frac{\partial}{\partial t} u' \frac{\partial}{\partial x_1} u' - \omega k_1 u'^2 \right) dx$$

where we have set

$$u'(t, x_1, x_2, x_3) = u(\gamma(x_1 - vt), x_2, x_3), \qquad \mathbf{k} = (k_1, 0, 0).$$

Then, performing the derivations, we get

$$P_1 = \int \left[\gamma^2 v \left(\frac{\partial u}{\partial x_1} \right)^2 + \omega k_1 u^2 \right]_{x_1 = \gamma(x_1 - vt)} dx.$$

By (31) we have

$$P_1 = v \int \gamma^2 \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \omega_0^2 u^2 \right]_{x_1 = \gamma(x_1 - vt)} dx.$$

Making the change of variables $y = (\gamma(x_1 - v_1 t), x_2, x_3)$, we get

$$P_1 = v\gamma \int ((\partial_1 u(y))^2 + \omega_0^2\, u(y)^2)\, dy.$$

Finally, it is immediate to check that

$$P_2 = P_3 = 0.$$

Thus the theorem is proved. □

The vector $\mathbf{P}(\psi_{\mathbf{v}})$ is parallel to $\mathbf{v}$. Next we will prove that its norm does not depend on the direction of $\mathbf{v}$.

To do this we need the following theorem.

Theorem 3. *Let $u \in H^1(\mathbb{R}^n)$ be a regular solution of*

$$-\Delta u + h(u) = 0 \tag{33}$$

where h is a continuous real function such that $h(0) = 0$. Assume that $H(u) \in L^1(\mathbb{R}^n)$ where

$$H(t) = \int_0^t h(s)\, ds.$$

Then

$$\int H(u)\, dx = \left(\frac{1}{n} - \frac{1}{2} \right) \int |\nabla u|^2\, dx, \tag{34}$$

$$\int |\partial_i u|^2\, dx = \frac{1}{n} \int |\nabla u|^2\, dx, \qquad i = 1, \ldots, n. \tag{35}$$

Proof. (34) is the Pohozaev–Derrick equality ([15], [11]). A proof of (35) (as well as (34)) can be found in [3] or in [2]. □

Since u solves equation (33) with $H(u) = W(u) - \frac{1}{2}\omega_0^2 u^2$, by Theorem 2 and (35), we have

$$\mathbf{P}(\psi_{\mathbf{v}}) = \mathbf{v}\gamma \int \left(\frac{1}{3}|\nabla u|^2 + \omega_0^2\, u^2 \right) dx_1\, dx_2\, dx_3. \tag{36}$$

Thus the mass of the solitary wave $\psi_{\mathbf{v}}$ is well defined and we have

$$m(\psi_{\mathbf{v}}) = \gamma \int \left(\frac{1}{3}|\nabla u|^2 + \omega_0^2\, u^2 \right) dx. \tag{37}$$

Hence we get a remarkable fact of the theory of relativity:

Theorem 4. *The mass of a solitary wave increases with velocity by the factor*

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

3.3. The Einstein equation

First we will give an explicit formula for the energy of a solitary wave:

Theorem 5. *Let $\psi_{\mathbf{v}}$ be defined by* (30)*; then its energy is given by*

$$\mathcal{E}(\psi_{\mathbf{v}}) = \gamma \int \left(\frac{1}{3}|\nabla u|^2 + \omega_0^2 u^2 \right) dx.$$

Proof. By (14) the energy of $\psi_{\mathbf{v}}$ is given by

$$\mathcal{E}(\psi_{\mathbf{v}}) = \int \left[\frac{1}{2}(\partial_t u')^2 + \frac{1}{2}|\nabla u'|^2 + \frac{1}{2}(\omega^2 + k^2)u'^2 + W(u') \right] dx$$

where we have set

$$u'(t, x_1, x_2, x_3) = u(\gamma(x_1 - vt), x_2, x_3), \qquad k = |\mathbf{k}|.$$

Then, performing the derivations, we get

$$\mathcal{E}(\psi_{\mathbf{v}}) = \int \left[\frac{1}{2}\left((\gamma^2 v^2 + \gamma^2)|\partial_1 u|^2 + \sum_{i\neq 1} |\partial_i u|^2 + \frac{1}{2}(\omega^2 + k^2)u^2 \right) + W(u) \right]_{x_1 = \gamma(x_1 - vt)} dx.$$

Making the change of variables $y = (\gamma(x_1 - v_1 t), x_2, x_3)$, we get

$$\begin{aligned}\mathcal{E}(\psi_{\mathbf{v}}) = {} & \frac{1}{2\gamma} \int \left[(\gamma^2 v^2 + \gamma^2)|\partial_1 u|^2 + \sum_{i\neq 1} |\partial_i u|^2 + (k^2 + \omega^2)u^2 \right] dy \\ & + \frac{1}{\gamma} \int W(u(y))\, dy. \end{aligned} \tag{38}$$

