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Generalized thermoelastic interaction in functional graded material with fractional order three-phase lag heat transfer

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Abstract: The present work is concerned with the solution of a problem on thermoelastic interactions in a functional graded material due to thermal shock in the context of the fractional order three-phase lag model. The governing equations of fractional order generalized thermoelasticity with three-phase lag model for functionally graded materials (FGM) (i.e., material with spatially varying material properties) are established. The analytical solution in the transform domain is obtained by using the eigenvalue approach. The inversion of Laplace transform is done numerically. The graphical results indicate that the fractional parameter has significant effects on all the physical quantities. Thus, we can consider the theory of fractional order generalized thermoelasticity an improvement on studying elastic materials.

Key words: three-phase lag model; functionally graded materials; fractional calculus; generalized thermoelasticity

1 Introduction

The linear theory of elasticity is of paramount importance in the stress analysis of steel, which is the commonest engineering structural material. However, the theory does not apply to the behavior of many of the new synthetic materials of the elastomer and polymer type. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time-fractional (non-integer order) derivatives. ABEL [1] who applied fractional calculus in the solution of an integral equation gave the first application of fractional derivatives. CAPUTO [2] gave the definition of fractional derivatives of order $0<\alpha\leq 1$ of continuous function. CAPUTO and MAINARDI [3−4] and CAPUTO [5] have employed the fractional order derivatives for the description of viscoelastic materials and have established the connection between fractional derivatives and the theory of linear viscoelasticity and found a good agreement with the experimental results. Among the few works devoted to applications of fractional calculus to thermoelasticity, we can refer to the works of POVSTENKO [6−7] who introduced a fractional heat conduction law and found the associated thermal stresses. SHERIEF et al [8], YOUSSEF [9] and EZZAT [10−11] introduced new models of thermoelasticity using a fractional heat conduction equation.

The generalization of the thermoelasticity theory is known as the dual-phase-lag thermoelasticity developed by TZOU [12] and CHANDRASEKHARIAH [13]. TZOU considered micro-structural effects in the delayed response in time in the macroscopic formulation by taking into account that increase of the lattice temperature is delayed due to photon-electron interactions on the macroscopic level. TZOU [12] introduced a two-phase-lag (2PHL) into both the heat flux vector and the temperature gradient. According to this model, classical Fourier's law $\boldsymbol{q}=-K\nabla T$ has been replaced by $\boldsymbol{q}(P,t+\tau_q)=-K\nabla T(P,t+\tau_T)$, where the temperature gradient ∇T at a point P of the material at time $t+\tau_T$ corresponds to the heat flux vector $\boldsymbol{q}$ at the same point in time $t+\tau_q$. Here K is the thermal conductivity of the material. The delay time τ_T is interpreted as that caused by the micro-structural interactions and is called the phase-lag of the temperature gradient. The other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the phase-lag of the heat flux.

The generalization is known as three-phase-lag (3PHL) thermoelasticity, which is due to Roychoudhuri [14]. According to this model,

$$\boldsymbol{q}(P,t+\tau_q)=-[K\nabla T(P,t+\tau_T)+K^*\nabla\nu(P,t+\tau_\nu)]$$

where $\nabla\nu$ $(\dot{\nu}=T)$ is the thermal displacement gradient

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and K^* is the additional material constant and τ_v is the phase lag for the thermal displacement gradient. To study some practical, relevant problems found in heat transfer problems involving very short time intervals and in the problems of very high heat fluxes, the hyperbolic equation gives significantly different results from the parabolic equation. According to this phenomenon, the logging behavior in the heat conduction in solid should not be ignored particularly when the elapsed times during a transient process are very small, i.e., about 10^{-7} s or the heat flux is very much high. Recently, EZZAT et al [15] investigated the fractional order theory in thermoelastic solid with three-phase lag heat transfer.

Functionally graded material (FGM) as a new kind of composites was initially designed as thermal barrier materials for aerospace structures, in which the volume fractions of different constituents of composites vary continuously from one side to another [16]. These novel nonhomogeneous materials have excellent thermo-mechanical properties to withstand high temperature and have extensive applications to important structures, such as pressure vessels, chemicals plants, aerospace, pipes and nuclear reactors. MALLIK and KANORIA [17], DAS and KANORIA [18] studied a periodically varying heat source in generalized thermoelastic functionally graded solid. Applying the above theories of generalized thermoelasticity, several problems have been solved by finite element method and analytical method [19−35].

