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Analytical solutions of transient heat conduction in multilayered slabs and application to thermal analysis of landfills

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Abstract: The study of transient heat conduction in multilayered slabs is widely used in various engineering fields. In this paper, the transient heat conduction in multilayered slabs with general boundary conditions and arbitrary heat generations is analysed. The boundary conditions are general and include various combinations of Dirichlet, Neumann or Robin boundary conditions at either surface. Moreover, arbitrary heat generations in the slabs are taken into account. The solutions are derived by basic methods, including the superposition method, separation variable method and orthogonal expansion method. The simplified double-layered analytical solution is validated by a numerical method and applied to predicting the temporal and spatial distribution of the temperature inside a landfill. It indicates the ability of the proposed analytical solutions for solving the wide range of applied transient heat conduction problems.

Key words: heat conduction; multilayered slab; heat generation; analytical solutions; landfill

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1 Introduction

In various engineering fields, the study of one-dimensional transient heat conduction with time-dependent boundary conditions and arbitrary heat generation is widely used. In the thermal analysis of building external walls, the exterior walls of a building are made up of composite materials. It is common for researchers to analyse the heat preservation performance of a building's external wall by a multilayered heat conduction equation [1–3]. The second application is the analysis of the transient response of multilayered materials with moving heat sources, such as machining [4], welding [5] and laser heating [6]. The consolidation of layered soils and multilayered diffusion model all have the similar form as the heat conduction equation. Therefore, the multilayered heat conduction model can be extended to calculate the consolidation of layered soil [7, 8] or multilayered diffusion [9–11]. The multilayered heat conduction model can also be used to predict the temperature distribution in a landfill. Heat generation occurs in municipal solid waste (MSW) landfills due to the biodegradation of the organic content of the waste. The waste layer and soil layer can be regarded as a multilayered heat transfer structure. It is important to predict the temporal and spatial distribution of temperature inside a landfill, which is helpful for the operation and management of the landfill [12, 13].

There are several different approaches that can be used to analyse the transient heat conduction in a

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multilayered medium such as: orthogonal and quasi-orthogonal expansion techniques [14–16], Green's function approach [17, 18], Laplace transform method [19–21], finite integral transform technique [22], distributed transfer function method [23], finite element method [24], and finite difference method [25]. These techniques can be divided into analytical methods [14–23] and numerical methods [24, 25]. Among the above listed approaches, the analytical solutions have the advantage of accuracy and efficiency. The analytical solutions can also provide deep physical insight. Furthermore, the analytical solutions can be used to analyse the inverse problem. Therefore, it is important to identify the analytical solutions [26].

Continued effort has been made recently to advance the analytical solutions of one-dimensional transient heat conduction in multilayered slabs. MONTE [27, 28] solved the double-layered and multilayered heat conduction problems using the orthogonal expansion method. The boundary conditions used were homogeneous Robin boundary conditions. SUN et al [29] solved the three-layered and multilayered heat conduction model with constant Dirichlet boundary conditions using the separation variables method. LU et al [30] used the Laplace transform method to solve the heat conduction model of a multilayered composite slab. The boundary conditions used were time-dependent Robin boundary conditions. ZHOU et al [31] solve the heat conduction problem in one-dimensional double-layered composite medium with homogeneous Robin boundary conditions by the natural eigenfunction expansion method. There were no heat generations in the governing equations of the above researches. BELGHAZI et al [32] presented an analytical approach of transient heat conduction in double-layered material with different heat generations in layers by the separation of variables method. Only the homogeneous Robin boundary conditions were taken into account in BELGHAZI et al's study. TIAN et al [33] obtained the solutions of transient heat conduction in multilayered slabs with homogeneous Neumann boundary conditions by the Green's function method. However, the heat generation in each layer was the same. FAKOOR-PAKDAMAN et al [34, 35] presented analytical solutions of heat diffusion inside a multilayered composite medium with arbitrary heat generations by separation of variables method. However, only space-dependent heat generation inside each layer was taken into consideration in his studies.

Although there are significant researches on the analytical solutions of the multilayered heat conduction models, few papers predict the thermal behaviour of a multilayered slab with arbitrary heat generation and general time-dependent boundary conditions. The purpose of this paper is to solve the multilayered heat conduction equation with general boundary conditions and arbitrary heat generation. The general boundary conditions include various combinations of Dirichlet, Neumann or Robin boundary condition at either surface. The solutions are solved by the superposition method, the orthogonal expansion method and the separation variable method. The double-layered analytical solution is validated by a numerical method and applied to predicting the temporal and spatial distribution of the temperature inside a landfill.

2 Mathematical formula

A composite slab consisting of a finite multilayer is shown in Figure 1. z_{i-1} and z_i are the upper depth and lower depth of the ith layer, where i=1, 2, ⋯, n. z_0 and z_n are the upper and lower boundaries of the entire multilayered slab.

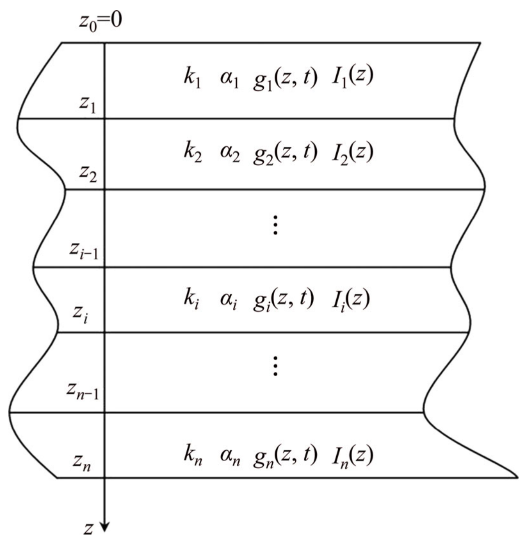

Figure 1 Schematic diagram of a multilayered slab

The assumptions made in deriving the mathematical formulation of this time-dependent heat conduction problem are [27–31, 34]:

1) The thermal conductivity and the thermal diffusivity are temperature independent and

uniform inside each layer;

2) The multilayered slab is large enough in the x and y directions in comparison to its thickness in the z direction;

3) The thermal contact resistance between the interfaces is negligible.

The heat conduction problem can be considered one-dimensional, due to the assumption 2). The governing equation of heat conduction in the ith layer is:

$$\frac{\partial T_i(z,t)}{\partial t}=\alpha_i\frac{\partial^2 T_i(z,t)}{\partial z^2}+\frac{\alpha_i q_i(z,t)}{k_i},\quad z_{i-1}\le z\le z_i,\ i=1,\ 2,\ \cdots,\ n \tag{1}$$

where $T_i(z, t)$ is the temperature of the ith layer, and k_i and α_i are the thermal conductivity and thermal diffusivity of the ith layer, respectively. $q_i(z, t)$ is heat generation in ith layer, which is a function of the position z and time t.

The initial condition of ith layer is:

$$T_i(z,0)=I_i(z),\ i=1,\ 2,\ \cdots,\ n \tag{2}$$

where $I_i(z)$ is a given initial temperature distribution through the ith layer.

The boundary conditions on the upper and lower surfaces of the multilayered slab are

$$z=z_0:\ T_1(z,t)=f_{D,1}(t)\ \text{or}\ k_1\frac{\partial T_1(z,t)}{\partial z}=f_{N,1}(t)\ \text{or}\ -k_1\frac{\partial T_1(z,t)}{\partial z}+h_1T_1(z,t)=f_{R,1}(t) \tag{3}$$

$$z=z_n:\ T_N(z,t)=f_{D,n}(t)\ \text{or}\ k_n\frac{\partial T_n(z,t)}{\partial z}=f_{N,n}(t)\ \text{or}\ k_n\frac{\partial T_n(z,t)}{\partial z}+h_nT_n(z,t)=f_{R,n}(t) \tag{4}$$

where $f_{D,1}(t)$, $f_{D,n}(t)$, $f_{N,1}(t)$, $f_{N,n}(t)$, $f_{R,1}(t)$, and $f_{R,n}(t)$ are the external conditions (prescribed temperature and/or heat flux) on the upper and lower surfaces of the multilayered slab. The subscripts D, N, and R represent the Dirichlet, Neumann and Robin boundary conditions, respectively. h_1 and h_n are the heat transfer coefficient of the upper and lower surfaces of the multilayered slab, respectively.