Since u solves (33) with $H(u) = W(u) - \frac{1}{2}\omega_0^2 u^2$, by (34) we have

$$\int W(u) = \frac{1}{2} \int \omega_0^2 u^2 - \frac{1}{6} \int |\nabla u|^2. \tag{39}$$

Moreover, by (35),

$$\int |\partial_i u|^2\,dx = \frac{1}{3}\int |\nabla u|^2\,dx, \qquad i = 1,2,3. \tag{40}$$

Substituting (40) and (39) in (38), we

$$\begin{aligned}\mathcal{E}(\psi_{\mathbf{v}}) &= \frac{1}{2\gamma}\int\left(\frac{\gamma^2v^2+\gamma^2+2}{3}|\nabla u|^2 + (k^2+\omega^2)u^2\right)dx + \frac{1}{\gamma}\int W(u)\,dx\\ &= \frac{1}{2\gamma}\int\left(\frac{\gamma^2v^2+\gamma^2+1}{3}|\nabla u|^2 + (k^2+\omega^2+\omega_0^2)u^2\right)dx.\end{aligned}$$

By the definition of γ, we have

$$\gamma^2v^2+\gamma^2+1 = \frac{v^2+1+1-v^2}{1-v^2} = 2\gamma^2,$$

and moreover, by (31),

$$k^2+\omega^2+\omega_0^2 = \gamma^2\omega_0^2v^2+\gamma^2\omega_0^2+\omega_0^2 = \omega_0^2(\gamma^2v^2+\gamma^2+1) = 2\omega_0^2\gamma^2.$$

Thus

$$\mathcal{E}(\psi_{\mathbf{v}}) = \frac{1}{2\gamma}\int\left(\frac{2\gamma^2}{3}|\nabla u|^2 + 2\omega_0^2\gamma^2u^2\right)dx = \gamma\int\left(\frac{1}{3}|\nabla u|^2+\omega_0^2u^2\right)dx. \qquad \square$$

So, by Theorem 5 and (37), we get the Einstein equation:

Theorem 6. *The energy of a solitary wave equals its mass:*

$$\mathcal{E}(\psi_{\mathbf{v}}) = m(\psi_{\mathbf{v}}).$$

By the above theorem, it turns out that the ergocenter $\mathbf{Q}$, defined by (20), is actually the center of mass. Thus, the conservation of $\dot{\mathbf{Q}}$ implies that the center of mass moves along a straight line.

Remark 7. Theorem 6 could have been deduced from (21) by just proving that $\dot{\mathbf{Q}} = \mathbf{v}$. We have proved it using Theorems 2 and 5 since we wanted to have an explicit formula for the momentum and energy of a solitary wave.

4. Gauge theories

4.1. The Maxwell equations in empty space

In Section 1 we introduced the Lagrangian (5) which gives rise to the simplest Poincaré invariant equation for a scalar field ψ, namely equation (4). In order to generalize this equation we will use the language of differential forms.

If we regard ψ as a 0-form ξ, then (5) takes the form

$$\mathcal{S}_0[\xi] = \int \mathcal{L}_0\,dx\,dt = -\int \langle d\psi, d\psi\rangle_M\,dx\,dt \tag{41}$$

where $\langle \xi,\eta\rangle_M$ is the Minkowskian scalar product on the space $\Lambda^k(\mathbb{R}^4)$ of k-forms.

Using the language of forms, the d'Alembert equation (4) becomes

$$\delta d\psi = 0 \tag{42}$$

where $\delta : \Lambda^k \to \Lambda^{k-1}$ $(k = 1, \ldots, 4)$ is the functional adjoint operator of $d : \Lambda^{k-1}(\mathbb{R}^4) \to \Lambda^k(\mathbb{R}^4)$, defined by the following identity:

$$\int \langle \xi, d\eta \rangle_M \; dx\, dt = \int \langle \delta\xi, \eta \rangle_M \; dx\, dt$$

for any $\xi \in \Lambda^k(\mathbb{R}^3)$ and any $\eta \in \Lambda^{k-1}(\mathbb{R}^4)$ with compact support.

The action $\mathcal{S}_0[\xi]$ is invariant for the "trivial gauge group" $\xi \mapsto \xi + c$ where $c \in \mathbb{C}$ is a constant: $\mathcal{S}_0[\xi] = \mathcal{S}_0[\xi + c]$. Thus if ξ is a solution of (42), then also $\xi + c$ solves (42).