The main object of the present work is to study the fractional order generalized thermoelasticity in a functionally graded material with three-phase lag in the presence of thermal shock by Laplace transform and eigenvalue approach. Then, the inversion of Laplace transform is carried out numerically by a method of numerical inversion of Laplace transform based on Stehfest technique [36]. Numerical results for all variables in physical space–time domain are represented graphically. The graphical results indicate that the effects of fractional parameter, nonhomogeneity parameter and different theories on all the physical quantities.

2 Basic equation

Following EZZAT et al [15], the basic equations of fractional order theory of thermoelasticity for a functionally graded material in the absence of body forces and heat sources are considered.

The equations of motion:

$$\sigma_{ij,j}=\rho\frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

The equation of heat conduction:

$$(K^*T_{,i})_{,i}+\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}(K\dot{T}_{,i})_{,i}+((K+\tau_\upsilon K^*)\dot{T}_{,i})_{,i}=\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)(\rho c_{\mathrm{e}}\ddot{T}+\gamma T_0\ddot{e}),0<\alpha\le 1 \tag{2}$$

The constitutive equation:

$$\sigma_{ij}=2\mu e_{ij}+[\lambda e-\gamma(T-T_0)]\delta_{ij} \tag{3}$$

where $e=e_{ii},\ i,j=x,y,z.$

Putting $\alpha=1,\ \tau_T=0,\tau_\upsilon=0$ and $\tau_q=0,$ Eq. (2) becomes

$$(K^*T_{,i})_{,i}+(K\dot{T}_{,i})_{,i}=\rho c_{\mathrm{e}}\ddot{T}+\gamma T_0\ddot{e} \tag{4}$$

which is the heat conduction equation of Green–Naghdi theory of type III [37−38] admitting damped thermoelastic wave solutions. α is the fractional parameter; λ, μ are the Lame's constants; ρ is the density of the medium; c_{e} is the specific heat capacity at constant strain; α_{t} is the coefficient of linear thermal expansion; t is the time; T is the temperature; T_0 is the reference temperature; K is the thermal conductivity; K^* is an additional material constant; τ_υ, delay time is called the phase-lag of thermal displacement gradient; τ_T, the other delay time, is called the phase-lag of the temperature gradient and τ_q is called the phase-lag of the heat flux; δ_{ij} is the Kronecker symbol; σ_{ij} are the components of stress tensor; u_i are the components of displacement vector. Thus, we replace λ, μ, γ, K, K^* and ρ by $\lambda_0 f(X)$, $\mu_0 f(X)$, $\gamma_0 f(X)$, $K_0 f(X)$, $K_0^* f(X)$ and $\rho_0 f(X)$ where λ_0, μ_0, γ_0, K_0, K_0^* and ρ_0 are assumed to be constants and $f(X)$ is a given dimensionless function of the space variable $X=(x, y, z)$. Then, Eqs. (1) to (3) take the following forms:

$$f(X)[2\mu_0 e_{ij}+[\lambda_0 e-\gamma_0(T-T_0)]\delta_{ij}]_{,j}+f(X)_{,j}[2\mu_0 e_{ij}+[\lambda_0 e-\gamma_0(T-T_0)]\delta_{ij}]=\rho_0 f(X)\frac{\partial^2 u_i}{\partial t^2} \tag{5}$$

$$(K_0^* f(X)T_{,i})_{,i}+\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}(K_0 f(X)\dot{T}_{,i})_{,i}+((K_0 f(X)+\tau_\upsilon K_0^* f(X))\dot{T}_{,i})_{,i}=\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\cdot(\rho_0 f(X)c_{\mathrm{e}}\ddot{T}+\gamma_0 f(X)T_0\ddot{e}),0<\alpha\le 1 \tag{6}$$

$$\sigma_{ij}=f(X)[2\mu_0 e_{ij}+[\lambda_0 e-\gamma_0(T-T_0)]\delta_{ij}] \tag{7}$$

