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Applications of theorems of Jean Leray to the Einstein-scalar field equations

Yvonne Choquet-Bruhat, James Isenberg and Daniel Pollack

Dedicated to the memory of Jean Leray

Abstract. The Einstein-scalar field theory can be used to model gravitational physics with scalar field sources. We discuss the initial value formulation of this field theory, and show that the ideas of Leray can be used to show that the Einstein-scalar field system of partial differential equations is well-posed as an evolutionary system. We also show that one can generate solutions of the Einstein-scalar field constraint equations using conformal methods.

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1. Introduction

In Newtonian theory, one models gravitational physics by studying a linear elliptic Poisson equation for the Newtonian potential on a fixed absolute background space and time, with the motion of material bodies governed by the Newtonian force equation on this fixed background. By contrast, in general relativity the gravitational field is modeled using Lorentzian spacetimes whose curvature reflects the material and field content of the spacetime. Mathematically, a Lorentzian spacetime is a pair $(M^{n+1}, \mathbf{g})$ where M^{n+1} is a smooth manifold of dimension $n+1$ (in everyday physics $n = 3$, but higher dimensions are sometimes considered for modeling electromagnetism and other interactions with gravity), and $\mathbf{g}$ is a pseudo-Riemannian metric of signature $(-, +, \ldots, +)$. The metric distinguishes timelike directions $\mathbf{g}(X, X) < 0$ for a tangent vector X (along the possible path for a massive physical object), null directions $\mathbf{g}(X, X) = 0$ (along the possible path for a massless physical particle), and spacelike directions $\mathbf{g}(X, X) < 0$. The physical time between a pair of events, as would be marked by a proper clock, corresponds to the $\mathbf{g}$-length of the timelike trajectory followed by that clock. The path followed by a test particle which is free of nongravitational forces corresponds to a timelike geodesic in the spacetime.

It is well known that theoretical studies of general relativity have predicted such strange and interesting phenomena as the expansion of the universe, black holes, gravitational lenses, and gravitational waves. These phenomena, all of which have now been confirmed either by direct or indirect observation, were originally discovered via studies of solutions of the Einstein field equations. It is also well known, at least among mathematicians, that Einstein's equations present a number of very challenging mathematical problems, such as the cosmic censorship conjectures and the question of the nonlinear stability of black holes.

In this brief review, we show that in two important ways, the ideas of Leray have played an important role in the study of Einstein's equations. In previous studies these ideas were applied in the absence of a scalar field. Here, since scalar fields are now viewed as possibly important for understanding the apparent acceleration of the expansion of the universe, and since including them results in some additional interesting features in the analysis, we work with the Einstein-scalar field system. We introduce this system in Section 2 and discuss its Cauchy formulation in Section 3. We then show in Section 4 how Leray's ideas play a role in understanding the evolutionary aspect of the Einstein-scalar field system. In Section 5 we prove the existence of solutions to the Einstein-scalar constraint equations via the conformal methods, where seminal ideas of Leray regarding solutions of nonlinear elliptic equations have played a crucial role.

We end this section with a note about notation. Wherever there may arise a possible confusion, we use bold faced symbols for spacetime variables and tensors. For initial data sets, we reserve the notation of over-barred symbols for physical variables, which satisfy the relevant constraint equations, and may or may not be "time-dependent" depending on the context. Unadorned symbols are used for free conformal data as described in Section 5. Finally, a "tilded" symbol (as in (5.4) below) appears when we need to introduce an intermediate quantity which is neither a free variable, nor a physical one.

2. Einstein-scalar field equations

For general source fields, the Einstein gravitational field equations take the tensorial form

$$\mathbf{G}(\mathbf{g}) = \mathbf{T}(\Phi, \mathbf{g}), \tag{2.1}$$

where $\mathbf{G}(\mathbf{g})$ is the Einstein tensor, which is a second order differential operator on the metric defined by $\mathbf{G}(\mathbf{g}) := \mathbf{Ric}(\mathbf{g}) - \frac{1}{2}\mathbf{R}\,\mathbf{g}$ with $\mathbf{Ric}(\mathbf{g})$ denoting the Ricci tensor of $\mathbf{g}$ and $\mathbf{R}$ denoting the scalar curvature of $\mathbf{g}$, and where $\mathbf{T}(\Phi, \mathbf{g})$ is the stress-energy or energy-momentum tensor,[1] a specified functional of the source fields Φ and the metric. The specific form that the stress-energy tensor takes depends upon

[1]Note that we have chosen units so that 8π times the Newtonian gravitational constant is set equal to one.

the source fields presumed to be present in the physical system being modeled. Here, we presume that Φ is a scalar field, which we label Ψ, with potential function $V(\Psi)$, and we set

$$\mathbf{T} = \partial\Psi \otimes \partial\Psi - \left[\frac{1}{2}|\partial\Psi|^2_{\mathbf{g}} + V(\Psi)\right]\mathbf{g}. \tag{2.2}$$

In addition to equation (2.1), the Einstein-scalar field theory includes a field equation for Ψ, which reads

$$\nabla^\alpha \partial_\alpha \Psi = \frac{dV}{d\Psi}, \tag{2.3}$$

where ∇ denotes the covariant derivative compatible with $\mathbf{g}$. While this extra equation may simply be added to the theory by hand, it also follows directly as a necessary consequence of equation (2.1) together with the geometric properties of the Einstein tensor: One readily verifies that the Bianchi identities for the curvature imply that the Einstein tensor satisfies the identity

$$\mathrm{div}_{\mathbf{g}}\, \mathbf{G}(\mathbf{g}) = 0 \tag{2.4}$$

where $\mathrm{div}_{\mathbf{g}}$ denotes the divergence operator for the metric $\mathbf{g}$. This condition together with (2.1) implies the conservation law

$$\mathrm{div}_{\mathbf{g}}\, \mathbf{T} = 0. \tag{2.5}$$

We readily verify that (2.5) applied to the scalar field stress-energy tensor (2.2) results in the field equation (2.3).

How does one choose the scalar field potential function $V(\Psi)$? While there are many possibilities, we note that $V(\Psi) = \frac{m}{2}\Psi^2$ corresponds to the massive Klein–Gordon field, while setting $V(\Psi) = \Lambda$ for a nonzero constant Λ and requiring that Ψ = constant produces the vacuum Einstein theory with nonzero cosmological constant Λ.