The inner boundary conditions (continuity conditions) are:

$$T_i(z_i,t)=T_{i+1}(z_i,t),\ i=1,\ 2,\ \cdots,\ n-1 \tag{5}$$

$$k_i\left.\frac{\partial T_i(z,t)}{\partial z}\right|_{z=z_i}=k_{i+1}\left.\frac{\partial T_{i+1}(z,t)}{\partial z}\right|_{z=z_i},\quad i=1,\ 2,\ \cdots,\ n-1 \tag{6}$$

3 Analytical solutions

The solution for combinations of Robin boundary conditions on the upper and lower surfaces of the multilayered slab is derived in this section. The solutions for other combinations of boundary conditions are also shown, except for the combinations of Neumann boundary conditions. The solution for the combinations of Neumann boundary conditions on the upper and lower surfaces of the multilayered slab is derived in Appendix B.

3.1 Homogenization of boundary condition

The nonhomogeneous boundary conditions can be homogenised by the superposition principle. The solution of $T_i(z, t)$ can be separated as follows:

$$T_i(z,t)=U_i(z,t)+W_i(z,t) \tag{7}$$

where $W_i(z, t)$ is the solution of the steady-state problem for the same region as $T_i(z, t)$, with no heat generation and nonhomogeneous boundary conditions at $z=z_0$ and $z=z_n$. $U_i(z, t)$ is the solution of the time-dependent heat conduction problem for the same region as $T_i(z, t)$, with heat generation, but subjected to homogeneous boundary conditions.

3.2. Solution of $W_i(z, t)$

$W_i(z, t)$ satisfies the Laplace's equation as shown in Eq. (8):

$$\frac{\partial^2 W_i(z,t)}{\partial z^2}=0,\ \ z_{i-1}\le z\le z_i,\ \ i=1,\ 2,\ \cdots,\ n \tag{8}$$

The upper, lower and inner boundary conditions of $W_i(z, t)$ are set as:

$$z=z_0:\ -k_1\frac{\partial W_1(z,t)}{\partial z}+h_1W_1(z,t)=f_{R,1}(t) \tag{9}$$

$$z=z_n:\ k_n\frac{\partial W_n(z,t)}{\partial z}+h_nW_n(z,t)=f_{R,n}(t) \tag{10}$$

$$W_i(z_i,t)=W_{i+1}(z_i,t),\ \ i=1,\ 2,\ \cdots,\ n-1 \tag{11}$$

$$k_i\left.\frac{\partial W_i(z,t)}{\partial z}\right|_{z=z_i}=k_{i+1}\left.\frac{\partial W_{i+1}(z,t)}{\partial z}\right|_{z=z_i},\quad i=1,\ 2,\ \cdots,\ n-1 \tag{12}$$

The solution for the function $W_i(z, t)$ is obtained from Eqs. (8)–(12) as follows:

$$W_i(z,t)=A_i(t)z+B_i(t) \tag{13}$$

with

$$A_1(t)=\left[f_{R,n}(t)-\frac{h_n}{h_1}f_{R,1}(t)\right]\bigg/\left[k_1+h_n\frac{k_1}{k_n}z_n+h_n\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p-(z_0h_1-k_1)\frac{h_n}{h_1}\right] \tag{14}$$

$$B_1(t)=\frac{f_{R,1}(t)}{h_1}-\frac{h_1z_0-k_1}{h_1}A_1(t) \text{ or } \frac{f_{R,n}(t)}{h_n}-\left(\frac{k_1}{h_n}+\frac{k_1z_n}{k_n}\right)A_1(t)-\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)h_pA_1(t) \tag{15}$$

$$A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t) \tag{16}$$

$$B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_qA_1(t) \tag{17}$$

Similarly, the formulas for homogenizing the other combinations of boundary conditions are shown in Table 1.

3.3 Solution of $U_i(z, t)$

The function $U_i(z, t)$ is the solution of the following time-dependent heat conduction problem with heat generation, but subjected to homogeneous boundary conditions as follows:

$$\frac{\partial U_i(z,t)}{\partial t}=\alpha_i\frac{\partial^2U_i(z,t)}{\partial z^2}+\frac{\alpha_iq_i^*(z,t)}{k_i},\quad z_{i-1}\le z\le z_i,\ i=1,\ 2,\ \cdots,\ n \tag{18}$$

with

$$q_i^*(z,t)=q_i(z,t)-\left[\frac{\mathrm{d}A_i(t)}{\mathrm{d}t}z+\frac{\mathrm{d}B_i(t)}{\mathrm{d}t}\right]\frac{k_i}{\alpha_i} \tag{19}$$

$U_i(z, t)$ is subjected to the upper and lower boundary conditions:

$$z=z_0:\ -k_1\frac{\partial U_1(z,t)}{\partial z}+h_1U_1(z,t)=0 \tag{20}$$

$$z=z_n:\ k_n\frac{\partial U_n(z,t)}{\partial z}+h_nU_n(z,t)=0 \tag{21}$$

and to the inner boundary conditions:

$$U_i(z_i,t)=U_{i+1}(z_i,t),\ i=1,\ 2,\ \cdots,\ n-1 \tag{22}$$

$$k_i\frac{\partial U_i(z,t)}{\partial z}\bigg|_{z=z_i}=k_{i+1}\frac{\partial U_{i+1}(z,t)}{\partial z}\bigg|_{z=z_i},\quad i=1,\ 2,\ \cdots,\ n-1 \tag{23}$$

The initial condition of $U_i(z,t)$ is expressed as:

$$U_i(z,0)=I_i^*(z)=I_i(z)-W_i(z,0),\ i=1,\ 2,\ \cdots,\ n \tag{24}$$

The orthogonal expansion technique is used to solve the homogeneous problem of $U_i(z, t)$. Let $H_i(z,t)$ be the solution of the following time-dependent heat conduction problem with no heat generation, which is used to obtain the characteristic function.

$$\frac{\partial H_i(z,t)}{\partial t}=\alpha_i\frac{\partial^2H_i(z,t)}{\partial z^2},\ z_{i-1}\le z\le z_i,\ i=1,\ 2,\ \cdots,\ n \tag{25}$$

The upper, lower and inner boundary conditions of $H_i(z, t)$ are as follows:

$$z=z_0:\ -k_1\frac{\partial H_1(z,t)}{\partial z}+h_1H_1(z,t)=0 \tag{26}$$

$$z=z_n:\ k_n\frac{\partial H_n(z,t)}{\partial z}+h_nH_n(z,t)=0 \tag{27}$$

$$H_i(z_i,t)=H_{i+1}(z_i,t),\ i=1,\ 2,\ \cdots,\ n-1 \tag{28}$$

$$k_i\frac{\partial H_i(z,t)}{\partial z}\bigg|_{z=z_i}=k_{i+1}\frac{\partial H_{i+1}(z,t)}{\partial z}\bigg|_{z=z_i},\quad i=1,\ 2,\ \cdots,\ n-1 \tag{29}$$

$H_i(z, t)$ can be separated into two parts as follows:

$$H_i(z,t)=Z_i(z)\gamma_i(t),\ i=1,\ 2,\ \cdots,\ n \tag{30}$$

Substituting Eq. (30) into Eq. (25), we have:

$$\frac{1}{\alpha_i\gamma_i(t)}\frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t}=\frac{1}{Z_i(z)}\frac{\mathrm{d}^2Z_i(z)}{\mathrm{d}z^2}=-\lambda_i^2,\ i=1,\ 2,\ \cdots,\ n \tag{31}$$

where λ_i is the separation constant. The separation given by Eq. (31) results in the following two ordinary differential equations:

$$\frac{\mathrm{d}^2Z_i(z)}{\mathrm{d}z^2}+\lambda_i^2Z_i(z)=0;\ \frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t}+\alpha_i\lambda_i^2\gamma_i(t)=0,\quad i=1,\ 2,\ \cdots,\ n \tag{32}$$

Based on Eq. (32), we can obtain:

$$\gamma_i(t)=\mathrm{e}^{-\alpha_i\lambda_i^2t},\ i=1,\ 2,\ \cdots,\ n \tag{33}$$

$$Z_i(z)=C_i\sin(\lambda_iz)+D_i\cos(\lambda_iz),\ i=1,\ 2,\ \cdots,\ n \tag{34}$$