3 Formulation of problem

Let us consider a functionally graded isotropic thermoelastic body at a uniform reference temperature T_0, occupying the region $x\ge 0$ where the x-axis is taken perpendicular to the bounding plane of the half-space pointing inwards. It assumed that the state of the medium depends only on x and the time variable t, so that the displacement vector $\boldsymbol{u}$ and temperature field T can be expressed in the following form:

$$\boldsymbol{u}=(u(x,t),0,0)\ ,\ T=T(x,t) \tag{8}$$

It is assumed that the material properties depend only on the x-coordinate. So, we take $f(X)$ as $f(x)$. In the context of the fractional order of generalized thermoelasticity theory based on three-phase-lag model, the equation of motion, heat equation, and constitutive equation can be written as

$$f(x)\left[(\lambda_0+2\mu_0)\frac{\partial^2 u}{\partial x^2}-\gamma_0\frac{\partial T}{\partial x}\right]+\frac{\partial f(x)}{\partial x}\cdot \left[(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\gamma_0 T\right]=\rho_0 f(x)\frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$\left(K_0^*+K_0\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}+(K_0+\tau_\upsilon K_0^*)\frac{\partial}{\partial t}\right)\left(f(x)\frac{\partial^2 T}{\partial x^2}+\frac{\partial f(x)}{\partial x}\frac{\partial T}{\partial x}\right)=f(x)\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho_0 c_{\mathrm{e}}\frac{\partial^2 T}{\partial t^2}+\gamma_0 T_0\frac{\partial^3 u}{\partial t^2\partial x}\right), 0<\alpha\le 1 \tag{10}$$

$$\sigma_{xx}=f(x)\left[(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\gamma_0(T-T_0)\right] \tag{11}$$

We define the following dimensionless quantities:

$$(x',u')=\frac{c}{\chi}(x,u)$$

$$T'=\frac{T-T_0}{T_0}$$

$$(t',\tau_q',\tau_T',\tau_\upsilon')=\frac{c^2}{\chi}(t,\tau_q,\tau_T,\tau_\upsilon)$$

$$\sigma_{xx}'=\frac{\sigma_{xx}}{\lambda_0+2\mu_0}$$

where $c^2=\dfrac{\lambda_0+2\mu_0}{\rho_0}$ and $\chi=\dfrac{K_0}{\rho_0 c_{\mathrm{e}}}$.

Upon introducing in Eqs. (9)−(11), and after suppressing the primes, we obtain

$$f(x)\left[\frac{\partial^2 u}{\partial x^2}-\xi\frac{\partial T}{\partial x}\right]+\frac{\partial f(x)}{\partial x}\left[\frac{\partial u}{\partial x}-\xi T\right]=f(x)\frac{\partial^2 u}{\partial t^2} \tag{12}$$

$$\left[\varepsilon_1+\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}+(1+\tau_\upsilon\varepsilon_1)\frac{\partial}{\partial t}\right]\left[f(x)\frac{\partial^2 T}{\partial x^2}+\frac{\partial f(x)}{\partial x}\frac{\partial T}{\partial x}\right]=f(x)\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\frac{\partial^2 T}{\partial t^2}+\varepsilon_2\frac{\partial^3 u}{\partial t^2\partial x}\right), 0<\alpha\le 1 \tag{13}$$

$$\sigma_{xx}=f(x)\left(\frac{\partial u}{\partial x}-\xi T\right) \tag{14}$$

where $\xi=\dfrac{T_0\gamma_0}{\lambda_0+2\mu_0}$, $\varepsilon_1=\dfrac{K_0^*}{\rho_0 c_{\mathrm{e}} c^2}$, $\varepsilon_2=\dfrac{\gamma_0}{\rho_0 c_{\mathrm{e}}}$.

4 Exponential variation of non-homogeneity

We take $f(x)=e^{nx}$, where n is a dimensionless constant. Then, Eqs. (12), (13) and (14) reduce to

$$\left[\frac{\partial^2 u}{\partial x^2}-\xi\frac{\partial T}{\partial x}\right]+n\left[\frac{\partial u}{\partial x}-\xi T\right]=\frac{\partial^2 u}{\partial t^2} \tag{15}$$

$$\left[\varepsilon_1+\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}+(1+\tau_\upsilon\varepsilon_1)\frac{\partial}{\partial t}\right]\left[\frac{\partial^2 T}{\partial x^2}+n\frac{\partial T}{\partial x}\right]=\left[1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\frac{\partial^2 T}{\partial t^2}+\varepsilon_2\frac{\partial^3 u}{\partial t^2\partial x}\right] \tag{16}$$

$$\sigma_{xx}=\mathrm{e}^{nx}\left[\frac{\partial u}{\partial x}-\xi T\right] \tag{17}$$