3. The Cauchy problem: constraints and evolution

The Einstein-scalar field system of partial differential equations on a 3+1 spacetime consists of eleven equations (2.1) and (2.3) for the eleven field variables $\mathbf{g}_{\mu\nu}$ and Ψ (for $n+1$ dimensions, there are $\frac{1}{2}(n+1)(n+2)+1$ equations for the same number of field variables). One of its most characteristic features, however, is that it is both an underdetermined and an overdetermined system, in the sense that if one formulates the Einstein-scalar-field system as a Cauchy problem, there are constraint equations which must be satisfied by any candidate set of initial data, and as the data evolves there are certain of the field variables whose evolution is entirely at one's discretion. Both of these features reflect the spacetime covariance of the theory (i.e., the theory has the spacetime diffeomorphism group as its gauge group).

To see these features explicitly, we now sketch out an $n+1$-decomposition of the Einstein-scalar field variables and equations. Given a spacetime $(M^{n+1}, \mathbf{g})$, we start by choosing an $n+1$-foliation of the spacetime manifold $F_t : \Sigma^n \to M^{n+1}$

($t \in \mathbb{R}$), for which each of the leaves $F_t(\Sigma^n)$ of the foliation is presumed spacelike. We also choose a threading of the spacetime by a congruence of timelike observer paths $T_p : \mathbb{R} \to M^{n+1}$ ($p \in \Sigma^n$). The choice of a foliation and a threading, together with a choice of coordinate patches for Σ^n, automatically determines local coordinates ($x^0 = t, x^1, \ldots, x^n$) and local coordinate bases ($\partial/\partial t, \partial/\partial x^1 \ldots, \partial/\partial x^n$) covering M^{n+1}. We may then, without loss of generality, write the metric (locally) in the form

$$\mathbf{g} = -N^2\theta^t \otimes \theta^t + \bar{\gamma}_{ij}\theta^i \otimes \theta^j$$

where ($\theta^t = dt, \theta^j = dx^j + \beta^j dt$) is the one-form basis dual to the surface-compatible tangent vector basis

$$\left(e_\perp = \frac{\partial}{\partial t} - \beta^j \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^1} \ldots, \frac{\partial}{\partial x^n}\right)$$

with $e_\perp$ a normal vector field to $F_t(\Sigma^n)$. Here N is the positive definite "lapse function", β^j are the components of the spacelike "shift vector", and $\bar{\gamma}_{ij}$ are the components of the spatial metric tensor. We note that for each choice of t, $\bar{\gamma}(t) = \bar{\gamma}_{ij}(t)dx^i \otimes dx^j$ is the induced Riemannian metric on the leaf $F_t(\Sigma^n)$. We use the notation $\bar{K}(t) = \bar{K}_{ij}(t)dx^i \otimes dx^j$ to denote the second fundamental form defined by the foliation.

For the scalar field Ψ, there is no need to do any space + time decomposition. However, we shall use the notation $\bar{\psi}$ to denote the restriction of Ψ to one of the leaves of the chosen foliation, and we shall use the definition

$$\bar{\pi} := \frac{1}{N}\left(\frac{\partial}{\partial t}\bar{\psi} - \beta^m \frac{\partial}{\partial x^m}\bar{\psi}\right)$$

for convenience in working with the time derivative of Ψ.

If we now apply the usual $n+1$-decomposition to express the spacetime curvature in terms of the time-dependent spatially covariant quantities $\bar{\gamma}, \bar{K}, N, \beta$ and their various derivatives and (spatial) curvature, we find that the Einstein-scalar field equations (2.1) and (2.3) split into two types: constraint equations which require any choice of initial data to satisfy certain identities, and evolution equations which describe how the spatial fields evolve from one leaf of the foliation to the others. Explicitly, we have the following:

Constraint equations

From the $\mathbf{G}_{\perp\perp}$ equation derived from (2.1), we obtain the *Hamiltonian constraint*

$$2N^{-2}\mathbf{G}_{\perp\perp} \equiv R_{\bar{\gamma}} - |\bar{K}|^2_{\bar{\gamma}} + (\operatorname{tr}\bar{K})^2 = \bar{\pi}^2 + |\bar{\nabla}\bar{\psi}|^2_{\bar{\gamma}} + 2V(\bar{\psi}). \tag{3.1}$$

From the $\mathbf{G}_{\perp j}$ equations derived from (2.1), we obtain the *momentum constraint*

$$-N^{-1}\mathbf{G}_{\perp j} \equiv \bar{\nabla}_m \bar{K}^m_j - \partial_j \operatorname{tr}\bar{K} = \bar{\pi}\partial_j\bar{\psi}. \tag{3.2}$$

Note that these equations constrain the choice of the data ($\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi}$); they do not involve the lapse and shift. We refer the interested reader to [2] for a survey on the constraint equations.

Evolution equations

From the $\mathbf{G}_{ij}$ equations derived from (2.1), we obtain

$$\frac{\partial}{\partial t}\bar{K}_{ij} = N\left(R_{ij} - 2\bar{K}_{im}\bar{K}_j^m + \operatorname{tr}\bar{K}\bar{K}_{ij} - \partial_i\bar{\psi}\partial_j\bar{\psi} + \frac{1}{n-1}\bar{\gamma}_{ij}V(\bar{\psi})\right) - \bar{\nabla}_i\partial_j N + \mathcal{L}_\beta\bar{K}_{ij}; \tag{3.3}$$

here and above $\bar{\nabla}$ is the covariant derivative associated to $\bar{\gamma}$, R_{ij} are the components of the spatial Ricci tensor calculated from $\bar{\gamma}$, and $\mathcal{L}$ denotes the Lie derivative operator. This is an evolution equation for $\bar{K}$. We obtain an evolution equation for $\bar{\pi}$ from the spacetime field equation (2.3) for $\bar{\psi}$:

$$\frac{\partial}{\partial t}\bar{\pi} = N\left(\Delta_{\bar{\gamma}}\bar{\psi} + \operatorname{tr}\bar{K}\bar{\pi} - \frac{dV}{d\bar{\psi}}\right) + \bar{\gamma}^{mn}\partial_m N\partial_n\bar{\psi} + \mathcal{L}_\beta\bar{\pi}, \tag{3.4}$$

where $\Delta_{\bar{\gamma}}$ is the Laplace–Beltrami operator for the metric $\bar{\gamma}$.