Substituting Eq. (34) into Eq. (26), the following result is obtained as:

$$D_1=\frac{k_1\lambda_1}{h_1}C_1 \tag{35}$$

The C_1 can be taken as 1 when $H_i(z, t)$ is given by:

$$H_i(z,t)=\sum_{j=1}^{\infty}G_j\mathrm{e}^{-\alpha_i\lambda_{i,j}^2t}Z_{i,j}(z),\ i=1,\ 2,\ \cdots,\ n \tag{36}$$

Table 1 Formula for homogenizing of boundary conditions

Boundary condition	Dirichlet boundary condition	Neumann boundary condition	Robin boundary condition
Dirichlet boundary condition	$A_1(t)=\left[f_{D,n}(t)-f_{D,1}(t)\right]\Big/\left[\frac{k_1 z_n}{k_n}+\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p-z_0\right]$ $B_1(t)=f_{D,1}(t)-A_1(t)z_0$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$	$A_1(t)=\frac{f_{N,1}(t)}{k_1}$ $B_1(t)=f_{D,n}(t)-\frac{k_1 z_n}{k_n}A_1(t)-\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p A_1(t)$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$	$A_1(t)=\left[h_1 f_{D,n}(t)-f_{R,1}(t)\right]\Big/\left[k_1+\frac{k_1 h_1 z_n}{k_n}+h_1\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p-h_1 z_0\right]$ $B_1(t)=\frac{f_{R,1}(t)}{h_1}-\frac{h_1 z_0-k_1}{h_1}A_1(t)$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$
Neumann boundary condition	$A_1(t)=\frac{f_{N,n}(t)}{k_1}$ $B_1(t)=f_{D,1}(t)-A_1(t)z_0$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$	see Appendix B	$A_1(t)=\frac{f_{N,n}(t)}{k_1}$ $B_1(t)=\frac{f_{R,1}(t)}{h_1}-\frac{h_1 z_0-k_1}{h_1}A_1(t)$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$
Robin boundary condition	$A_1(t)=[f_{R,n}(t)-h_n f_{D,1}(t)]/\left[k_1+h_n\frac{k_1}{k_n}z_n+h_n\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p-z_0 h_n\right]$ $B_1(t)=f_{D,1}(t)-A_1(t)z_0$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$	$A_1(t)=\frac{f_{N,1}(t)}{k_1}$ $B_1(t)=\frac{f_{R,n}(t)}{h_n}-\left(\frac{k_1}{h_n}+\frac{k_1 z_n}{k_n}\right)A_1(t)-\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)h_p A_1(t)$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$	$A_1(t)=[f_{R,n}(t)-\frac{h_n}{h_1}f_{R,1}(t)]/\left[k_1+h_n\frac{k_1}{k_n}z_n+h_n\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)z_p-(z_0 h_1-k_1)\frac{h_n}{h_1}\right]$ $B_1(t)=\frac{f_{R,1}(t)}{h_1}-\frac{h_1 z_0-k_1}{h_1}A_1(t)$ or $\frac{f_{R,n}(t)}{h_n}-\left(\frac{k_1}{h_n}+\frac{k_1 z_n}{k_n}\right)A_1(t)-\sum_{p=1}^{n-1}\left(\frac{k_1}{k_p}-\frac{k_1}{k_{p+1}}\right)h_p A_1(t)$ $A_{i+1}(t)=\frac{k_1}{k_{i+1}}A_1(t)$ $B_{i+1}(t)=B_1(t)+\sum_{q=1}^{i}\left(\frac{k_1}{k_q}-\frac{k_1}{k_{q+1}}\right)z_q A_1(t)$

Substituting Eq. (34) into Eq. (28) and Eq. (29) yields:

$$C_{i,j}\sin\left(\lambda_{i,j}z_{i,j}\right)+D_{i,j}\cos\left(\lambda_{i,j}z_{i,j}\right)=C_{i+1,j}\sin\left(\lambda_{i+1,j}z_{i,j}\right)+D_{i+1,j}\cos\left(\lambda_{i+1,j}z_{i,j}\right),\quad i=1,\ 2,\ \cdots,\ n-1 \tag{37}$$

$$k_i\lambda_{i,j}C_{i,j}\cos\left(\lambda_{i,j}z_i\right)-k_i\lambda_{i,j}D_{i,j}\sin\left(\lambda_{i,j}z_i\right)=k_{i+1}\lambda_{i+1,j}C_{i+1j}\cos\left(\lambda_{i+1,j}z_i\right)-k_{i+1}\lambda_{i+1,j}D_{i+1,j}\sin\left(\lambda_{i+1,j}z_i\right),\quad i=1,\ 2,\ \cdots,\ n-1 \tag{38}$$

$$\lambda_{i,j}=\sqrt{\alpha_1/\alpha_i}\,\lambda_{1,j},\quad i=1,\ 2,\ \cdots,\ n \tag{39}$$

Substituting Eq. (34) to Eq. (27) results in:

$$C_{n,j}\left[k_n\lambda_{n,j}z_n\cos\left(\lambda_{n,j}z_n\right)+h_n\sin\left(\lambda_{n,j}z_n\right)\right]+D_{n,j}\left[-k_n\lambda_{n,j}z_n\sin\left(\lambda_{n,j}z_n\right)+h_n\cos\left(\lambda_{n,j}z_n\right)\right]=0 \tag{40}$$

From Eq. (37) and Eq. (38), the recursive relationship between C_i, D_i and C_{i+1}, D_{i+1} (i=1, 2, ⋯, n–1) are obtained as follows:

$$\begin{bmatrix} C_{i+1,j} \\ D_{i+1,j} \end{bmatrix} = \begin{bmatrix} E_{i,j}G_{i,j} + \dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}} F_{i,j}H_{i,j} & F_{i,j}G_{i,j} - \dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}} E_{i,j}H_{i,j} \\ E_{i,j}H_{i,j} - \dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}} F_{i,j}G_{i,j} & F_{i,j}H_{i,j} + \dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}} E_{i,j}G_{i.j} \end{bmatrix} \cdot \begin{bmatrix} C_{i,j} \\ D_{i,j} \end{bmatrix} \tag{41}$$

with

$$E_{i,j} = \sin\left(\lambda_{i,j} z_i\right);$$
$$F_{i,j} = \cos\left(\lambda_{i,j} z_i\right);$$
$$G_{i,j} = \sin\left(\lambda_{i+1,j} z_i\right);$$
$$H_{i,j} = \cos\left(\lambda_{i+1,j} z_i\right) \tag{42}$$

Based on Eq. (41), the relationship between $C_{1,j}$, $D_{2,j}$ and $C_{n,j}$, $D_{n,j}$ can be derived as follows:

$$\begin{bmatrix} C_{n,j} \\ D_{n,j} \end{bmatrix} = S_{n-1,j} \begin{bmatrix} C_{1,j} \\ D_{1,j} \end{bmatrix} \tag{43}$$

with

$$S_{n-1,j} = \prod_{l=1}^{n-1} \begin{bmatrix} E_{l,j}G_{l,j} + \dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}} F_{l,j}H_{l,j} & F_{l,j}G_{l,j} - \dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}} E_{l,j}H_{l,j} \\ E_{l,j}H_{l,j} - \dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}} F_{l,j}G_{l,j} & F_{l,j}H_{l,j} + \dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}} E_{l,j}G_{l,j} \end{bmatrix} \tag{44}$$

Substituting Eq. (39) and Eq. (43) into Eq. (40), a transcendental equation is obtained. The eigenvalues are the solutions of the transcendental equation.

For other combinations of boundary conditions, $C_{1,j}$, $D_{1,j}$ and the relational expression of $C_{n,j}$ and $D_{n,j}$ are shown in Table 2.