One of the most natural generalizations of (41) is given by

$$\mathcal{S}_1[\xi] = \int \mathcal{L}_1 \, dx\, dt = -\frac{1}{2} \int \langle dA, dA \rangle_M \; dx\, dt \tag{43}$$

where A is a 1-form

$$A = \sum_{j=0}^{3} A_j dx^j.$$

The variation of the action (43) gives the Euler–Lagrange equation

$$\delta dA = 0. \tag{44}$$

This simple generalization gives a much richer structure; in fact, the action (43) is invariant for the gauge transformation $A \mapsto A + d\chi$ where $\chi \in \mathfrak{G} = \mathcal{C}^2(\mathbb{R}^4, \mathbb{R})$, i.e. the gauge group is an infinite-dimensional group. However, in most physical interpretations of this theory it is assumed that A and $A + d\chi$ give the same experimental results, i.e. χ has no physical meaning. For this reason, we can introduce the quantity

$$F = dA \tag{45}$$

which does not depend on χ (since $dd\chi = 0$) and which is considered the physically measurable quantity.

By equation (44), and the fact $ddA = 0$, we see that F satisfies

$$dF = 0, \tag{46}$$

$$\delta F = 0. \tag{47}$$

These are the Maxwell equations in the empty space.

Now let us write equations (46), (47) using vector notation. We denote by

$$\mathrm{j} : \mathbb{R}^{3+1} \to \Lambda^1(\mathbb{R}^3)$$

the duality map which associates to a 4-vector $(v_0, \mathbf{v})$ the 1-form $\mathrm{j}(v_0, \mathbf{v})$ defined by

$$\mathrm{j}(v_0, \mathbf{v})[(w_0, \mathbf{w})] = -v_0 w_0 + \mathbf{v} \cdot \mathbf{w}.$$

Then we set $(\varphi, \mathbf{A}) = \mathrm{j}^{-1}(A)$, i.e.

$$\varphi := A^0 = -A_0, \quad \mathbf{A} := (A^1, A^2, A^3) = (A_1, A_2, A_3). \tag{48}$$

Then

$$\begin{aligned}\langle dA, dA\rangle_M &= \frac{1}{2}\Big[\sum_{i,j=1}^{3}(\partial_i A_j - \partial_j A_i)^2 - \sum_{j=1}^{3}(\partial_0 A_j - \partial_j A_0)^2 - \sum_{i=1}^{3}(\partial_i A_0 - \partial_0 A_i)^2\Big] \\ &= \frac{1}{2}\Big[\sum_{i,j=1}^{3}(\partial_i A^j - \partial_j A^i)^2 - \sum_{j=1}^{3}(\partial_t A^j + \partial_j \varphi)^2 - \sum_{i=1}^{3}(\partial_i \varphi + \partial_t A^i)^2\Big] \\ &= |\nabla \times \mathbf{A}|^2 - |\partial_t \mathbf{A} + \nabla\varphi|^2.\end{aligned}$$

Thus the action (43) becomes

$$\mathcal{S}_1[(\varphi, \mathbf{A})] = \frac{1}{2}\int (|\partial_t \mathbf{A} + \nabla\varphi|^2 - |\nabla \times \mathbf{A}|^2)\, dx\, dt.$$

Taking the variation of $\mathcal{S}$ with respect to φ and $\mathbf{A}$ we get

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla\varphi) = 0, \tag{49}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\partial_t \mathbf{A} + \nabla\varphi) = 0. \tag{50}$$

If we make the change of variables

$$\mathbf{E} = -(\partial_t \mathbf{A} + \nabla\varphi), \tag{51}$$

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{52}$$

we obtain

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \tag{53}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{54}$$

and (49), (50) become

$$\nabla \cdot \mathbf{E} = 0,$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{E} = 0.$$

Thus we have obtained the Maxwell equations in the usual 3-vector notation. In this case, the action (43) can be written as follows:

$$\mathcal{S}_1 = \frac{1}{2}\int (\mathbf{E}^2 - \mathbf{H}^2)\, dx\, dt. \tag{55}$$

Moreover we can give a physical meaning to the 2-form

$$F = \sum_{i<j} F_{ij}\, dx^i\, dx^j.$$

By duality, we can associate to it an antisymmetric 4-tensor $\{F^{ij}\}$. Direct computations show that

$$\{F^{ij}\} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & H^3 & -H^2 \\ E^2 & -H^3 & 0 & H^1 \\ E^3 & H^2 & -H^1 & 0 \end{bmatrix}. \tag{56}$$

Thus the electromagnetic field $\{F^{ij}\}$ is a 4-tensor composed of the 3-vector "electric field"

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

and an antisymmetric 3-tensor "magnetic field"

$$\mathcal{H} = \begin{bmatrix} 0 & -H_3 & H_2 \\ H_3 & 0 & -H_1 \\ -H_2 & H_1 & 0 \end{bmatrix}.$$

It is well known that it is possible to associate a pseudovector $\mathbf{H}$ to any antisymmetric 3-tensor $\mathcal{H}$ by the equation

$$\mathcal{H}\mathbf{v} = \mathbf{H} \times \mathbf{v}, \qquad \mathbf{v} \in \mathbb{R}^3.$$

In this case, $\mathbf{H}$ takes the form

$$\mathbf{H} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}.$$

Remark 8. While the d'Alembert equation is the simplest equation which satisfies **A-1, A-2** and **A-3** with a global gauge group, the Maxwell equations are the simplest equations which satisfy **A-1, A-2** and **A-3** with a nontrivial gauge group.