We have evolution equations for $\bar{K}$ and $\bar{\pi}$. What about $\bar{\gamma}$, N, β and $\bar{\psi}$? The evolution equation for $\bar{\psi}$ comes from the definition for $\bar{\pi}$:

$$\frac{\partial}{\partial t}\bar{\psi} = N\bar{\pi} + \mathcal{L}_\beta\bar{\psi}. \tag{3.5}$$

The evolution equation for $\bar{\gamma}$ comes from the definition of the second fundamental form:

$$\frac{\partial}{\partial t}\bar{\gamma}_{ij} = -2N\bar{K}_{ij} + \mathcal{L}_\beta\bar{\gamma}_{ij}. \tag{3.6}$$

For the other field variables, N and β, there are no evolution equations. This freedom to choose N and β and their evolution any way one wishes reflects the gauge invariance of the field equations under the action of the diffeomorphism group.

To summarize, the Cauchy formulation of the Einstein-scalar field equations asks that one choose the initial data set $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ subject to the constraint equations (3.1)–(3.2). One then chooses N and β freely in time, to fix the gauge, and finally one proceeds to evolve $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ via the evolution equations just listed. Note that if one chooses the constraints to hold initially, in any accurate evolution they must remain satisfied for all time.

4. Evolution system

Does an initial data set satisfying the constraint equations (3.1)–(3.2) always generate (via the evolution equations (3.3)–(3.6)) a spacetime solution of the Einstein-scalar field equations (2.1)–(2.3)? To show this, one needs to prove that the system is well-posed in some appropriate sense. Here we discuss the well-posedness results of Leray–Ohya and of Leray, as applied to the Einstein-scalar field system. Note that our focus here is solely on the evolution system. One must use the Bianchi identity to show that the constraints are preserved in the evolution (they satisfy a hyperbolic system, see [9]), thus yielding a local existence result for the full set of field equations. We refer the reader to [1] where this issue (again for vacuum solutions) is addressed in a similar setting to the one considered here.

To check if a given system is hyperbolic in the Leray–Ohya sense, one seeks to diagonalize the matrix of the principal parts (highest derivatives) of the system. If this diagonalization can be done, the system is a causal Leray–Ohya hyperbolic system if in addition to being Leray–Ohya hyperbolic, each operator has a characteristic cone which contains the metric cone. As a consequence, it can be shown that it is well-posed in Gevrey classes of functions. These are spaces of C^∞ functions whose successive derivatives satisfy certain inequalities which are generally too weak to imply the convergence of the corresponding Taylor series. Well-posedness means that a set of initial data in such classes will generate a spacetime solution with evolving data in the same classes. It also implies causal propagation, with the domain of dependence of the data determined by the causal cones of the spacetime metric, as well as continuous dependence of the evolved solution on the choice of data. Note that the principal parts of a Leray–Ohya system may generally have multiple characteristics. Also note that in verifying the criteria for a Leray–Ohya system, one need not have the same order for each of the various evolution equations which make up the system.

If in fact the operators in the principal part of a Leray–Ohya system do not have multiple characteristics—i.e., if there exists a cone in the cotangent plane such that each straight line passing through a point in its interior intersects the characteristic cone in N distinct points if the operator is of order N, then the system is called *Leray hyperbolic.* It can be shown that such a system is well-posed in Sobolev spaces as well in Gevrey class spaces.

For a more detailed discussion of Leray–Ohya and Leray hyperbolicity, see [13, 14, 4].

We now apply these ideas to the Einstein-scalar field system. In doing so, we work with the Einstein-scalar field system in mixed first order (for $\bar{\gamma}$ and $\bar{K}$) and second order (for $\bar{\psi}$), to mesh with the extant treatments.

4.1. Leray–Ohya hyperbolic system for $\bar{\gamma}, \bar{K}$ and $\bar{\psi}$

We set $\bar{\partial}_0 := \partial/\partial t - \mathcal{L}_\beta$ and we consider the system

$$\bar{\partial}^2_{00} R_{ij} - \bar{\partial}_0 \bar{\nabla}_i R_{j0} - \bar{\partial}_0 \bar{\nabla}_j R_{i0} + \bar{\nabla}_i \bar{\nabla}_j R_{00} = F_{ij}$$

which is derived by taking linear combinations of equations from the Einstein-scalar field system (see [1] for a similar derivation in the vacuum case). If $\bar{\partial}_0 \bar{\gamma}_{ij}$ is replaced by its value in terms of $\bar{K}$,

$$\bar{\partial}_0 \bar{\gamma}_{ij} = -2N\bar{K}_{ij}, \tag{4.1}$$

this system reads as a third order system for $\bar{K}$ of the form

$$\bar{\partial}_0(-N^{-2}\bar{\partial}_0^2 + \bar{\nabla}^h \bar{\nabla}_h)\bar{K}_{ij} = f_{ij}\,(2 \text{ in } \bar{\gamma},\, 2 \text{ in } \bar{K},\, 3 \text{ in } N,\, 3 \text{ in } \bar{\psi}) + \tilde{f}_{ij}, \tag{4.2}$$

with

$$\tilde{f}_{ij} := N\bar{\nabla}_i \partial_j (N^{-2}\bar{\partial}_0^2 - \bar{\nabla}^h \partial_h)N.$$

The numbers appearing above with f_{ij} (and below with h) denote the order of the highest derivatives of the unknowns which occur in that term. The additional term $N\bar{\nabla}_i\partial_j(N^{-2}\bar{\partial}_0^2 - \bar{\nabla}^h\partial_h)N$ is clearly third order in $\bar{\gamma}$ (and fourth order in N).