5 Application

We assume that the medium is initially at rest. Then, we have

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=\frac{\partial^2 u(x,0)}{\partial t^2}=\frac{\partial^3 u(x,0)}{\partial t^3}=0,$$

$$T(x,0)=\frac{\partial T(x,0)}{\partial t}=\frac{\partial^2 T(x,0)}{\partial t^2}=\frac{\partial T^3(x,0)}{\partial t^3}=0 \tag{18}$$

We consider the problem of a thick plate of finite high l. Choosing the x-axis perpendicular to the surface of the plate with the origin coinciding with the lower plate, the region Ω under consideration becomes:

$$\Omega=\{(x,y,x):0\le x\le l,-\infty<y<\infty,-\infty<z<\infty\}$$

The surface of the plate is taken to be traction free. The lower plate is subjected to a thermal shock. The upper plate is kept at zero temperature. These can be written mathematically:

$$\sigma_{xx}(0,t)=0,\ T(0,t)=T_1 H(t) \tag{19}$$

$$\sigma_{xx}(l,t)=0,\ T(l,t)=0 \tag{20}$$

where $H(t)$ denotes the Heaviside unit step function.

6 Governing equations in Laplace transform domain

Applying the Laplace transform for Eqs. (15)−(20) by the formula:

$$\bar{f}(s)=L[f(t)]=\int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t \tag{21}$$

Hence, we obtain the following system of differential equations:

$$\left[\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}x^2}-\xi\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right]+n\left[\frac{\mathrm{d}\overline{u}}{\mathrm{d}x}-\xi\overline{T}\right]=s^2\overline{u} \tag{22}$$

$$\left[\varepsilon_1+\frac{\tau_T^{\alpha}}{\alpha!}s^{\alpha+1}+(1+\tau_\upsilon\varepsilon_1)s\right]\left[\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+n\frac{\mathrm{d}T}{\mathrm{d}x}\right]=$$
$$\left[s^2+\frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+2}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+2}\right]\left[T+\varepsilon_2\frac{\mathrm{d}u}{\mathrm{d}x}\right] \tag{23}$$

$$\overline{\sigma}_{xx}=\mathrm{e}^{nx}\left[\frac{\mathrm{d}\overline{u}}{\mathrm{d}x}-\xi\overline{T}\right] \tag{24}$$

$$\overline{\sigma}_{xx}(0,s)=0,\ \overline{T}(0,s)=\frac{T_1}{s} \tag{25}$$

$$\overline{\sigma}_{xx}(l,s)=0,\ \overline{T}(l,s)=0 \tag{26}$$

Equations (22) and (23) can be written in a vector-matrix differential equation as follows [39]:

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}x}=\boldsymbol{A}\boldsymbol{V} \tag{27}$$

where

$$\boldsymbol{V}=\left[\overline{u}\ \overline{T}\ \frac{\mathrm{d}\overline{u}}{\mathrm{d}x}\ \frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right]^{\mathrm{T}},\quad \boldsymbol{A}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\s^2&n\xi&-n&\xi\\0&\beta&\varepsilon_2\beta&-n\end{bmatrix},$$

$$\beta=\frac{s^2+\frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+2}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+2}}{\varepsilon_1+\frac{\tau_T^{\alpha}}{\alpha!}s^{\alpha+1}+(1+\tau_\upsilon\varepsilon_1)s}.$$

7 Solution of vector-matrix differential equation

Let us now proceed to solve Eq. (27) by the eigenvalue approach proposed in Ref. [39]. The characteristic equation of the matrix $\boldsymbol{A}$ takes the form:

$$p^4-m_1p^3+m_2p^2+m_3p+m_4=0 \tag{28}$$

where

$$m_1=-2n,\ m_2=n^2-s^2-\beta-\xi\beta\varepsilon_2,\ m_3=-n\beta-n\xi\varepsilon_2\beta-s^2n,\ m_4=s^2\beta. \tag{29}$$