4.2. Abelian gauge theories

A gauge theory provides a very elegant way to combine the actions (41) and (43) and to obtain an action invariant for $\mathfrak{G}$.

Let G be a subgroup of $U(N)$, the unitary group in $\mathbb{C}^N$, and denote by $\Lambda^k(\mathbb{R}^4, \mathfrak{g})$ the set of k-forms defined in $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of the group G. A 1-form

$$\Gamma = \sum_{j=0}^{3} \Gamma_j dx_j \in \Lambda^1(\mathbb{R}^4, \mathfrak{g})$$

is called a *connection form*. The operator

$$d_\Gamma : \Lambda^k(\mathbb{R}^4, \mathfrak{g}) \to \Lambda^{k+1}(\mathbb{R}^4, \mathfrak{g})$$

defined by

$$d_\Gamma = d + \Gamma = \sum_{j=0}^{3} (\partial_j + \Gamma_j) dx^j$$

is called the *covariant differential* and the operators

$$D_j = \frac{\partial}{\partial x^j} + \Gamma_j : \mathcal{C}^1(\mathbb{R}^4, \mathbb{C}^N) \to \mathcal{C}^0(\mathbb{R}^4, \mathbb{C}^N), \qquad j = 0, \ldots, 3,$$

are called the *covariant derivatives*. The 2-form

$$\mathcal{F} = d_\Gamma \Gamma = \sum_{i,j=0}^{3} (\partial_i \Gamma_j + [\Gamma_i, \Gamma_j]) dx^i \wedge dx^j$$

is called the *curvature*.

We set

$$\mathcal{L}_0 = \frac{1}{2}\langle d_\Gamma \psi, d_\Gamma \psi\rangle_M,$$

where $\psi \in \mathbb{C}^N$. Moreover, we set

$$\mathcal{L}_G = \frac{1}{2q^2}\langle d_\Gamma \Gamma, d_\Gamma \Gamma\rangle_M,$$

where q is a parameter which controls the coupling of $\mathcal{L}_G$ with $\mathcal{L}_0$.

A *gauge field* (see e.g. [22], [17]) is a critical point of the action functional

$$\mathcal{S} = \int \mathcal{L}\, dxdt, \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_G - W(\psi), \tag{57}$$

where $W : \mathbb{C}^N \to \mathbb{R}$ is a function which is assumed to be G-invariant, i.e.

$$W(g\psi) = W(\psi), \qquad g \in G. \tag{58}$$

We are interested in the Abelian gauge theory, i.e. in the case in which

$$G = U(1) = S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

In this case the $\Gamma_j(x,t)$ are imaginary numbers; if we set

$$A_j = -\frac{1}{iq}\Gamma_j, \qquad j = 0, \dots, 3, \tag{59}$$

it turns out that $A = \sum_{j=0}^{3} A_j dx^j$ is a real-valued 1-form. Moreover, since in this case $[\Gamma_i, \Gamma_j] = 0$, we have

$$d_\Gamma \Gamma = \sum_{i,j=0}^{3} \partial_i \Gamma_j dx^i \wedge dx^j = d\Gamma = -iqdA.$$

In this case, it turns out that

$$\mathcal{L}_G = \frac{1}{2q^2}\langle d_\Gamma \Gamma, d_\Gamma \Gamma\rangle_M = -\frac{1}{2}\langle dA, dA\rangle_M = \mathcal{L}_1$$

as defined in (43). By (48) and (59), the covariant derivatives take the form

$$D_t = \frac{\partial}{\partial t} + iq\varphi, \qquad D_j = \frac{\partial}{\partial x_j} - iqA_j$$

and, for $q = 0$, they reduce to the usual derivatives. Using the above notation, the Lagrangian density $\mathcal{L}_0$ can be written as follows:

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}|D_t\psi|^2 - \frac{1}{2}|\mathbf{D}_x\psi|^2 \\ &= \frac{1}{2}\left[\left|\left(\frac{\partial}{\partial t} + iq\varphi\right)\psi\right|^2 - |(\nabla - iq\mathbf{A})\psi|^2\right]\end{aligned}$$

where $\mathbf{D}_x\psi = (D_1\psi, D_2\psi, D_3\psi)$ and, using (55), the action (57) takes the form

$$\mathcal{S} = \frac{1}{2}\int \left[|D_t\psi|^2 - \frac{1}{2}|\mathbf{D}_x\psi|^2 + \left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{2}|\nabla \times \mathbf{A}|^2\right] dx\, dt - \int W(\psi)\, dx\, dt.$$