Based on the orthogonal expansion method and the characteristic function obtained in the above, the $U_i(z, t)$ is expressed as:

$$U_i(z,t) = \sum_{j=1}^{\infty} \chi_j(t) Z_{i,j}(z) \tag{45}$$

As shown in the Appendix A, the characteristic function Eq. (34) satisfies the following orthogonal relationship:

$$\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} Z_{i,j}(z) Z_{i,m}(z) \frac{k_i}{\alpha_i} \mathrm{d}z = \begin{cases} 0, & j \neq m \\ \displaystyle\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} Z_{i,j}^2(z) \frac{k_i}{\alpha_i} \mathrm{d}z, & j = m \end{cases} \tag{46}$$

The orthogonal expansion of $q_i^*(z, t)$ is expressed as follows:

$$q_i^*(z,t) = \sum_{j=1}^{\infty} \phi_j(t) Z_{i,j}(z) \frac{k_i}{\alpha_i} \tag{47}$$

Due to the orthogonal relationship of Eq. (46), the $\phi_j(t)$ is:

$$\phi_j(t) = \frac{\displaystyle\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} Z_{i,j}(z) q_i^*(z,t) \mathrm{d}z}{\displaystyle\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} Z_{i,j}^2(z) \frac{k_i}{\alpha_i} \mathrm{d}z} \tag{48}$$

Substituting Eq. (45) and Eq. (47) into Eq. (18), with the orthogonal relationship Eq. (46), we get the following ordinary differential equation:

$$\frac{\mathrm{d}\chi_j(t)}{\mathrm{d}t} + \alpha_i \lambda_{i,j}^2 \chi_j(t) = \phi_j(t) \tag{49}$$

The solution of the ordinary differential equation Eq. (49) is obtained as:

$$\chi_j(t) = \int_0^t \mathrm{e}^{-\alpha_i \lambda_{i,j}^2 (t-\tau)} \phi_j(\tau) \mathrm{d}\tau + \mu_j \mathrm{e}^{-\alpha_i \lambda_{i,j}^2 t} \tag{50}$$

Substituting Eq. (45) and Eq. (50) into the initial condition Eq. (24), μ_j is obtained as:

$$\mu_j = \frac{\displaystyle\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} I_i^*(z) Z_{i,j}(z) \frac{k_i}{\alpha_i} \mathrm{d}z}{\displaystyle\sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} Z_{i,j}^2(z) \frac{k_i}{\alpha_i} \mathrm{d}z} \tag{51}$$

Table 2 C_1, D_1 and relational expression of C_n and D_n under different boundary conditions

Boundary condition	Dirichlet boundary condition	Neumann boundary condition	Robin boundary condition
Upper boundary	$C_{1,j}$=1, $D_{1,j}$=0	$C_{1,j}$=0, $D_{1,j}$=1	$C_{1,j}$=1, $D_{1,j} = \dfrac{k_1\lambda_{1,j}}{h_1}$
Lower boundary	$C_{n,j}\sin\left(\lambda_{n,j}z_n\right) + D_{n,j}\cos\left(\lambda_{n,j}z_n\right) = 0$	$C_{n,j}\cos\left(\lambda_{n,j}z_n\right) - D_{n,j}\sin\left(\lambda_{n,j}z_n\right) = 0$	$C_{n,j}\left[k_n\lambda_{n,j}z_{n,j}\cos\left(\lambda_{n,j}z_n\right) + h_n\sin\left(\lambda_{n,j}z_n\right)\right] + D_{n,j}\left[-k_n\lambda_{n,j}z_{n,j}\sin\left(\lambda_{n,j}z_n\right) + h_n\cos\left(\lambda_{n,j}z_n\right)\right] = 0$

Finally, the complete solution of $T_i(z,t)$ can be expressed as:

$$T_i(z,t)=\sum_{j=1}^{\infty}\left[C_{i,j}\sin\left(\lambda_{i,j}z\right)+D_{i,j}\cos\left(\lambda_{i,j}z\right)\right]\chi_j(t)+A_i(t)z+B_i(t),\quad i=1,\ 2,\ \ldots,\ n \tag{52}$$

4 Example analysis and numerical verification

The temperature rises in landfills because of the heat generated during biodegradation of the organic compounds. Elevated temperatures affect the engineering properties of liners, covers, and foundation soil. In this paper, the analytical solution of transient heat conduction in multilayered slab is used to predict the spatial and temporal distribution of temperature in a landfill. The schematic diagram of a landfill is shown in Figure 2.

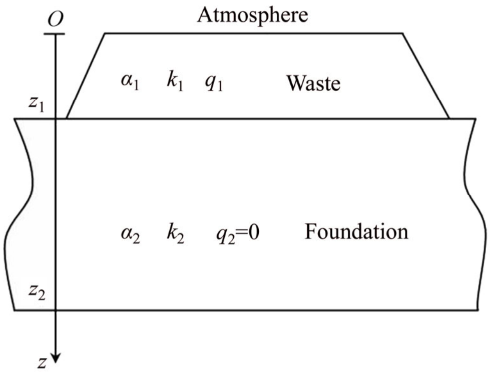

Figure 2 Schematic diagram of model landfill

In the surface layer of a landfill, the temperature is similar to that of the atmosphere due to the heat transfer effects between the surface layer and the atmosphere. The upper boundary condition is the temperature outside of the landfill which is expressed as a sine function.

$$T_1(0,t)=T_{\mathrm{m}}+A_{\mathrm{s}}\sin\left(\frac{2\pi t}{365}\right) \tag{53}$$

The temperature in the lower boundary is relatively constant, as expressed by Eq. (54).

$$T_2(z_2,t)=T_{\mathrm{m}} \tag{54}$$

The inner boundary conditions (continuity conditions) are:

$$T_1(z_1,t)=T_2(z_1,t) \tag{55}$$

$$k_1\left.\frac{\partial T_1(z,t)}{\partial z}\right|_{z=z_1}=k_2\left.\frac{\partial T_2(z,t)}{\partial z}\right|_{z=z_1} \tag{56}$$

Heat is generated in the waste layer of a landfill. As a landfill takes many years to fill to its capacity, waste at the bottom of the landfill is expected to have a different heat production rate from the waste close to the top surface. To account for the different heat production rates at different depths in a landfill with a linearly depositing rate, the heat production rate of the waste layer is defined as a modified single peak function [13]:

$$q_1(z,t)=\frac{A}{B}\left(t+\frac{z}{z_1}t_{\mathrm{f}}\right)\mathrm{e}^{-\frac{1}{B}\left(t+\frac{z}{z_1}t_{\mathrm{f}}\right)} \tag{57}$$

where A and B are the shape factors; t_{f} is the total time to fill the landfill to capacity; z is the depth of waste measured from the surface.

The lower layer is foundation soil, and there is no heat generation as follows:

$$q_2(z,t)=0 \tag{58}$$

The initial condition is the mean annual temperature outside the landfill as follows:

$$T_1(z,0)=T_2(z,0)=T_{\mathrm{m}} \tag{59}$$

The solution of the temperature distribution of the landfill is derived as:

$$T_i(z,t)=\sum_{j=1}^{\infty}\left[C_{i,j}\sin\left(\lambda_{i,j}z\right)+D_{i,j}\cos\left(\lambda_{i,j}z\right)\right]\chi_j(t)+A_i(t)z+B_i(t),\quad i=1,\ 2 \tag{60}$$

where

$$A_1(t)=\frac{k_2}{k_1}A_2(t);\quad B_1(t)=T_{\mathrm{m}}+A_{\mathrm{s}}\sin\left(\frac{2\pi t}{365}\right) \tag{61}$$

$$A_2(t)=\frac{A_{\mathrm{s}}\sin\left(\frac{2\pi t}{365}\right)}{\left(1-\frac{k_2}{k_1}\right)z_1-z_2};\quad B_2(t)=T_{\mathrm{m}}-A_2(t)z_2 \tag{62}$$

$$C_{1,j}=1;\quad D_{1,j}=0 \tag{63}$$

$$C_{2,j}=\frac{\sin\left(\lambda_{1,j}z_1\right)-\cos\left(\lambda_{2,j}z_1\right)D_{2,j}}{\sin\left(\lambda_{2,j}z_1\right)} \tag{64}$$

$$D_{2,j}=\left[k_2\lambda_{2,j}\sin\left(\lambda_{1,j}z_1\right)\cos\left(\lambda_{2,j}z_1\right)-k_1\lambda_{1,j}\cos\left(\lambda_{1,j}z_1\right)\sin\left(\lambda_{2,j}z_1\right)\right]/(k_2\lambda_{2,j}) \tag{65}$$

with

$$\lambda_{2,j}=\sqrt{\alpha_1/\alpha_2}\lambda_{1,j} \tag{66}$$

$$C_{2,j}\sin\left(\lambda_{2,j}z_2\right)+D_{2,j}\cos\left(\lambda_{2,j}z_2\right)=0 \tag{67}$$

Substituting Eqs. (64)–(66) into Eq. (67), a transcendental equation is obtained. The eigenvalues $\lambda_{1,j}$ and $\lambda_{2,j}$ are obtained on the base of the transcendental equation.