The wave equation for $\bar{\psi}$ reads, in terms of $\bar{\gamma}$, $\bar{K}$ and the presumably specified variables N and β, as follows:

$$-N^{-1}\partial_0(N^{-1}\partial_0\bar{\psi}) + N^{-1}\bar{\gamma}^{ij}\bar{\nabla}_i(N\partial_j\bar{\psi}) + \bar{K}_i^i N^{-1}\partial_0\bar{\psi} = \frac{dV}{d\bar{\psi}}.$$

Applying the operator ∂_0 and using (4.1) gives an equation of the form

$$\partial_0(-N^{-2}\bar{\partial}_0^2 + \bar{\nabla}^h\partial_h)\bar{\psi} = h\,(1 \text{ in } \bar{\gamma},\ 1 \text{ in } \bar{K},\ 2 \text{ in } N,\ 2 \text{ in } \bar{\psi}). \tag{4.3}$$

The principal matrix of a system of partial differential equations $E_B(u_A) = 0$, with unknowns u_A, is obtained by assigning to each unknown an integer $m(u_A)$ and to each equation an integer $n(E_B)$ such that the highest derivatives of u_A appearing in E_B are at most of order $m(u_A) - n(E_B)$. The principal part relative to u_A in the equation $E_B = 0$ then consists of the terms of order $m(u_A) - n(E_B)$ in u_A. It is zero if there are only terms of smaller order. These integers $m(u_A)$ and $n(E_B)$ are collectively called the *Leray–Volevich indices* for the system.

For the system of PDEs (4.1), (4.2) and (4.3) with the field variables $\bar{\gamma}(t)$, $\bar{K}(t)$ and $\bar{\psi}(t)$, we choose the Leray–Volevich indices

$$m(\bar{\gamma}) = 3, \quad m(K) = 3, \quad m(\Psi) = 4, \tag{4.4}$$

$$n(4.1) = 2, \quad n(4.2) = 0, \quad n(4.3) = 1. \tag{4.5}$$

The principal matrix is then a triangular matrix, with the elements in the diagonal being the derivative $\bar{\partial}_0$ along the normal $e_\perp$ to the space sections and the product of this operator by the wave operator in the spacetime matrix. These operators are causal and hyperbolic. Both are such that their characteristic cones at a point contain the metric null cone. However, the nondiagonal form of the principal matrix does not permit one to conclude that it is Leray hyperbolic.

If we replace $\bar{K}$ in (4.2) by its value in terms of $\bar{\gamma}$ using (4.1), and give to $\bar{\gamma}$ the index 4, we obtain for (4.2) and (4.3) a diagonal system with principal operators the wave operator and the operator

$$\bar{\partial}_0^2(-N^{-2}\bar{\partial}_0^2 + \bar{\nabla}^h\bar{\nabla}_h).$$

This operator has a double characteristic, the spacelike hyperplane, and the system is only Leray–Ohya hyperbolic, relative to the Gevrey class of index 2. The Cauchy problem for this system is well posed in this class; the domain of dependence of the solution is determined by the light cone of the spacetime metric.

4.2. Leray hyperbolic system for $\bar{\gamma}$, $\bar{K}$ and $\bar{\psi}$ with a lapse condition

The system for $\bar{\gamma}$, $\bar{K}$ and $\bar{\psi}$ can be put into Leray hyperbolic form if we impose a condition on the lapse function N which makes it a quasi-diagonal system for $\bar{\gamma}$, $\bar{K}$, $\bar{\psi}$ and now also N.

To remove the term $\tilde{f}_{ij}$, which introduces nondiagonal elements into the principal matrix, we require that N satisfy a wave equation with source term, with that source term being an arbitrarily specified function F:

$$-N^{-2}\partial^2_{00}N + \bar{\nabla}^i\partial_i N = F,$$

and we then insert this equation into (4.2). We now consider the system consisting of (4.1)–(4.3) with this change, together with the equation

$$\bar{\partial}_0((-N^{-2}\bar{\partial}^2_0 + \bar{\nabla}^i\partial_i)N) = \bar{\partial}_0 F, \tag{4.6}$$

obtained by taking the $\bar{\partial}_0$ derivative of (4.2), and using (4.1). We choose for N and (4.6) the Leray–Volevich indices

$$m(N) = 4, \quad n(4.6) = 1. \tag{4.7}$$

The system is now quasi-diagonal with hyperbolic diagonal elements given by $\bar{\partial}_0(-N^{-2}\bar{\partial}^2_0 + \bar{\nabla}^i\bar{\nabla}_i)$ and $\bar{\partial}_0$. This leads to the following result.

Theorem 4.1. *The system* (4.1), (4.2), (4.3), *and* (4.6) *is a Leray causal hyperbolic system for* $\bar{\gamma}$, $\bar{K}$, $\bar{\psi}$ *and* N.

5. The constraints

The Einstein-scalar field constraints consist of the $n+1$ equations (3.1)–(3.2), to be satisfied by the initial data $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ on an n-dimensional manifold Σ. Locally, due to the symmetry of these tensors, this initial data can be regarded as a set of $n(n+1)+2$ functions, which makes the underdetermined nature of the constraint equations apparent. We recall that $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ denotes a set of initial data which satisfies the constraint equations; we use the same quantities without the over-bars to denote functions which we choose freely in order to construct the data $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$. Also for convenience here, we restrict our considerations to the so-called cosmological case, for which Σ is a compact manifold (see [6] for a treatment of the asymptotically flat case). Even in vacuum there are infinitely many solutions of the constraints, depending on arbitrary transverse-traceless (divergence and trace free) tensors, which can be interpreted as "radiation data".

5.1. The conformally formulated constraints

The conformal method involves decomposing the data $(\bar{\gamma}, \bar{K}, \bar{\psi}, \bar{\pi})$ into certain parts which are chosen freely, and other parts which are determined by solving equations which we derive from the constraint equations. We consider the case $n \geq 3$. The most basic piece of the freely chosen data is a choice of a Riemannian metric γ, or rather the conformal class of metrics represented by γ. The physical metric $\bar{\gamma}$ is required to be conformally related to γ. One sets

$$\bar{\gamma} \equiv \varphi^{\frac{4}{n-2}}\gamma$$

for a positive function φ on Σ. The following identity then holds between the scalar curvatures of $\bar{\gamma}$ and γ:

$$R(\bar{\gamma}) = -\varphi^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_{\gamma}\varphi - R(\gamma)\varphi\right). \tag{5.1}$$

On the other hand, the divergences of traceless contravariant symmetric 2-tensors are related by the identity

$$\operatorname{div}_{\bar{\gamma}} \bar{P} = \varphi^{-\frac{2(n+2)}{n-2}} \operatorname{div}_{\gamma} P \tag{5.2}$$

if

$$\bar{P} = \varphi^{-\frac{2(n+2)}{n-2}} P.$$

Using (5.2) (applied to the traceless part of $\bar{K}$) together with (5.1), the Einstein-scalar field constraints may be conformally reformulated as follows.