Taking the variation of $\mathcal{S}$ with respect to ψ, φ and $\mathbf{A}$ we get the following system of equations:

$$D_t^2\psi - \mathbf{D}_x^2\psi + W'(\psi) = 0, \tag{60}$$

$$\nabla\cdot\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) = q\left(\operatorname{Im}\frac{1}{\psi}\frac{\partial\psi}{\partial t} + q\varphi\right)|\psi|^2, \tag{61}$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) = q\left(\operatorname{Im}\frac{\nabla\psi}{\psi} - q\mathbf{A}\right)|\psi|^2. \tag{62}$$

The evolution problem for equations (60)–(62) has been studied in [13], where the existence of a global solution has been proved.

4.3. The Maxwell equations and matter

The Abelian gauge theory, i.e. equations (60)–(62), provides an elegant way to couple the Maxwell equation with matter if we interpret ψ as a matter field.

In order to give a more meaningful form to these equations, we will write ψ in polar form

$$\psi(x,t) = u(x,t)e^{iS(x,t)}, \qquad u \geq 0,\ S \in \mathbb{R}/2\pi\mathbb{Z}.$$

So (57) takes the form

$$\begin{aligned}\mathcal{S}(u,S,\varphi,\mathbf{A}) = &\iint\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}|\nabla u|^2 - W(u)\right]dx\,dt\\ &+ \frac{1}{2}\iint\left[\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - |\nabla S - q\mathbf{A}|^2\right]u^2\,dx\,dt\\ &+ \frac{1}{2}\iint\left(\left|\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - |\nabla\times\mathbf{A}|^2\right)dx\,dt\end{aligned}$$

and equations (60)–(62) take the form

$$\Box u + W'(u) + \left[|\nabla S - q\mathbf{A}|^2 - \left(\frac{\partial S}{\partial t} + q\varphi\right)^2\right]u = 0, \tag{63}$$

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial S}{\partial t} + q\varphi\right)u^2\right] - \nabla\cdot[(\nabla S - q\mathbf{A})u^2] = 0, \tag{64}$$

$$\nabla\cdot\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) = q\left(\frac{\partial S}{\partial t} + q\varphi\right)u^2, \tag{65}$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right) = q(\nabla S - q\mathbf{A})u^2. \tag{66}$$

These are the complete Maxwell equations. In order to write them in the usual form, we make the change of variables (51), (52) obtaining (53) and (54). Moreover, setting

$$\rho = -\left(\frac{\partial S}{\partial t} + q\varphi\right)qu^2, \tag{67}$$

$$\mathbf{j} = (\nabla S - q\mathbf{A})qu^2, \tag{68}$$

it turns out that (65) and (66) are the second couple of the Maxwell equations with respect to a matter distribution with charge ρ and current density $\mathbf{j}$:

$$\nabla \cdot \mathbf{E} = \rho, \tag{69}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}. \tag{70}$$

Equation (63) can be written as follows:

$$\Box u + W'(u) + \frac{\mathbf{j}^2 - \rho^2}{q^2 u} = 0 \tag{71}$$

and finally (64) is the charge continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0. \tag{72}$$

Notice that equation (72) is a consequence of (69) and (70). In conclusion, an Abelian gauge theory, via equations (69)–(71), provides a model of interaction of the matter field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$. In fact it is possible to show that equations (69)–(71) are equivalent to (63)–(66).

The gauge group is given by $\mathfrak{G} \cong \mathcal{C}^\infty(\mathbb{R}^4)$ with the additive structure and it acts on the variables $\psi, \varphi, \mathbf{A}$ as follows:

$$T_\chi \psi = \psi e^{i\chi},$$
$$T_\chi \varphi = \varphi - \frac{\partial \chi}{\partial t},$$
$$T_\chi \mathbf{A} = \mathbf{A} + \nabla \chi,$$

with $\chi \in \mathcal{C}^\infty(\mathbb{R}^4)$. Our equations are gauge invariant due to the way they have been constructed. However, if we use the variables $u, \rho, \mathbf{j}, \mathbf{E}$, $\mathbf{H}$, this fact can be checked directly since these variables are gauge invariant.