$\chi_j(t)$ is obtained as follows:

$$\chi_j(t)=\int_0^t \mathrm{e}^{-\alpha_i\lambda_{i,j}^2(t-\tau)}\phi_j(\tau)\mathrm{d}\tau+\mu_j\mathrm{e}^{-\alpha_i\lambda_{i,j}^2 t},\quad i=1,\ 2 \tag{68}$$

where

$$\phi_j(t)=\left\{\int_0^{z_1}\sin(\lambda_{1,j}z)q_1^*(z,t)\mathrm{d}z+\int_{z_1}^{z_2}\left[C_{2,j}\sin(\lambda_{2,j}z)+D_{i,j}\cos(\lambda_{2,j}z)\right]q_2^*(z,t)\mathrm{d}z\right\}\Big/\left\{\int_0^{z_1}\sin^2(\lambda_{1,j}z)\frac{k_1}{\alpha_1}\mathrm{d}z+\int_{z_1}^{z_2}\left[C_{2,j}\sin(\lambda_{2,j}z)+D_{2,j}\cos(\lambda_{2,j}z)\right]^2\frac{k_2}{\alpha_2}\mathrm{d}z\right\} \tag{69}$$

with

$$q_1^*(z,t)=\frac{A}{B}\left(t+\frac{z}{z_1}t_\mathrm{f}\right)\mathrm{e}^{-\frac{1}{B}\left(t+\frac{z}{z_1}t_\mathrm{f}\right)}-\left[\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}z+\frac{\mathrm{d}B_1(t)}{\mathrm{d}t}\right]\frac{k_1}{\alpha_1} \tag{70}$$

$$q_i^*(z,t)=-\left[\frac{\mathrm{d}A_2(t)}{\mathrm{d}t}z+\frac{\mathrm{d}B_2(t)}{\mathrm{d}t}\right]\frac{k_2}{\alpha_2} \tag{71}$$

$$I_1^*(z)=T_\mathrm{m}-A_1(0)z-B_1(0)=0 \tag{72}$$

$$I_2^*(z)=T_\mathrm{m}-A_2(0)z-B_2(0)=0 \tag{73}$$

$$\mu_j=\left\{\int_0^{z_1}\sin(\lambda_{1,j}z)I_1^*(z,t)\frac{k_1}{\alpha_1}\mathrm{d}z+\int_{z_1}^{z_2}\left[C_2\sin(\lambda_{2,j}z)+D_2\cos(\lambda_{2,j}z)\right]I_2^*(z,t)\frac{k_2}{\alpha_2}\mathrm{d}z\right\}\Big/\left\{\int_0^{z_1}\sin^2(\lambda_{1,j}z)\frac{k_1}{\alpha_1}\mathrm{d}z+\int_{z_1}^{z_2}\left[C_2\sin(\lambda_{2,j}z)+D_2\cos(\lambda_{2,j}z)\right]^2\frac{k_2}{\alpha_2}\mathrm{d}z\right\}=0 \tag{74}$$

The parameters of the model landfill are shown in Table 3. The solution was compared with the finite element software Comsol Multiphysics. The results calculated by Comsol Multiphysics and the analytical method are shown in Table 4. There were small differences between the two methods.

Variations of temperature in the different depths are shown in Figure 3. In the shallow depths, the temperature was influenced by the ambient temperature. When the depth was more than 10 m, the temperature of the air temperature had little effect on the temperature of the waste Figure 4 shows the variations of temperatures at different time. The highest temperature was 52.9 °C, which was similar to the data from HANSON et al's study [13]. The temperature gradient in the interface of waste and the foundation soil changed because of the difference of the heat transfer coefficient and the thermal diffusion coefficient in the two layers.

Table 3 Parameters of landfill

Parameter	Value
A/(J·(d·m·K)$^{-1}$)	2.48×10^5
B/d	540.5
T_m/°C	14.6
A_s/°C	17.3
z_1/m	33
z_2/m	108
k_1/(J·(d·m·K)$^{-1}$)	86400
k_2/(J·(d·m·K)$^{-1}$)	216000
α_1/(m^2·d^{-1})	0.0432
α_2/(m^2·d^{-1})	0.0778
t_f/d	1000

Table 4 Comparison between analytical and numerical solutions

Time/d	Method	Temperature at different depths/°C					
		2 m	10 m	20 m	30 m	40 m	50 m
0	Analytical	14.60	14.60	14.60	14.60	14.60	14.60
	numerical	14.60	14.60	14.60	14.60	14.60	14.60
500	Analytical	30.54	35.38	34.11	25.98	15.98	14.69
	numerical	30.13	35.28	34.07	25.93	15.97	14.69
1000	Analytical	22.29	47.50	45.29	30.79	18.37	15.31
	numerical	22.42	47.31	45.23	30.75	18.37	15.31
2000	Analytical	30.14	47.76	49.66	33.80	22.07	17.39
	numerical	29.92	47.58	49.57	33.72	22.06	17.39
3000	Analytical	24.15	39.77	44.62	33.26	24.07	19.29
	numerical	23.90	39.59	44.55	33.26	24.07	19.29
4000	Analytical	12.34	33.13	38.62	31.62	24.91	20.63
	numerical	12.23	32.85	38.52	31.64	24.90	20.63
5000	Analytical	15.23	28.33	33.77	29.86	25.04	21.44
	numerical	15.41	28.04	33.60	29.77	25.01	21.43
6000	Analytical	23.78	25.01	30.09	28.14	24.77	21.84
	numerical	23.53	24.80	29.93	28.02	24.73	21.83

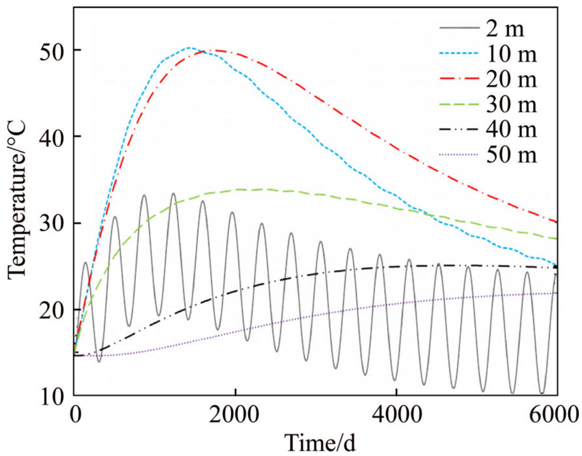

Figure 3 Variation of temperatures at different depths

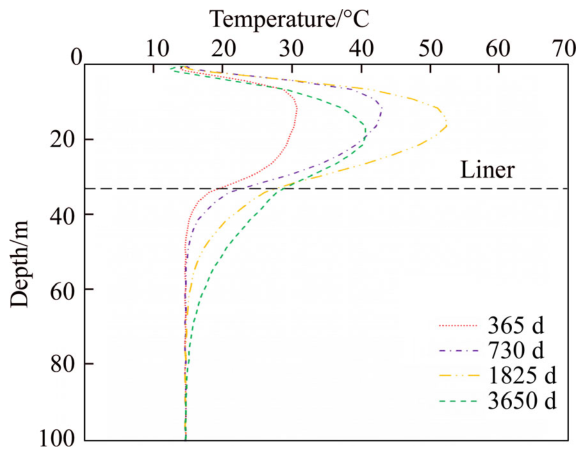

Figure 4 Variation of temperatures at different periods

5 Conclusions

The transient heat conduction in multilayered slabs with general boundary conditions and arbitrary heat generations was analytically investigated. The boundary conditions are general and include various combinations of Dirichlet, Neumann or Robin boundary conditions at either surface, and the governing equations contain arbitrary time and space dependent heat generations. The solutions are obtained using the superposition method, separation variable method and orthogonal expansion method. The Sturm-Liouville theory is used to prove the orthogonality of the characteristic function. The solutions have a wide range of applications. As an applied example of the solutions, the simplified double-layered slab solution is applied to predict the spatial and temporal distribution of the temperature in the landfill. The results are validated by Comsol Multiphysics, which verifies the correctness of the solution. Although one-dimensional multilayered slabs are investigated in this paper, the proposed method can be extended to multidimensional transient heat conduction in multilayered slabs or transient heat conduction in multilayered cylinders and spheres.