5.1.1. The momentum constraint. The momentum constraint may be expressed with respect to the background metric φ by

$$\operatorname{div}_{\gamma} \tilde{K} = \frac{n-1}{n}\varphi^{\frac{2n}{n-2}}\nabla\tau + \varphi^{\frac{2(n+2)}{n-2}}\bar{J}, \tag{5.3}$$

where $\tau = \operatorname{tr}_{\gamma}\tilde{K} = \operatorname{tr}_{\bar{\gamma}}\bar{K}$ is the mean curvature, the physical extrinsic curvature (second fundamental form) $\bar{K}$ is related to $\tilde{K}$ (as contravariant tensors) by

$$\bar{K} = \varphi^{-\frac{2(n+2)}{n-2}}\tilde{K} + \frac{\tau}{n}\bar{\gamma}^{-1}$$

with $\bar{\gamma}^{-1}$ denoting the contravariant form of the metric $\bar{\gamma}$, and where $\bar{J} := -\pi\nabla\psi$. We have shown in [6], following ideas originating from York [18], that it is useful to associate to the background conformal metric γ a function $\tilde{N}$ which is related to the original lapse by the equation[2]

$$N(\operatorname{Det}\bar{\gamma})^{-1/2} = \tilde{N}(\operatorname{Det}\gamma)^{-1/2},$$

or simply

$$N = \varphi^{\frac{2n}{n-2}}\tilde{N}.$$

Then the "physical" scalar field initial data $(\bar{\psi}, \bar{\pi})$ consists of $\bar{\psi} = \psi$ and

$$\bar{\pi} = rN^{-1}\overline{\partial}_0\psi = \varphi^{-\frac{2n}{n-2}}\pi, \quad \text{where} \qquad \pi = \tilde{N}^{-1}\partial_0\psi,$$

and it follows that

$$\bar{J} = -\varphi^{-\frac{2(n+2)}{n-2}}\pi\nabla\psi = \varphi^{-\frac{2(n+2)}{n-2}}J, \quad \text{where} \quad J = -\pi\nabla\psi.$$

Hence *the Einstein-scalar field momentum constraint equation does not contain* φ *if* $\nabla\tau = 0$. Setting

$$\tilde{K} = \mathcal{L}_{\gamma,\mathrm{conf}}X + U, \qquad \Delta_{\gamma,\mathrm{conf}} := \operatorname{div}_{\gamma}\mathcal{L}_{\gamma,\mathrm{conf}} \tag{5.4}$$

[2]This relation consists in requiring that each metric has an associated initial lapse with the same "densitized lapse".

with $\mathcal{L}_{\gamma,\mathrm{conf}}X$ the conformal Lie derivative of γ (or conformal Killing operator) with respect to a vector field X and with U a freely specified traceless 2-tensor, the system may be regarded as a self-adjoint linear elliptic system for X, as follows:

$$\Delta_{\gamma,\mathrm{conf}}X = -\operatorname{div}_\gamma U + \frac{n-1}{n}\varphi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi. \tag{5.5}$$

In summary, if we begin with a choice of "free" initial data $(\gamma, U, \tau, \psi, \pi)$ and solve the conformally formulated momentum constraint equation (5.5) to determine $\tilde{K}$ as indicated in (5.4) then $\tilde{K}$ satisfies the momentum constraint equation (5.3).

5.1.2. The Hamiltonian constraint. If we specify the initial data $(\gamma, U, \tau, \psi, \pi)$ and use the identities (5.2) and (5.1), then the Hamiltonian constraint equation (3.1) becomes a semilinear elliptic equation, called the *Lichnerowicz equation*, for φ. We have shown [6, 7] that it takes the form

$$\mathcal{H} \equiv \Delta_\gamma\varphi - f(\varphi) = 0 \tag{5.6}$$

with

$$f(\varphi) := \mathcal{R}_{\gamma,\psi}\,\varphi - \mathcal{A}_{\gamma,\tilde{K},\pi}\,\varphi^{-\frac{3n-2}{n-2}} + \mathcal{B}_{\tau,\psi}\,\varphi^{\frac{n+2}{n-2}}$$

where we set $c_n := \frac{n-2}{4(n-1)}$, and we let

$$\mathcal{R}_{\gamma,\psi} := c_n(R(\gamma) - |\nabla\psi|^2_\gamma), \qquad \mathcal{A}_{\gamma,\tilde{K},\pi} := c_n(|\tilde{K}|^2_\gamma + \pi^2)$$

and

$$\mathcal{B}_{\tau,\psi} := c_n\left(\frac{n-1}{n}\tau^2 - 4V(\psi)\right).$$

We observe that $\mathcal{A}_{\gamma,\tilde{K},\pi} \geq 0$, while the sign of $\mathcal{B}_{\tau,\psi}$ depends on the relative values of τ and $V(\psi)$. Note that in the constant mean curvature case the system of equations are "semi-decoupled" in that we may first solve the momentum constraint equation (5.3) (which does not involve φ) and then use the resulting $\tilde{K}$ to formulate the Lichnerowicz equation as described above. If we can find a positive solution φ to this equation then this determines the physical metric $\bar{\gamma}$ and second fundamental form $\bar{K}$ as well as the physical initial data $(\bar{\psi}, \bar{\pi})$ for the scalar field.

In [6, 7] we establish a number of results regarding the existence or nonexistence of solutions for the system consisting of (5.5) and (5.6). We present here some of the existence results, in a low regularity setting, on manifolds with τ constant (constant mean curvature, or "CMC", initial data). The assertion that these results hold in a low regularity setting follows from methods established by Choquet-Bruhat [3] and Maxwell [15, 16] for the vacuum Einstein constraint equations.