4.4. Existence of solitary waves

Now let us consider the problem of the existence of solitary waves and solitons for an Abelian gauge theory. The Lagrangian $\mathcal{L}$ is invariant for the following representation of the Lorentz group:

$$\psi_{\mathbf{v}}(t,x) = \psi(t', x'),$$
$$\varphi_{\mathbf{v}}(t,x) = \gamma[\varphi(t', x') + \mathbf{v} \cdot \mathbf{A}(t', x')],$$
$$\mathbf{A}_{\mathbf{v}}(t,x) = \gamma[\mathbf{A}(t', x') + \varphi(t', x')\mathbf{v}],$$

thus, similarly to the case of (8), in order to produce solitary waves, it is sufficient to find stationary solutions of (63)–(66) and to make a Lorentz transformation. By definition, a stationary solution of (63)–(66) is a solution of the form

$$\psi(t,x) = u(x)e^{i(S(x)-\omega t)}, \qquad u > 0,\ \omega \in \mathbb{R},\ S \in \mathbb{R}/2\pi\mathbb{Z},$$
$$\partial_t \mathbf{A} = 0, \qquad \partial_t \varphi = 0.$$

A stationary solution solves the following system of equations:

$$-\Delta u + [|\nabla S - q\mathbf{A}|^2 - (q\varphi - \omega)^2]\, u + W'(u) = 0, \tag{73}$$

$$-\nabla \cdot [(\nabla S - q\mathbf{A})u^2] = 0, \tag{74}$$

$$\Delta\varphi = q(q\varphi - \omega)u^2, \tag{75}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(\nabla S - q\mathbf{A})u^2. \tag{76}$$

Equations (73)–(76) are the Euler–Lagrange equation of the *reduced action functional*, i.e. their solutions are the critical points of the functional

$$\begin{aligned}\mathcal{J}(u, S, \varphi, \mathbf{A}) = \frac{1}{2}\int (|\nabla u|^2 - |\nabla\varphi|^2 + |\nabla \times \mathbf{A}|^2)\, dx\, dt \\ + \frac{1}{2}\int [|\nabla S - q\mathbf{A}|^2 - (q\varphi - \omega)^2]u^2\, dx\, dt + \int W(u)\, dx\, dt.\end{aligned} \tag{77}$$

Clearly when $u = 0$, the only finite energy gauge potentials are the trivial ones $\mathbf{A} = \mathbf{0}$, $\varphi = 0$.

It is possible to have three types of stationary nontrivial solutions of (73)–(76):

- electrostatic solutions: $\mathbf{A} = 0$, $\varphi \neq 0$;
- magnetostatic solutions: $\mathbf{A} \neq 0$, $\varphi = 0$;
- electromagnetostatic solutions: $\mathbf{A} \neq 0$, $\varphi \neq 0$.

Under suitable assumptions, all these types of solutions exist. In this section we will discuss a theorem relating to electrostatic solutions. In this case we make the following Ansatz:

$$u = u(x), \qquad S = -\omega t, \qquad \mathbf{A} = 0, \qquad \varphi = \varphi(x).$$

With this Ansatz, the equations (64) and (66) are identically satisfied, while (63) and (65) become

$$-\Delta u + (q\varphi - \omega)^2 u + W'(u) = 0, \tag{78}$$

$$\Delta\varphi = q(q\varphi - \omega)u^2; \tag{79}$$

and the functional (77) reduces to

$$\mathcal{J}(u, \varphi) = \frac{1}{2}\int [|\nabla u|^2 - |\nabla\varphi|^2 - (q\varphi - \omega)^2 u^2]\, dx\, dt. \tag{80}$$

The following theorem can be proved

Theorem 9. *Assume that*

$$W(u) = \frac{1}{2}\Omega^2 u^2 - \frac{1}{p}u^p \qquad \textit{with } 4 < p < 6,\ \Omega > 0. \tag{81}$$

Then, if $|\omega| < \Omega$*, there exist infinitely many solutions* (u, φ) *of* (78), (79) *such that*

$$u \in H^1(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} |\nabla\varphi|^2\, dx < \infty.$$

The proof of this theorem is contained in [1]. The existence and nonexistence of electrostatic solutions for a system like (78), (79) has been proved under different growth conditions on W (see [7]–[10], [20]). The analysis of the stability of such solutions has not been carried out; however, some results in this direction are contained in [14].

4.5. Existence of vortices

In this section, we will discuss the existence of magnetostatic and electromagnetostatic solutions, in particular we shall study the existence of vortices in the sense of the definition stated below. We set

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$$

and we define the map

$$\theta : \mathbb{R}^3 \setminus \Sigma \to \mathbb{R}/2\pi\mathbb{Z}, \qquad \theta(x) = \operatorname{Im}\log(x_1 + ix_2).$$

A solution of (73)–(76) is called a *vortex* if it has the form

$$\psi(x,t) = u(x)e^{i(k\theta(x)-\omega t)}, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{82}$$

Observe that $\theta \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}/2\pi\mathbb{Z})$ and $\nabla\theta \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3)$, namely

$$\nabla\theta(x) = \left(\frac{x_2}{x_1^2 + x_2^2}, \frac{-x_1}{x_1^2 + x_2^2}, 0\right).$$