Appendix A:

Proof of orthogonality of characteristic function

The Sturm-Liouville theory was used to prove the orthogonality of the characteristic function. Let $\lambda_{i,j}$ and $\lambda_{i,m}$ be the jth and mth eigenvalue of the ith layer, respectively.

The following results can be obtained based on Eq. (32) as follows:

$$\int_{z_{i-1}}^{z_i} Z_{i,j}(z)\frac{\mathrm{d}^2 Z_{i,m}(z)}{\mathrm{d}z^2}\mathrm{d}z+\lambda_{i,j}^2\int_{z_{i-1}}^{z_i} Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z=0,\quad i=1,\ 2,\ \cdots,\ n \tag{A1}$$

$$\int_{z_{i-1}}^{z_i} Z_{i,m}(z)\frac{\mathrm{d}^2 Z_{i,j}(z)}{\mathrm{d}z^2}\mathrm{d}z+\lambda_{i,m}^2\int_{z_{i-1}}^{z_i} Z_{i,m}(z)Z_{i,j}(z)\mathrm{d}z=0,\quad i=1,\ 2,\ \cdots,\ n \tag{A2}$$

Subtracting Eq. (A1) from Eq. (A2) gives:

$$\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i} Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z,\quad i=1,\ 2,\ \cdots,\ n$$
$$=-\frac{k_i}{\alpha_i}\frac{1}{\lambda_{i,j}^2-\lambda_{i,m}^2}\cdot\int_{z_{i-1}}^{z_i}\left[Z_{i,m}\frac{\mathrm{d}^2 Z_{i,j}(z)}{\mathrm{d}z^2}-Z_{i,j}\frac{\mathrm{d}^2 Z_{i,m}(z)}{\mathrm{d}z^2}\right]\mathrm{d}z \tag{A3}$$

Equation (A3) can be converted into Eq. (A4) using integration by parts.

$$\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i} Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z,\ i=1,\ 2,\ \cdots,\ n$$
$$=-\frac{k_i}{\alpha_i}\frac{1}{\lambda_{i,j}^2-\lambda_{i,m}^2}\left[-Z_{i,j}(z_{i-1})\frac{\mathrm{d}Z_{i,m}(z_{i-1})}{\mathrm{d}z}+Z_{i,m}(z_{i-1})\frac{\mathrm{d}Z_{i,j}(z_{i-1})}{\mathrm{d}z}\right]-\frac{k_i}{\alpha_i}\frac{1}{\lambda_{i,j}^2-\lambda_{i,m}^2}\cdot\left[Z_{i,j}(z_i)\frac{\mathrm{d}Z_{i,m}(z_i)}{\mathrm{d}z}-Z_{i,m}(z_i)\frac{\mathrm{d}Z_{i,j}(z_i)}{\mathrm{d}z}\right] \tag{A4}$$

Using the continuity conditions Eq. (28) and Eq. (29) and Eq. (39), Eq. (A4) can be expressed as:

$$\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i} Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z,\quad i=1,\ 2,\ \cdots,\ n-1$$

$$=-\frac{k_i}{\alpha_i}\frac{1}{\lambda_{i,j}^2-\lambda_{i,m}^2}\left[-Z_{i,j}\left(z_{i-1}\right)\frac{\mathrm{d}Z_{i,m}\left(z_{i-1}\right)}{\mathrm{d}z}+Z_{i,m}\left(z_{i-1}\right)\frac{\mathrm{d}Z_{i,j}\left(z_{i-1}\right)}{\mathrm{d}z}\right]-\frac{k_{i+1}}{\alpha_{i+1}}\frac{1}{\lambda_{i+1,j}^2-\lambda_{i+1,m}^2}\cdot\left[Z_{i+1,j}\left(z_i\right)\frac{\mathrm{d}Z_{i+1,m}\left(z_i\right)}{\mathrm{d}z}-Z_{i+1,m}\left(z_i\right)\frac{\mathrm{d}Z_{i+1,j}\left(z_i\right)}{\mathrm{d}z}\right] \tag{A5}$$

Using the recurring relationship in Eq. (A5), we obtain:

$$\sum_{i=1}^{n-1}\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i}Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z=-\frac{k_1}{\alpha_1}\frac{1}{\lambda_{1,j}^2-\lambda_{1,m}^2}\left[-Z_{1,j}\left(z_0\right)\frac{\mathrm{d}Z_{1,m}\left(z_0\right)}{\mathrm{d}z}+Z_{1,m}\left(z_0\right)\frac{\mathrm{d}Z_{1,j}\left(z_0\right)}{\mathrm{d}z}\right]-\frac{k_n}{\alpha_n}\frac{1}{\lambda_{n,j}^2-\lambda_{n,m}^2}\cdot\left[Z_{n,j}\left(z_{n-1}\right)\frac{\mathrm{d}Z_{n,m}\left(z_{n-1}\right)}{\mathrm{d}z}-Z_{n,m}\left(z_{n-1}\right)\frac{\mathrm{d}Z_{n,j}\left(z_{n-1}\right)}{\mathrm{d}z}\right] \tag{A6}$$

Based on Eq. (A5), we get:

$$\frac{k_n}{\alpha_n}\int_{z_{n-1}}^{z_n}Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z=-\frac{k_n}{\alpha_n}\frac{1}{\lambda_{n,j}^2-\lambda_{n,m}^2}\left[-Z_{n,j}\left(z_{n-1}\right)\frac{\mathrm{d}Z_{n,m}\left(z_{n-1}\right)}{\mathrm{d}z}+Z_{n,m}\left(z_{n-1}\right)\frac{\mathrm{d}Z_{n,j}\left(z_{n-1}\right)}{\mathrm{d}z}\right]-\frac{k_n}{\alpha_n}\frac{1}{\lambda_{n,j}^2-\lambda_{n,m}^2}\cdot\left[Z_{n,j}\left(z_n\right)\frac{\mathrm{d}Z_{n,m}\left(z_n\right)}{\mathrm{d}z}-Z_{n,m}\left(z_n\right)\frac{\mathrm{d}Z_{n,j}\left(z_n\right)}{\mathrm{d}z}\right] \tag{A7}$$

When we add Eq. (A6) and Eq. (A7) together, Eq. (A8) is obtained as:

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i}Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z=-\frac{k_1}{\alpha_1}\frac{1}{\lambda_{1,j}^2-\lambda_{1,m}^2}\left[Z_{1,j}\left(z_0\right)\frac{\mathrm{d}Z_{1,m}\left(z_0\right)}{\mathrm{d}z}+Z_{1,m}\left(z_0\right)\frac{\mathrm{d}Z_{1,j}\left(z_0\right)}{\mathrm{d}z}\right]-\frac{k_n}{\alpha_n}\frac{1}{\lambda_{n,j}^2-\lambda_{n,m}^2}\cdot\left[Z_{n,j}\left(z_n\right)\frac{\mathrm{d}Z_{n,m}\left(z_n\right)}{\mathrm{d}z}-Z_{n,m}\left(z_n\right)\frac{\mathrm{d}Z_{n,j}\left(z_n\right)}{\mathrm{d}z}\right] \tag{A8}$$

As $Z_{i,j}(z)$ and $Z_{i,m}(z)$ are subjected to the general homogeneous boundary conditions at $z=z_0$ and $z=z_n$, Eq. (A8) finally becomes:

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{z_{i-1}}^{z_i}Z_{i,j}(z)Z_{i,m}(z)\mathrm{d}z=-\frac{k_1}{\alpha_1}\frac{1}{\lambda_{1,j}^2-\lambda_{1,m}^2}\cdot 0-\frac{k_n}{\alpha_n}\frac{1}{\lambda_{n,j}^2-\lambda_{n,m}^2}\cdot 0 \tag{A9}$$