The roots of the characteristic Eq. (28) which are also the eigenvalues of matrix $\boldsymbol{A}$ are of the form $p=p_1$, $p=p_2$, $p=p_3$, $p=p_4$. The eigenvector $\boldsymbol{X}=[x_1,x_2,x_3,x_4]^{\mathrm{T}}$, corresponding to eigenvalue η, can be calculated as

$$x_1=-n\xi-p\xi,\ x_2=s^2-p(n+p),\ x_3=-p(n\xi+p\xi),\ x_4=p[s^2-p(n+p)] \tag{30}$$

From Eq. (29), we can easily calculate the eigenvector $\boldsymbol{X}_j$, corresponding to eigenvalue p_j, j=1, 2, 3, 4. For further reference, we shall use the following notations:

$$\boldsymbol{X}_1=[X]_{p=p_1},\ \boldsymbol{X}_2=[X]_{p=p_2},\ \boldsymbol{X}_3=[X]_{p=p_3},\ \boldsymbol{X}_4=[X]_{p=p_4} \tag{31}$$

The solution of Eq. (27) can be written from as follows:

$$\boldsymbol{V}=\sum_{j=1}^{4}B_j\boldsymbol{X}_j\mathrm{e}^{p_ix}=B_1\boldsymbol{X}_1\mathrm{e}^{p_1x}+B_2\boldsymbol{X}_2\mathrm{e}^{p_2x}+B_3\boldsymbol{X}_3\mathrm{e}^{p_3x}+B_4\boldsymbol{X}_4\mathrm{e}^{p_4x} \tag{32}$$

where B_1, B_2, B_3 and B_4 are constants to be determined from the boundary condition of the problem. Thus, the field variables can be written for x and s as

$$\overline{u}(x,\mathrm{s})=\sum_{j=1}^{4}B_jx_3^j\mathrm{e}^{p_jx} \tag{33}$$

$$\overline{T}(x,s)=\sum_{j=1}^{4}B_jx_4^j\mathrm{e}^{p_jx} \tag{34}$$

$$\overline{\sigma}_{xx}(x,\mathrm{s})=\sum_{j=1}^{4}(p_jx_3^j-\xi x_4^j)B_j\mathrm{e}^{(\mathrm{p}_j+n)x} \tag{35}$$

To complete the solution, we have to know the constants B_1, B_2, B_3 and B_4, by using the boundary conditions (25) and (26):

$$\begin{pmatrix}B_1\\B_2\\B_3\\B_4\end{pmatrix}=\begin{pmatrix}H_{11}&H_{12}&H_{13}&H_{14}\\H_{21}&H_{22}&H_{23}&H_{24}\\H_{31}&H_{23}&H_{33}&H_{34}\\H_{41}&H_{42}&H_{43}&H_{44}\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\\frac{T_1}{s}\\0\end{pmatrix} \tag{36}$$

where

$$H_{11}=p_1x_3^1-\xi x_4^1,\ H_{12}=p_2x_3^2-\xi x_4^2,\ H_{13}=p_3x_3^3-\xi x_4^3,$$
$$H_{14}=p_4x_3^4-\xi x_4^4,\ H_{21}=(p_1x_3^1-\xi x_4^1)\mathrm{e}^{p_1l},$$
$$H_{22}=(p_2x_3^2-\xi x_4^2)\mathrm{e}^{p_2l},\ H_{23}=(p_3x_3^3-\xi x_4^3)\mathrm{e}^{p_3l},$$
$$H_{24}=(p_4x_3^4-\xi x_4^4)\mathrm{e}^{p_4l},\ H_{31}=x_4^1, H_{32}=x_4^2,$$
$$H_{33}=x_4^3, H_{34}=x_4^4,\ H_{41}=x_4^1e^{p_1l},\ H_{42}=x_4^2\mathrm{e}^{p_2l},$$
$$H_{43}=x_4^3\mathrm{e}^{p_3l},\ H_{44}=x_4^4\mathrm{e}^{p_4l}.$$