5.2. Existence theorems

We denote by W^p_s the usual Sobolev space on (Σ, γ), consisting (for s a positive integer) of functions with all weak derivatives of order less than or equal to s lying in L^p, and by M^p_s the space of W^p_s Riemannian metrics (which is an open cone in the space of all W^p_s 2-tensors if $s \geq 2$ and $p > n/2$, or if $p = 2$ and $s > n/2$). We denote W^2_s by H_s and M^2_s by M_s.

5.2.1. Solving the momentum constraint. Given a traceless tensor U on Σ, the conformally formulated momentum constraint equation (5.5) is a linear elliptic equation for the vector field X. The kernel of $\Delta_{\gamma,\mathrm{conf}}$ consists of the space of conformal Killing vector fields. The inhomogeneous term $\mathrm{div}_\gamma U$ is orthogonal to this space. The following theorem is a consequence of known theorems for linear elliptic systems. We suppose that s is an integer.[3] We let F denote the left hand side of (5.5) in the CMC setting, so that

$$F = -\,\mathrm{div}_\gamma U - \pi\nabla\psi.$$

Theorem 5.1. *The conformally formulated momentum constraint equation* (5.5) *with* $\gamma \in M_2^p$, $p > n/2$, *has a solution* $X \in W_2^q$, $1 < q \le p$, *if* $F \in L^q$ *and if* $J = \pi\nabla\psi$ *is orthogonal in the* L^2 *sense to the space of conformal Killing (CK) vector fields on* (Σ,γ).

Moreover the solution is uniquely determined up to the addition of a conformal Killing vector field. There exists a constant $c(\gamma)$, *depending only on* γ, *such that the unique solution which is orthogonal to the space of CK vectors satisfies*

$$\|X\|_{W_2^q} \le c(\gamma)\|F\|_{L^q}. \tag{5.7}$$

Corollary 5.2. *If* $\gamma \in M_s^2$, $s > n/2$ *and* $F \in H_{s-2}$, *then* $X \in H_s$.

Proof. We first remark that if $s > n/2$, then the Sobolev embedding theorem $W_2^p \subset H_s$ for $p \le 2n/(n-2s+4)$ implies that if $s > n/2$ there exists $p > n/2$ such that the embedding holds.

The corollary is then established in the usual way, by differentiating the equation and using the Sobolev multiplication and interpolation properties. □

5.2.2. Solving the Hamiltonian constraint. Satisfying the Hamiltonian constraint is equivalent to finding a positive solution of the Lichnerowicz equation (5.6). To prove the existence of positive solutions we use the method of sub- and supersolutions. The early approaches (see [10]) to solving the Lichnerowicz equation made use of the Leray–Schauder degree. The method employed here uses estimates for linear elliptic equations together with the Arzelà–Ascoli theorem; see for example [11]. The first solutions were found in Hölder spaces, then in Sobolev spaces H_s, $s > (n+1)/2+1$. The regularity has since been reduced to W_2^p, $p > n/2$ (see [3]) and to H_s, $s > n/2$ (see [16]), in the absence of the scalar field.

We give a general theorem, adapted to equations of the type of the Lichnerowicz equation, with or without a scalar field. Consider the semilinear equation

$$\Delta_\gamma\varphi = f(x,\varphi) \equiv \sum_{i=1}^{N} a_i(x)\varphi^{p_i}, \tag{5.8}$$

[3] The case of noninteger $s > n/2$ is treated by Maxwell [16] for the vacuum Einstein constraint equations. This of course requires working with distributional solutions if $n = 3$. For integral choices of $s > n/2$ we in particular have $s \ge 2$ when $n \ge 3$. Thus the formulas presented here involve pointwise almost everywhere defined derivatives.

on the compact manifold (Σ, γ), where $x \in \Sigma$, and $p_i \in \mathbb{R}$. We say that φ_- is a *subsolution* of (5.8) if $\Delta_\gamma \varphi_- \geq f(x, \varphi_-)$, and φ_+ is a *supersolution* of (5.8) if $\Delta_\gamma \varphi_+ \leq f(x, \varphi_+)$.

Theorem 5.3. *Equation* (5.8) *admits a positive solution* $\varphi \in W_2^p$, $p > n/2$, *provided the following conditions are satisfied:*

(a) $\gamma \in M_2^p$, $p > n/2$, *and* $a_i \in L^p$ *for* $i = 1, \ldots, N$.
(b) *The equation admits a strictly positive subsolution* φ_- *and a supersolution* φ_+, *both in* W_2^p, *with* $0 < \varphi_- \leq \varphi_+ < \infty$.

The solution φ *then satisfies* $\varphi_- \leq \varphi \leq \varphi_+$, *and is unique if* $f(x, y)$ *is increasing in* y *for each* $x \in \Sigma$. *On the other hand, if the* a_i *are all of the same sign then no positive solution exists.*

Corollary 5.4. *If in addition* $\gamma \in M_s$, $s > n/2$, $a_i \in H_{s-2}$, $i = 1, \ldots, N$, *then* $\varphi \in H_s$.

It can be proved by using conformal invariance that, in the case of the Lichnerowicz equation, the uniqueness of the solution is independent of the sign of $\mathcal{R}_{\gamma,\psi}$ (see Theorem 7.12 of [4]).

In the original analysis of the Lichnerowicz equation for vacuum CMC data on compact manifolds, the full Yamabe theorem[4] is employed to fix the sign of the linear zero order term [11]. A verification that one only needs to control the *sign* of the scalar curvature (a much easier result), even in the low regularity setting, is provided by results of [3] and [16]. The following result provides the analog of this control in the presence of a scalar field (see Proposition 1 of [7]).