Using this Ansatz equations (73), (75), (76) become

$$-\Delta u + \left[|k\nabla\theta - q\mathbf{A}|^2 - (q\varphi - \omega)^2\right]u + W'(u) = 0, \tag{83}$$

$$\Delta\varphi = q(q\varphi - \omega)u^2, \tag{84}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(k\nabla\theta - q\mathbf{A})u^2. \tag{85}$$

The following existence result for vortex solutions holds:

Theorem 10. *Assume that W is defined by* (81) *with $2 < p < 6$ and set*

$$\omega_p = \Omega\sqrt{\min\left(1, \frac{p-2}{2}\right)}, \qquad \Omega > 0.$$

Then for any $\omega \in (-\omega_p, \omega_p)$ and any $k \in \mathbb{Z}$ equations (83)–(85) *admit a finite energy solution $(u, \varphi, \mathbf{A})$ with $u \neq 0$. Moreover*

(i) *if $\omega \neq 0$ and $k = 0$, then $\varphi \neq 0$ and $\mathbf{A} = 0$ (electrostatic solutions),*
(ii) *if $\omega = 0$ and $k \neq 0$, then $\varphi = 0$ and $\mathbf{A} \neq 0$ (magnetostatic vortices),*
(iii) *if $\omega \neq 0$ and $k \neq 0$, then $\varphi \neq 0$ and $\mathbf{A} \neq 0$ (electromagnetostatic vortices).*

The proof of this result, in a slightly more general form, is contained in [4].

Remark 11. By using (84) it is easy to see that, if $\omega_1 \neq \omega_2$, then the corresponding solutions $(u_{\omega_1}, \varphi_{\omega_1})$, $(u_{\omega_2}, \varphi_{\omega_2})$ are different. In an analogous manner equation (85) implies that if $k_1 \neq k_2$, then the corresponding solutions $(u_{k_1}, \mathbf{A}_{k_1})$, $(u_{k_2}, \mathbf{A}_{k_2})$ are different.

Remark 12. If $(u, \varphi, \mathbf{A})$ with $u \neq 0$ solves (83)–(85), then assertions (i)–(iii) in Theorem 10 follow immediately from (84), (85).

4.6. The case of positive energy density

Theorems 9 and 10 allow one to show the existence of solitary waves in gauge theories; however, they are not suitable for physical models, as in these theorems W is not positive. Thus there exist field configurations for which the energy density and hence, by Theorem 6, the matter density is not positive, which is a very unpleasant fact.

In order to check this, we compute the formula for the energy.

Theorem 13. *If* $(u, S, \mathbf{E}, \mathbf{H})$ *is a solution of the field equations* (63)–(66), *its energy is given by*

$$\mathcal{E}(u, S, \mathbf{E}, \mathbf{H}) = \int \left[\frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} |\nabla u|^2 + W(u) + \frac{\rho^2 + \mathbf{j}^2}{2q^2 u^2} + \frac{\mathbf{E}^2 + \mathbf{H}^2}{2} \right] dx$$

where ρ *and* $\mathbf{j}$ *are defined in* (67), (68).

Proof. We recall the well known expression for the energy density (see e.g. [12]):

$$\frac{\partial \mathcal{L}}{\partial (\frac{\partial u}{\partial t})} \cdot \frac{\partial u}{\partial t} + \frac{\partial \mathcal{L}}{\partial (\frac{\partial S}{\partial t})} \cdot \frac{\partial S}{\partial t} + \frac{\partial \mathcal{L}}{\partial (\frac{\partial \varphi}{\partial t})} \cdot \frac{\partial \varphi}{\partial t} + \frac{\partial \mathcal{L}}{\partial (\frac{\partial \mathbf{A}}{\partial t})} \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L},$$

where

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} |\nabla u|^2 - W(u) \\ & + \frac{1}{2} \left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - \frac{1}{2} |\nabla S - q\mathbf{A}|^2 u^2 \\ & + \frac{1}{2} \left| \frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right|^2 - \frac{1}{2} |\nabla \times \mathbf{A}|^2 . \end{aligned}$$

Now we will compute each term. We have

$$\frac{\partial \mathcal{L}}{\partial (\frac{\partial u}{\partial t})} \cdot \frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t} \right)^2 \tag{86}$$

and

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial (\frac{\partial S}{\partial t})} \cdot \frac{\partial S}{\partial t} &= \left(\frac{\partial S}{\partial t} + q\varphi \right) \frac{\partial S}{\partial t} u^2 \\ &= \left(\frac{\partial S}{\partial t} + q\varphi \right) \frac{\partial S}{\partial t} u^2 + \left(\frac{\partial S}{\partial t} + q\varphi \right) q\varphi u^2 - \left(\frac{\partial S}{\partial t} + \varphi \right) q\varphi u^2 \\ &= \left(\frac{\partial S}{\partial t} + q\varphi \right)^2 u^2 - \left(\frac{\partial S}{\partial t} + q\varphi \right) q\varphi u^2 \\ &= \frac{\rho^2}{q^2 u^2} + \rho\varphi . \end{aligned}$$