8 Numerical inversion of Laplace transforms

For the final solution of temperature, displacement and stress distributions in the time domain, we adopt a numerical inversion method based on the STEHFEST [36]. In this method, the inverse $f(t)$ of the Laplace transform $f(s)$ is approximated by the relation:

$$f(t)=\frac{\ln 2}{t}\sum_{j=1}^{N}V_jF\left(\frac{\ln 2}{t}j\right) \tag{37}$$

where V_j is given by

$$V_i=(-1)^{\left(\frac{N}{2}+1\right)}\sum_{k=\frac{i+1}{2}}^{\min\left(i,\frac{N}{2}\right)}\frac{k^{\left(\frac{N}{2}+1\right)}(2k)!}{\left(\frac{N}{2}-k\right)!k!(i-k)!(2k-1)!} \tag{38}$$

and the parameter N is the number of terms used in the summation in Eq. (36) and should be optimized by trial and error. Increasing N increases the accuracy of the result up to a point, and then the accuracy declines because of increasing round-off errors. Thus, the solutions of all variables in physical space–time domain are given by

$$u(x,t)=\frac{\ln 2}{t}\sum_{i=1}^{N}V_i\bar{u}\left(x,\frac{\ln 2}{t}i\right) \tag{39}$$

$$T(x,t)=\frac{\ln 2}{t}\sum_{i=1}^{N}V_i\bar{T}\left(x,\frac{\ln 2}{t}i\right) \tag{40}$$

$$\sigma_{xx}(x,t)=\frac{\ln 2}{t}\sum_{i=1}^{N}V_i\bar{\sigma}_{xx}\left(x,\frac{\ln 2}{t}i\right) \tag{41}$$

9 Numerical results and discussion

The copper material was chosen for purposes of numerical evaluations and the constants of the problem were taken as follows [20]: λ_0=7.76×10^{10} kg/(m·s^2), μ_0=3.86×10^{10} kg/(m·s^2), T_0= 293 K, K_0=3.86×10^2 kg·m/(K·s^3), c_e=3.831×10^2 m^2/(K·s^2), T_1=1, l=4, ρ_0=8.954×10^3 kg/m^3, α_t=17.8×10^{-6} K^{-1}, τ_T=0.1, τ_q=0.15.

Figures 1−3 display the temperature, displacement and the stress distributions for wide range of x $0\le x\le l$ for two different theories (three-phase lag model 3PHL and Green–Naghdi model of type III GNIII) in presented two values of time t=0.5 and t=1 with n=0.5 and α=0.5. In Fig. 1, we display the temperature for different values of t to show its effect on the field in the two types (3PHL) and (GNIII) and we have noticed that 1) in the (GNIII) theory, the speed of the wave propagation of the conductive temperature vanished at larger distance than in the (3PHL) theory; 2) the time t has a significant effect on the temperature for the two theories.

In Fig. 2, we display the displacement u for different values of t to show its effect on the filed in the two types (3PHL) and (GNIII) and we have noticed that 1) in the (GNIII) theory, the speed of the wave propagation of the displacements u and v vanished at larger distant than in the (3PHL) theory; 2) the time t has a significant effect on the displacement u for the two theories.

In Fig. 3, we display the stress σ_{xx} for different values of t to show its effect on the filed in the two types (3PHL) and (GNIII) and we have noticed that 1) in the (GNIII) theory, the speed of the wave propagation of the stress σ_{xx} vanished at larger distant than in the (3PHL) theory; 2) the time t has a significant effect on the stress σ_{xx} for the two theories.

Figures 4−6 represent the variation of the physical quantities versus the distance x without fraction parameter α=1 and with two values of the fraction parameter α=0.1, 0.5 for 3PHL model and we have