When $j\neq m$, we get $\lambda_{1,j}\neq\lambda_{1,m}$ and $\lambda_{n,j}\neq\lambda_{n,m}$. Thus the orthogonality of the characteristic function can be proven as follows:

$$\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}Z_{i,j}(z)Z_{i,m}(z)\frac{k_i}{\alpha_i}\mathrm{d}z=0 \tag{A10}$$

Appendix B:

Solution for combinations of Neumann boundary conditions

The solution of $T_i(z, t)$ for the combinations of Neumann boundary conditions can be separated as follows:

$$T_i(z,t)=V_i(z,t)+M_i(z,t) \tag{B1}$$

The function $W_i(z, t)$ is used for homogenising of the boundary conditions and set as follows:

$$M_i(z,t)=R_i(t)z^2+S_i(t)z \tag{B2}$$

with

$$\begin{bmatrix} R_1(t)\\ S_1(t)\\ R_2(t)\\ S_2(t)\\ \vdots\\ S_{i-1}(t)\\ R_{i-1}(t)\\ R_i(t)\\ S_i(t)\\ R_{i+1}(t)\\ S_{i+1}(t)\\ \vdots\\ R_{n-1}(t)\\ S_{n-1}(t)\\ R_n(t)\\ S_n(t) \end{bmatrix}_{2n\times 1}=\begin{bmatrix} 2z_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ z_1 & 1 & -z_1 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 2k_1z_1 & k_1 & -k_2z_1 & -k_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & z_{i-1} & 1 & -z_{i-1} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \cdots & 2k_{i-1}z_{i-1} & k_{i-1} & -k_iz_{i-1} & -k_i & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & z_i & 1 & -z_i & -1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2k_iz_i & k_{i-1} & -k_{i+1}z_i & -k_{i+1} & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & z_n & 1 & -z_n & -1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2k_{n-1}z_{n-1} & k_{n-1} & -k_nz_{n-1} & -k_n\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2z_n & 1 \end{bmatrix}_{2n\times 2n}^{-1}\begin{bmatrix} f_{N,1}(t)\\ 0\\ 0\\ 0\\ \vdots\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ \vdots\\ 0\\ 0\\ 0\\ f_{N,n}(t) \end{bmatrix}_{2n\times 1} \tag{B3}$$

The governing equation of $V_i(z, t)$ is:

$$\frac{\partial V_i(z,t)}{\partial t}=\alpha_i\frac{\partial^2 V_i(z,t)}{\partial z^2}+\frac{\alpha_i\overline{q}_i(z,t)}{k_i},\quad z_{i-1}\le z\le z_i,\quad i=1,\ 2,\ \cdots,\ n \tag{B4}$$

with

$$\overline{q}_i(z,t)=q_i(z,t)-\left[\frac{\mathrm{d}R_i(t)}{\mathrm{d}t}z^2+\frac{\mathrm{d}S_i(t)}{\mathrm{d}t}z\right]\frac{k_i}{\alpha_i}+2k_iR_i(t) \tag{B5}$$

$V_i(z, t)$ is subjected to the upper and lower boundary conditions:

$$z=z_0:\ k_1\frac{\partial V_1(z,t)}{\partial z}=0 \tag{B6}$$

$$z=z_n:\ k_n\frac{\partial V_n(z,t)}{\partial z}=0 \tag{B7}$$

and to the inner boundary conditions:

$$V_i(z_i,t)=V_{i+1}(z_i,t),\quad i=1,\ 2,\ \cdots,\ n-1 \tag{B8}$$

$$k_i\left.\frac{\partial V_i(z,t)}{\partial z}\right|_{z=z_i}=k_{i+1}\left.\frac{\partial V_{i+1}(z,t)}{\partial z}\right|_{z=z_i},\quad i=1,\ 2,\ \cdots,\ n-1 \tag{B9}$$

The initial condition of $V_i(z, t)$ gives:

$$V_i(z,0)=\overline{I}_i(z)=I_i(z)-M_i(z,0),\quad i=1,\ 2,\ \cdots,\ n \tag{B10}$$

$V_i(z, t)$ is obtained by using separation variable method and orthogonal expansion method as follows:

$$V_i(z,t)=\sum_{j=0}^{\infty}\varphi_j(t)Z_{i,j}(z) \tag{B11}$$

where

$$Z_{i,j}(z)=C_{i,j}\sin(\lambda_{i,j}z)+D_{i,j}\cos(\lambda_{i,j}z),\quad i=1,\ 2,\ \cdots,\ n \tag{B12}$$

with

$$C_{1,j}=0;\quad D_{1,j}=1 \tag{B13}$$

$$\begin{bmatrix}C_{i+1,j}\\ D_{i+1,j}\end{bmatrix}=\begin{bmatrix}E_{i,j}G_{i,j}+\dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}}F_{i,j}H_{i,j} & F_{i,j}G_{i,j}-\dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}}E_{i,j}H_{i,j}\\ E_{i,j}H_{i,j}-\dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}}F_{i,j}G_{i,j} & F_{i,j}H_{i,j}+\dfrac{k_{i,j}\lambda_{i,j}}{k_{i+1,j}\lambda_{i+1,j}}E_{i,j}G_{i.j}\end{bmatrix}\cdot\begin{bmatrix}C_{i,j}\\ D_{i,j}\end{bmatrix} \tag{B14}$$

$$E_{i,j}=\sin(\lambda_{i,j}z_i);\quad F_{i,j}=\cos(\lambda_{i,j}z_i);\quad G_{i,j}=\sin(\lambda_{i+1,j}z_i);\quad H_{i,j}=\cos(\lambda_{i+1,j}z_i) \tag{B15}$$

$$C_{n,j}\cos(\lambda_{n,j}z_n)-D_{n,j}\sin(\lambda_{n,j}z_n)=0 \tag{B16}$$

$$\lambda_{i,j}=\begin{cases}0, & j=0\\ \sqrt{\alpha_1/\alpha_i}\lambda_{1,j}, & j=1,\ 2,\ \cdots\end{cases} \tag{B17}$$

The relationship between $C_{1,j}$, $D_{2,j}$ and $C_{n,j}$, $D_{n,j}$ is obtained as follows:

$$\begin{bmatrix}C_{n,j}\\ D_{n,j}\end{bmatrix}=S_{n-1,j}\begin{bmatrix}C_{1,j}\\ D_{1,j}\end{bmatrix} \tag{B18}$$

with

$$S_{n-1,j}=\prod_{l=1}^{n-1}\begin{bmatrix}E_{l,j}G_{l,j}+\dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}}F_{l,j}H_{l,j} & F_{l,j}G_{l,j}-\dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}}E_{l,j}H_{l,j}\\ E_{l,j}H_{l,j}-\dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}}F_{l,j}G_{l,j} & F_{l,j}H_{l,j}+\dfrac{k_{l,j}\lambda_{l,j}}{k_{l+1,j}\lambda_{l+1,j}}E_{l,j}G_{l,j}\end{bmatrix} \tag{B19}$$

Substituting Eq. (B17) and Eq. (B18) into Eq. (B16), a transcendental equation is obtained. The eigenvalues $\lambda_{i,j}$ (j=1, 2, ⋯) are the solutions of the transcendental equation.

$\varphi_j(t)$ is obtained as follows:

$$\varphi_j(t)=\int_0^t \mathrm{e}^{-\alpha_i\lambda_{i,j}^2(t-\tau)}\eta_j(\tau)\mathrm{d}\tau+\nu_j\mathrm{e}^{-\alpha_i\lambda_{i,j}^2 t} \tag{B20}$$

with

$$\eta_j(t)=\frac{\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}Z_{i,j}(z)\overline{q}_i(z,t)\mathrm{d}z}{\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}Z_{i,j}^2(z)\dfrac{k_i}{\alpha_i}\mathrm{d}z} \tag{B21}$$

$$\nu_j=\frac{\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}\overline{I}_i(z)Z_{i,j}(z)\dfrac{k_i}{\alpha_i}\mathrm{d}z}{\sum_{i=1}^{n}\int_{z_{i-1}}^{z_i}Z_{i,j}^2(z)\dfrac{k_i}{\alpha_i}\mathrm{d}z} \tag{B22}$$

The complete solution of $T_i(z, t)$ for the combinations of Neumann boundary conditions is obtained as follows:

$$T_i(z,t)=\sum_{j=0}^{\infty}\left[C_{i,j}\sin(\lambda_{i,j}z)+D_{i,j}\cos(\lambda_{i,j}z)\right]\varphi_j(t)+R_i(t)z^2+S_i(t)z \tag{B23}$$