Theorem 5.5 (The Yamabe-scalar field conformal invariant). *The functional on* $H_1(\Sigma)$ *(for given* $\gamma \in M_2^p$ *and* $\psi \in W_2^p$*) defined by*

$$Q_{\gamma,\psi}(u) = \frac{c_n^{-1} \int_\Sigma [|\nabla u|_\gamma^2 + \mathcal{R}_{\gamma,\psi} u^2] \, d\mathrm{vol}_\gamma}{(\int_\Sigma u^{2n/(n-2)} \, d\mathrm{vol}_\gamma)^{(n-2)/n}} \tag{5.9}$$

admits an infimum, $\mathcal{Y}_\psi([\gamma]) > -\infty$, *which is a conformal invariant. Its sign determines the Yamabe-scalar field classes of pairs* (γ, ψ). *A pair* (γ, ψ) *with* $\mathcal{Y}_\psi([\gamma]) < 0$ *(respectively* $\mathcal{Y}_\psi([\gamma]) = 0$, *or* $\mathcal{Y}_\psi([\gamma]) > 0$*) can be conformally transformed to a pair such that* $\mathcal{R}_{\gamma,\psi} < 0$ *(respectively* $\mathcal{R}_{\gamma,\psi} = 0$, *or* $\mathcal{R}_{\gamma,\psi} > 0$*) on* Σ, *and moreover if* $\mathcal{R}_{\gamma,\psi}$ *maintains a fixed sign on* Σ *it is necessarily of the same sign at* $\mathcal{Y}_\psi([\gamma])$.

The proof of the following existence theorem relies on the construction of sub- and supersolutions φ_- and φ_+. We have supposed that τ is a constant and that V is a smooth function of $\psi \in W_2^p \subset C^0(\Sigma)$, since $p > n/2$. We therefore also have $\mathcal{B}_{\tau,\psi} \in C^0(\Sigma) \subset L^\infty$.

---

[4]This says that every metric on a compact manifold is conformal to one with constant scalar curvature. The proof of this theorem was completed by Schoen [17] after essential contributions by Yamabe, Trudinger and Aubin. We refer the interested reader to [12, 5] and the references contained therein.

The expression $\mathcal{A}_{\gamma,\tilde{K},\pi} = c_n(|\tilde{K}|^2_\gamma + \pi^2)$ satisfies $\mathcal{A}_{\gamma,\tilde{K},\pi} \geq 0$, and, for a solution $\tilde{K}$ of the momentum constraint, $\mathcal{A}_{\gamma,\tilde{K},\pi} \in L^p$ since

$$W_1^p \times W_1^p \subset L^p \quad \text{when } p > n/2.$$

Here we present results in the case that $\mathcal{B}_{\tau,\psi} = c_n(\frac{n-1}{n}\tau^2 - 4V(\psi)) \geq 0$. We refer the interested reader to [7] for a more general treatment.

Theorem 5.6. *Suppose that $\gamma \in M_2^p$, $p > n/2$, $\psi \in W_2^p$, $\tilde{K}, \pi \in W_1^p$ and $\mathcal{B}_{\tau,\psi} \geq 0$. Then the Lichnerowicz equation* (5.6) *admits a positive solution $\varphi > 0$, $\varphi \in W_2^p$, in the following cases:*

(1) *(γ, ψ) is in the positive Yamabe-scalar field class and $\mathcal{A}_{\gamma,\tilde{K},\pi} \not\equiv 0$, or*
(2) *(γ, ψ) is in the zero Yamabe-scalar field class and* $\inf_\Sigma \mathcal{B}_{\tau,\psi} > 0$.

Proof. If (γ, ψ) is in the positive or zero Yamabe-scalar field class, a constant supersolution can be constructed directly as follows. First note that, as indicated in Theorem 5.5, we may assume that $\mathcal{R}_{\gamma,\psi} \geq 0$. This may require that we make a preliminary conformal transformation of our initial data, and find a new solution of the conformally formulated momentum constraint equation (see [7]). We consider the function of the single variable y defined by

$$F(y) = \underline{\mathcal{B}}_{\tau,\psi} y^{\frac{n}{n-2}} + \underline{\mathcal{R}}_{\gamma,\psi} y^{\frac{n-1}{n-2}} - \underline{\mathcal{A}}_{\gamma,\tilde{K},\pi}, \tag{5.10}$$

where $\underline{f}$ denotes the mean value of a function f on (Σ, γ):

$$\underline{f} \equiv \frac{1}{\mathrm{Vol}(\Sigma, \gamma)} \int_\Sigma f \, d\mathrm{vol}_\gamma.$$

Note that by setting $y(x) = \varphi(x)^4$ we see that $f(x, \phi) = y^{-\frac{3n-2}{4(n-2)}} F(y)$, provided that we do not replace the coefficients by their average values.

Under the stated hypothesis one easily sees that $F(y)$ is increasing on $\mathbb{R}_+$ and has exactly one positive root. We let $y_0 = \varphi_0^4$ denote this root, so that $F(y_0) = 0$. Now consider the linear equation

$$\Delta_\gamma v = \mathcal{R}_{\gamma,\psi}\varphi_0 - \mathcal{A}_{\gamma,\tilde{K},\pi}\varphi_0^{-\frac{3n-2}{n-2}} + \mathcal{B}_{\tau,\psi}\varphi_0^{\frac{n+2}{n-2}}. \tag{5.11}$$

By our choice of φ_0 the right hand side of this equation has mean value zero and is therefore orthogonal to the constants. Thus we may consider the function $v \in W_2^p$, with mean value zero on Σ, which solves (5.11). The function

$$\varphi_+ \equiv \varphi_0 + v - \inf_\Sigma v \geq \varphi_0, \qquad \Delta_\gamma \varphi_+ \equiv \Delta_\gamma v,$$

is a supersolution if $\mathcal{R}_{\gamma,\psi} \geq 0$, because

$$\Delta_\gamma\varphi_+ - f(\cdot, \varphi_+) = \mathcal{R}_{\gamma,\psi}(\varphi_0 - \varphi_+) - \mathcal{A}_{\gamma,\tilde{K},\pi}(\varphi_0^{-\frac{3n-2}{n-2}} - \varphi_+^{-\frac{3n-2}{n-2}}) + \mathcal{B}_{\tau,\psi}(\varphi_0^{\frac{n+2}{n-2}} - \varphi_+^{\frac{n+2}{n-2}}).$$

Hence if $\mathcal{R}_{\gamma,\psi} \geq 0$, then since $\mathcal{A}_{\gamma,\tilde{K},\pi} \geq 0$ and $\mathcal{B}_{\tau,\psi} \geq 0$, we have

$$\Delta_\gamma\varphi_+ - f(\cdot, \varphi_+) \leq 0$$

because $\varphi_+ \geq \varphi_0$. In the case of the zero Yamabe-scalar field class the same type of argument holds, but we must use, in addition, the hypothesis $\mathcal{B}_{\tau,\psi} \not\equiv 0$ to ensure that $\varphi_0 > 0$.