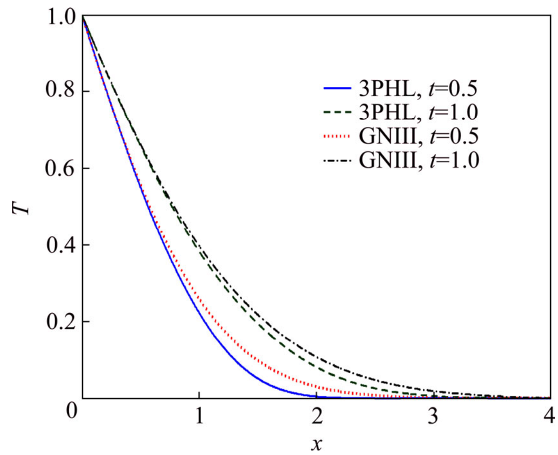

Fig. 1 Temperature distribution with distance x for different theories and time

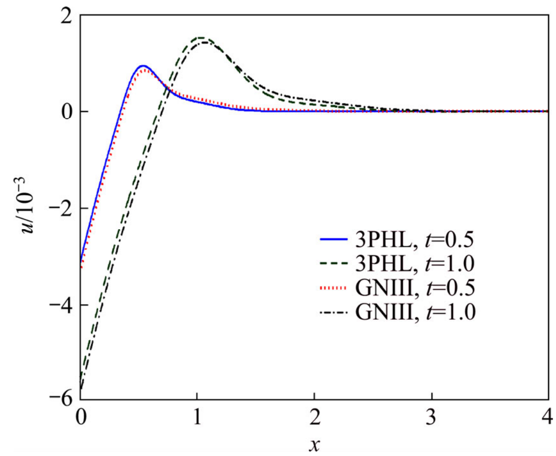

Fig. 2 Displacement distribution with distance x for different theories and time

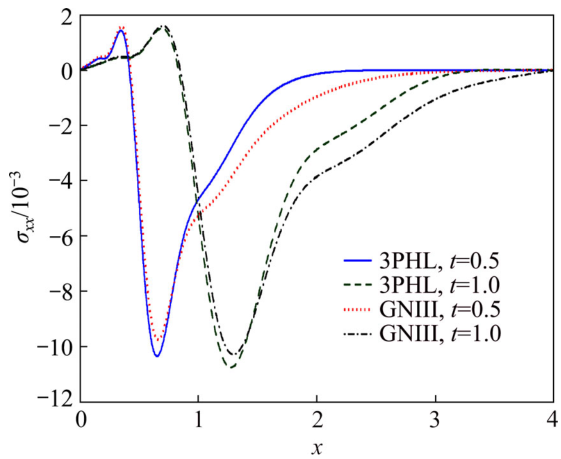

Fig. 3 Stress distribution with distance x for different theories and time

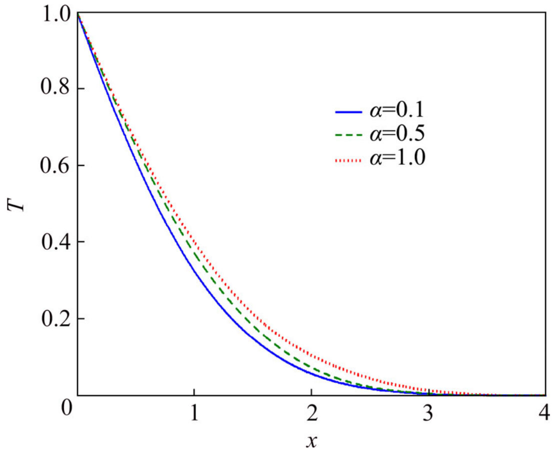

Fig. 4 Temperature distribution with distance x for different α values

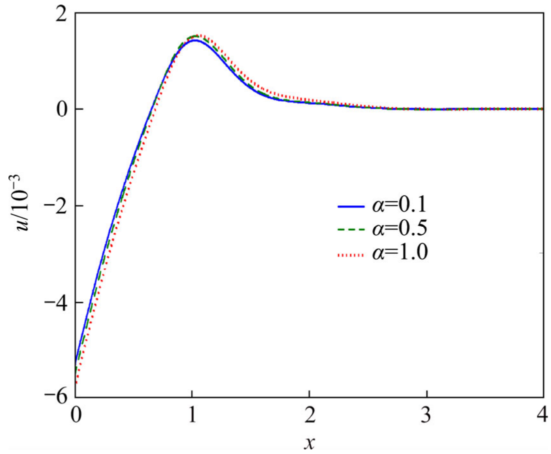

Fig. 5 Displacement distribution with distance x for different α values

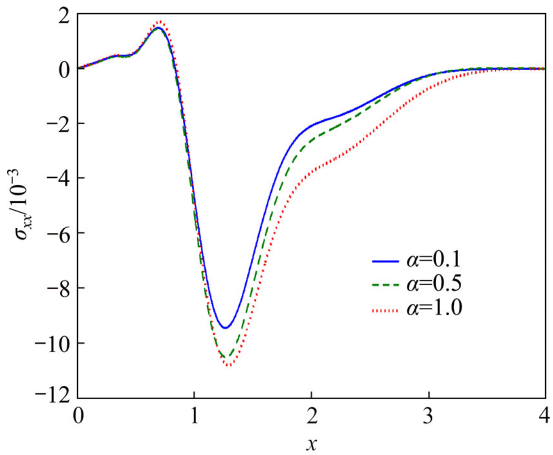

Fig. 6 Stress distribution with distance x for different α values

noticed that as expected, the fractional order has a great effect on the distribution of the field quantities.