By the Gauss equation (69), multiplying by φ and integrating, we get

$$-\int \mathbf{E}\cdot\nabla\varphi = \int \rho\varphi.$$

Thus, replacing this expression in the above formula, we get

$$\int \frac{\partial \mathcal{L}}{\partial(\frac{\partial S}{\partial t})}\cdot\frac{\partial S}{\partial t} = \int\left(\frac{\rho^2}{q^2u^2} - \mathbf{E}\cdot\nabla\varphi\right). \tag{87}$$

Also we have

$$\frac{\partial \mathcal{L}}{\partial(\frac{\partial \varphi}{\partial t})}\cdot\frac{\partial \varphi}{\partial t} = 0 \tag{88}$$

and

$$\frac{\partial \mathcal{L}}{\partial(\frac{\partial \mathbf{A}}{\partial t})}\cdot\frac{\partial \mathbf{A}}{\partial t} = \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right)\cdot\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}. \tag{89}$$

Moreover, using the notation (51), (52), (67), (68), we have

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}|\nabla u|^2 - W(u) + \frac{\rho^2 - \mathbf{j}^2}{2q^2u^2} + \frac{\mathbf{E}^2 - \mathbf{H}^2}{2}.$$

Then, by (86)–(89) and the above expression for $\mathcal{L}$ we get

$$\begin{aligned}
\mathcal{E}(u,S,\varphi,\mathbf{A}) &= \int\left[\frac{\partial \mathcal{L}}{\partial(\frac{\partial u}{\partial t})}\cdot\frac{\partial u}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\frac{\partial S}{\partial t})}\cdot\frac{\partial S}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\frac{\partial \mathbf{A}}{\partial t})}\cdot\frac{\partial \mathbf{A}}{\partial t} - \mathcal{L}\right]\\
&= \int\left[\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\rho^2}{q^2u^2} - \mathbf{E}\cdot\nabla\varphi - \mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t} - \mathcal{L}\right]\\
&= \int\left[\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\rho^2}{q^2u^2} + \mathbf{E}^2\right]\\
&\quad - \int\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}|\nabla u|^2 - W(u) + \frac{\rho^2 - \mathbf{j}^2}{2q^2u^2} + \frac{\mathbf{E}^2 - \mathbf{H}^2}{2}\right]\\
&= \int\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}|\nabla u|^2 + W(u) + \frac{\rho^2 + \mathbf{j}^2}{2q^2u^2} + \frac{\mathbf{E}^2 + \mathbf{H}^2}{2}\right]. \qquad \square
\end{aligned}$$

The above theorem implies that the interesting case occurs when $W \geq 0$. The study of equations (60)–(62) with $W \geq 0$ is just at the beginning. Now we will state a very recent first result in this direction.

We assume that W is a C^2 function satisfying the following assumptions:

(W_1) $W \geq 0$, $W(0) = W'(0) = 0$.
(W_2) $W''(0) = \Omega_0^2 > 0$.
(W_3) There exist $\Omega_1, c > 0$ with $\Omega_1 < \Omega_0$ such that

$$W(s) \leq \frac{1}{2}\Omega_1^2 s^2 + c \quad \text{for all } s \in \mathbb{R}.$$

(W_4) For all $s \in \mathbb{R}$,

$$0 \leq \frac{1}{2} W'(s)s \leq W(s).$$

(W_5) W'' is bounded.

The following theorem holds [5]:

Theorem 14. *Assume that W satisfies (W_1)–(W_5). Then there exists $q_* > 0$ such that, for $0 < q < q_*$, equations* (78) *and* (79) *have solutions (u, φ) such that $u \in H^1(\mathbb{R}^3)$, $\int |\nabla\varphi|^2 < \infty$ and $\omega_0 \in (\Omega_1, \Omega_0)$.*

Remark 15. The assumption that q must be sufficiently small is essential in the proof of the theorem. This fact has the following physical interpretation: if q is too big, the electric force becomes too strong with respect to the forces that keep the solitary wave concentrated. Then solitary waves cannot form.

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Vieri Benci
Dipartimento di Matematica Applicata "U. Dini"
Università di Pisa
and
Centro Interdisciplinare per lo Studio dei Sistemi Complessi (CISSC)
Via Filippo Buonarroti 1
56126 Pisa, Italy
e-mail: `benci@dma.unipi.it`

Donato Fortunato
Dipartimento di Matematica
Università di Bari
Via Orabona 4
70125 Bari, Italy
e-mail: `fortunat@dm.uniba.it`