References

[1] GALIP O, SAHIN A Z. Eigenvalue analysis of temperature distribution in composite walls [J]. International Journal of Energy Research, 2001, 25(13): 1189–1196.
[2] LÜ X, LU T, VILJANEN M. A new analytical method to simulate heat transfer process in buildings [J]. Applied Thermal Engineering, 2006, 26(16): 1901–1909.
[3] DAOUAS N. Impact of external longwave radiation on optimum insulation thickness in Tunisian building roofs based on a dynamic analytical model [J]. Applied Energy, 2016, 177: 136–148.
[4] ZHANG S J, LIU Z Q. An analytical model for transient temperature distributions in coated carbide cutting tools [J]. International Communications in Heat and Mass Transfer, 2008, 35(10): 1311–1315.
[5] FASSANI R N S, TREVISAN O V. Analytical modeling of multipass welding process with distributed heat source [J]. Journal of the Brazilian Society of Mechanical Sciences and Engineering, 2003, 25(3): 302–305.
[6] BELGHAZI H, EL GANAOUI M, LABBE J C. Analytical solution of unsteady heat diffusion within a porous copper layer deposited on alumina substrate and subjected to a moving laser beam [J]. Defect and Diffusion Forum, 2008, 273-276(3): 52–57.
[7] LIU J C, GRIFFITHS D V. A general solution for 1D consolidation induced by depth- and time-dependent changes in stress [J]. Géotechnique, 2014, 65(1): 66–72.
[8] XIE K H, XIA C Q, AN R, YING H W, WU H. A study on one-dimensional consolidation of layered structured soils [J]. International Journal for Numerical and Analytical Methods in Geomechanics, 2016, 40(7): 1081–1098.
[9] HICKSON R I, BARRY S I, MERCER G N. Critical times in multilayer diffusion. Part 1: Exact solutions [J]. International Journal of Heat & Mass Transfer, 2009, 52(25, 26): 5776–5783.
[10] WU X, SHI J Y, HE J. Analytical solutions for diffusion of organic contaminant through GCL triple-layer composite liner considering degradation in liner [J]. Environmental Earth Sciences, 2016, 75(20): 1371.
[11] WU X, SHI J Y, HE J. Analytical solutions for diffusion of organic contaminant through composite liner considering degradation in leachate and soil liner [J]. International Journal of Environment & Pollution, 2017, 61(2): 166–185.
[12] HANSON J L, LIU W L, YEŞILLER N. Analytical and numerical methodology for modeling temperatures in landfills [J]. Geotechnical Special Publication, 2008, 78(177): 2599–2604.
[13] HANSON J L, YEŞILLER N, ONNEN M T, LIU W L, OETTLE N K, MARINOS J A. Development of numerical model for predicting heat generation and temperatures in MSW landfills [J]. Waste Management, 2013, 33(10): 1993–2000.
[14] HAJI-SHEIKH A, BECK J V, AGONAFER D. Steady-state heat conduction in multi-layer bodies [J]. International Journal of Heat and Mass Transfer, 2003, 46(13): 2363–2379.
[15] SINGH S, JAIN P K, RIZWAN-UDDIN. Analytical solution to transient heat conduction in polar coordinates with multiple layers in radial direction [J]. International Journal of Thermal Sciences, 2008, 47(3): 261–273.
[16] DÜLK I, KOVÁCSHÁZY T. A method for computing the analytical solution of the steady-state heat equation in multilayered mdia [J]. Journal of Heat Transfer, 2014, 136(9): 091303.
[17] HAJI-SHEIKH A, BECK J V. Temperature solution in multi-dimensional multi-layer bodies [J]. International Journal of Heat & Mass Transfer, 2002, 45(9): 1865–1877.
[18] YEN D H Y, BECK J V. Green's functions and three-dimensional steady-state heat-conduction problems in a two-layered composite [J]. Journal of Engineering Mathematics, 2004, 49(3): 305–319.
[19] LÜ X, TERVOLA P, VILJANEN M. A new analytical method to solve the heat equation for a multi-dimensional composite slab [J]. Journal of Physics A: General Physics, 2005, 38(13): 2873.
[20] LÜ X, TERVOLA P, VILJANEN M. Transient analytical solution to heat conduction in multi-dimensional composite cylinder slab [J]. International Journal of Heat and Mass Transfer, 2006, 49(5, 6): 1107–1114.
[21] LI Min, LAI A C K. Analytical solution to heat conduction in finite hollow composite cylinders with a general boundary condition [J]. International Journal of Heat and Mass Transfer, 2013, 60(1): 549–556.
[22] SINGH S, JAIN P K, RIZWAN-UDDIN. Finite integral transform method to solve asymmetric heat conduction in a multilayer annulus with time-dependent boundary conditions [J]. Nuclear Engineering and Design, 2011, 241(1): 144–154.
[23] YANG B G, LIU S B. Closed-form analytical solutions of transient heat conduction in hollow composite cylinders with any number of layers [J]. International Journal of Heat and Mass Transfer, 2017, 108: 907–917.
[24] SINGH I V. Heat transfer analysis of composite slabs using meshless element Free Galerkin method [J]. Computational Mechanics, 2006, 38(6): 521–532.
[25] WANG H J, DAI W Z, HEWAVITHARANA L G. A finite difference method for studying thermal deformation in a double-layered thin film with imperfect interfacial contact exposed to ultrashort pulsed lasers [J]. International Journal of Thermal Sciences, 2008, 47(1): 7–24.
[26] OZISIK M N. Heat conduction [M]. New York: John Wiley & Sons, 1993.
[27] de MONTE F. Transient heat conduction in one-dimensional composite slab. A 'natural' analytic approach [J]. International Journal of Heat & Mass Transfer, 2000, 43(19): 3607–3619.
[28] de MONTE F. An analytic approach to the unsteady heat conduction processes in one-dimensional composite media [J]. International Journal of Heat and Mass Transfer, 2002, 45(6): 1333–1343.
[29] SUN Y Z, WICHMAN I S. On transient heat conduction in a one-dimensional composite slab [J]. International Journal of Heat and Mass Transfer, 2004, 47(6, 7): 1555–1559.
[30] LÜ X, TERVOLA P. Transient heat conduction in the composite slab-analytical method [J]. Journal of Physics A General Physics, 2004, 38(1): 81.

[31] ZHOU L, BAI M, CUI W. Theoretical solution of transient heat conduction problem in one-dimensional double-layer composite medium [J]. Journal of Central South University of Technology, 2010, 17(6): 1403–1408.
[32] BELGHAZI H, GANAOUI M E, LABBE J C. Analytical solution of unsteady heat conduction in a two-layered material in imperfect contact subjected to a moving heat source [J]. International Journal of Thermal Sciences, 2010, 49(2): 311–318.
[33] TIAN M, ZHU S, CHEN Q, PAN N. Effects of layer stacking sequence on temperature response of multi-layer composite materials under dynamic conditions [J]. Applied Thermal Engineering, 2012, 33: 219–226.
[34] FAKOOR-PAKDAMAN M, AHMADI M, BAGHERI F, BAHRAMI M. Dynamic heat transfer inside multilayered packages with arbitrary heat generations [J]. Journal of Thermophysics & Heat Transfer, 2014, 28(4): 687–699.
[35] FAKOOR-PAKDAMAN M, AHMADI M, BAGHERI F, BAHRAMI M. Optimal time-varying heat transfer in multilayered packages with arbitrary heat generations and contact resistance [J]. Journal of Heat Transfer, 2015, 137(8): 081401.

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中文导读

层状结构中热传导的解析解及其在填埋场热分析中的应用

摘要：层状结构中瞬态热传导模型广泛应用于不同工程领域。本文建立层状结构中瞬态热传导模型，模型的边界条件为 Dirichlet、Neumann 或 Robin 边界的不同组合，模型考虑不同层中不同的产热函数。通过叠加法、分离变量法和正交展开法得到模型的解析解。运用两层模型的解析解分析填埋场中的温度分布并通过数值解验证解答的正确性。表明本文模型及其解析解在瞬态热传导问题中的适用性。

关键词：热传导；层状结构；产热；解析解；填埋场