In order to find a positive subsolution first note that any number $\ell < 1$ such that

$$\ell < \frac{\inf_\Sigma \mathcal{A}_{\gamma,\tilde{K},\pi}}{\sup_\Sigma(\mathcal{R}_{\gamma,\psi} + \mathcal{B}_{\tau,\psi})}$$

is a constant subsolution. It is positive only if $\inf_\Sigma \mathcal{A}_{\gamma,\tilde{K},\pi} > 0$.

One may relax this hypothesis on $\mathcal{A}_{\gamma,\tilde{K},\pi}$ by instead constructing a non-constant subsolution using the conformal invariance of the Lichnerowicz equation [10, 11, 16, 7]. We ignore for the purposes of this theorem the connection between the coefficient $\mathcal{A}_{\gamma,\tilde{K},\pi}$ and the tensor $\tilde{K}$ arising from the solution to the conformally formulated momentum constraint equation (5.5). In order to state the conformal invariance properly one must, in addition to conformally rescaling the coefficients, solve (5.5) with the conformally transformed data before posing the conformally transformed Lichnerowicz equation. We refer the reader to Proposition 2 of [7] for details. Since we are only concerned here with the Lichnerowicz equation we may state the conformal invariance as follows:

$$\begin{aligned}\Delta_\gamma\varphi - \mathcal{R}_{\gamma,\psi}\varphi + \mathcal{A}_{\gamma,\tilde{K},\pi}\varphi^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\varphi^{\frac{n+2}{n-2}} \\ = \theta^{\frac{n+2}{n-2}}(\Delta_{\gamma'}\varphi' - \mathcal{R}'_{\gamma,\psi}\varphi' + \mathcal{A}'_{\gamma,\tilde{K},\pi}\varphi'^{-\frac{3n-2}{n-2}} - \mathcal{B}'_{\tau,\psi}\varphi'^{\frac{n+2}{n-2}}),\end{aligned}$$

with

$$\gamma' = \theta^{\frac{4}{n-2}}\gamma, \quad \varphi' = \theta^{-1}\varphi, \quad \mathcal{A}'_{\gamma,\tilde{K},\pi} = \mathcal{A}_{\gamma,\tilde{K},\pi}\theta^{-\frac{4}{n-2}}, \quad \mathcal{B}'_{\tau,\psi} = \mathcal{B}_{\tau,\psi}.$$

Now suppose $\mathcal{A}_{\gamma,\tilde{K},\pi} \geq 0$, $\mathcal{A}_{\gamma,\tilde{K},\pi} \not\equiv 0$. We set

$$k = \mathcal{R}_{\gamma,\psi} + \lambda\mathcal{B}_{\tau,\psi}$$

with $\lambda = 0$ in the positive Yamabe-scalar field case, while we take $\lambda > (\inf_\Sigma \mathcal{B}_{\tau,\psi})^{-1}$ in the case $\mathcal{R}_{\gamma,\psi} = 0$. Then there exists a $\theta > 0$, $\theta \in W_2^p$, such that

$$\Delta_\gamma\theta - k\theta = -\mathcal{A}_{\gamma,\tilde{K},\pi}.$$

Then $\Delta_\gamma\theta - \mathcal{R}_{\gamma,\psi}\theta = -\theta^{\frac{n+2}{n-2}}\mathcal{R}'_{\gamma,\psi}$ implies

$$\mathcal{R}'_{\gamma,\psi} = \theta^{-\frac{n+2}{n-2}}(\mathcal{A}_{\gamma,\tilde{K},\pi} + (\mathcal{R}_{\gamma,\psi} - k)\theta).$$

The "primed Lichnerowicz equation" then admits the positive constant subsolution ℓ if

$$-\mathcal{R}'_{\gamma,\psi}\ell + \mathcal{A}'_{\gamma,\tilde{K},\pi}\ell^{-\frac{3n-2}{n-2}} - \mathcal{B}'_{\tau,\psi}\ell^{\frac{n+2}{n-2}} \geq 0,$$

or, equivalently,

$$\theta^{-\frac{n+2}{n-2}}\{-(\mathcal{A}_{\gamma,\tilde{K},\pi} + \lambda\mathcal{B}_{\tau,\psi})\theta\}\ell + \mathcal{A}_{\gamma,\tilde{K},\pi}\theta^{-\frac{4}{n-2}}\ell^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\ell^{\frac{n+2}{n-2}} \geq 0.$$

Any number ℓ such that

$$\ell \leq \min(\inf_\Sigma \lambda^{\frac{n-2}{4}}\theta^{-1}, \ \inf_\Sigma \theta^{\frac{n-2}{4(n-1)}})$$

is a positive subsolution of this transformed equation, and this shows that $\theta^{-1}\ell$ is a positive subsolution of the original equation. □

An analogous method allows for the construction of sub- and supersolutions in the negative Yamabe-scalar field class. We refer the interested reader to [3] and to [16] for the vacuum case and [6] for the Einstein-scalar field system.

Remark 5.7. The existence and nonexistence results presented here cover all cases when $\mathcal{B}_{\tau,\psi} \geq 0$ is a constant,[5] since then either $\inf_\Sigma \mathcal{B}_{\tau,\psi} > 0$ or $\mathcal{B}_{\tau,\psi} \equiv 0$, and one is therefore studying the equation

$$\Delta\varphi \geq 0 \quad [\text{or } \leq 0] \quad \text{and } \not\equiv 0,$$

which has no solution on a compact manifold.

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Yvonne Choquet-Bruhat
Académie des Sciences
23 Quai de Conti
75270 Paris Cedex 06, France
e-mail: `YCB@CCR.jussieu.fr`

James Isenberg
Department of Mathematics
University of Oregon
Eugene, OR 97403, USA
e-mail: `jim@newton.uoregon.edu`

Daniel Pollack
Department of Mathematics
University of Washington
Seattle, WA 98195-4350, USA
e-mail: `pollack@math.washington.edu`