In order to study the effect of non-homogeneity on temperature, displacement and stress, we now present our results in the form of graphs (Figures 7−9) at (α=0.5, t=0.5). Figure 7 show that the variation of temperature with distance x. It is seen from Fig. 7 that as the value of

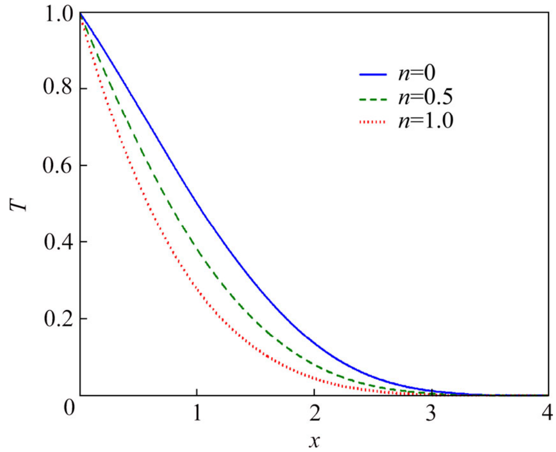

Fig. 7 Temperature distribution with distance x for different n values (α=0.5, t=0.5)

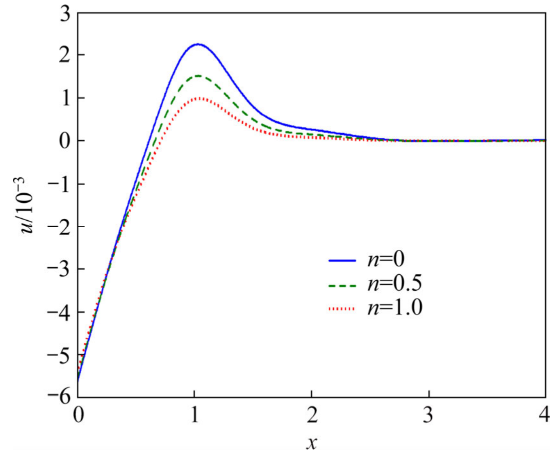

Fig. 8 Displacement distribution with distance x for different n values (α=0.5, t=0.5)

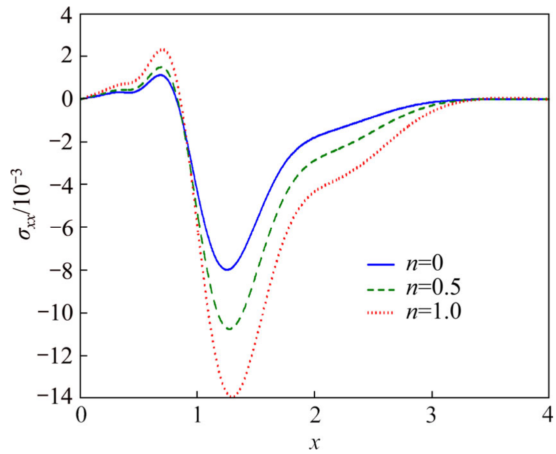

Fig. 9 Stress distribution with distance x for different n values (α=0.5, t=0.5)

n increases the magnitude of the temperature decreases for fixed x and ultimately the temperature approaches to zero value. Figure 8 depicts the variation of displacement versus distance. It is observed that as the value of the non-homogeneity parameter n decreases the peak of thermal displacement also decreases. Figure 9 shows the variation of the stress with distance x. It is clear from Fig. 9 that the variations of stress increase initially for $0\le x\le 0.8$, decrease for $0.8\le x\le 1.35$ and then increase continuously for other values until become steady with the effect of different values of n.

10 Conclusions

1) The time t has significant effects on all the field quantities.

2) Comparisons with predictions are also made in which there is a three-phase lag parameters term, and we have found that this parameters has a significant effect on all the fields and on the speed of the wave propagation.

3) The fractional parameter α has significant effects on all the field quantities.

4) The non-homogeneity parameter n has significant effects on all the field quantities.

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