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RESEARCH ARTICLE

Random gradient-free method for online distributed optimization with strongly pseudoconvex cost functions

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Abstract

This paper focuses on the online distributed optimization problem based on multi-agent systems. In this problem, each agent can only access its own cost function and a convex set, and can only exchange local state information with its current neighbors through a time-varying digraph. In addition, the agents do not have access to the information about the current cost functions until decisions are made. Different from most existing works on online distributed optimization, here we consider the case where the cost functions are strongly pseudoconvex and real gradients of the cost functions are not available. To handle this problem, a random gradient-free online distributed algorithm involving the multi-point gradient estimator is proposed. Of particular interest is that under the proposed algorithm, each agent only uses the estimation information of gradients instead of the real gradient information to make decisions. The dynamic regret is employed to measure the proposed algorithm. We prove that if the cumulative deviation of the minimizer sequence grows within a certain rate, then the expectation of dynamic regret increases sublinearly. Finally, a simulation example is given to corroborate the validity of our results.

Keywords Multi-agent system · Online distributed optimization · Pseudoconvex optimization · Random gradient-free method

1 Introduction

In recent years, distributed optimization has attracted extensive attention in various fields [1–6]. This is due to its wide range of practicability in numerous fields such as distributed resource allocation [1], distributed economic dispatch [2], distributed machine learning [3], distributed coordination control [4], etc.

Distributed optimization in static environments has been widely studied in [7–9]. However, in practical applications, the scenarios that distributed optimization occurs are often dynamic. In recent years, online distributed optimization has been extensively studied [10–14]. For example, in [10], an online distributed push-sum algorithm is proposed for the unconstrained problem, and an online distributed coordinated algorithm based on the gradient descent method is developed in [11]. In [12], an online distributed saddle point algorithm is developed for optimization problem with a global set constraint, an online distributed mirror descent algorithm is proposed in [13], and an online distributed dual averaging algorithm is designed in [14].

It is worth pointing out that all the above works rely on real gradient information. However, it is not feasible or costly to calculate the gradient information accurately in practical applications. For example, in the Internet of Things [15], fog computing can not get the closed expression of delay since its online decision-making needs to adapt to the user preferences and the availability of resources is temporarily unpredictable. Moreover, in bandit optimization [16], agents

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can only observe the values of the cost functions, not the specific cost functions. In these cases, using zero-order information is desirable for distributed optimization. Recently, zeroth-order random online distributed optimization has been investigated in [17–20]. In [17, 18], zeroth-order random online distributed algorithms are proposed, where the zeroth-order information of cost functions is used. Furthermore, in [19, 20], two different random gradient-free algorithms are proposed for online distributed optimization under time-varying networks.

It is worth noting that most of the mentioned articles are applied to convex optimization problems. However, the problems of pseudoconvex optimization exist widely in reality. For example, in computer vision [21], the geometric expressions are usually modeled by pseudoconvex functions. Also, in portfolio planning [22] and fractional programming problems [23], cost functions are commonly formulated as pseudoconvex functions. Pseudoconvex optimization has a wider application range than convex optimization, as it can also be applied to some nonconvex cases. In fact, distributed optimization problems with pseudoconvex functions have only been studied in [24, 25], where real gradient information of cost functions is required.

Motivated by [19, 20, 24–26], we study the online distributed optimization problems with strongly pseudoconvex cost functions and random gradient-free method in this paper. Compared with [24, 25], where agents need to achieve real gradient information, here agents only use estimation of gradients instead of real gradient information to make decisions. To solve this problem, an online distributed algorithm with random gradient-free method is proposed, where a multi-point gradient estimator is used to estimate the gradients of local cost functions. Different from [11–17], which are based on the fact that the cost functions are convex, here the cost functions are considered to be strongly pseudoconvex. In [19, 20], gradient-free method and the convexity of cost functions are used to analyze the convergence of the proposed algorithms. Different from them, here we employ the strong pseudomonotonicity of cost functions and the Karush–Kuhn–Tucker (KKT) condition associated with pseudoconvex optimization to guarantee the effectiveness. We prove that if the graph is B-strongly connected, and the cumulative deviation of the minimizer sequence grows with a certain rate, then the expectation of dynamic regret increases sublinearly.

This paper is organized as follows. In Sect. 2, we formulate the problem and propose an algorithm. In Sect. 3, the main results are presented and the detailed proofs are given. A simulation example is given in Sect. 4. Section 5 is the conclusion of the full paper.

Notations We use $\nabla\psi(u)$ to denote the gradient of function ψ at point u. $\lfloor T\rfloor$ is defined as set $\{0, 1, \dots, T\}$ for any $T \in \mathbb{N}$. For vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$ and matrix $\boldsymbol{W} \in \mathbb{R}^{m\times m}$, we denote $[\boldsymbol{u}]_i$ represents the ith element of $\boldsymbol{u}$, $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}}$, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_W = \langle \boldsymbol{W}\boldsymbol{u}, \boldsymbol{v}\rangle$, $\|\boldsymbol{u}\|_{\boldsymbol{W}}^2 = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{u}$. We use $\mathbb{E}\{\boldsymbol{u}\}$ to denote the expectation of random variable $\boldsymbol{u}$.

2 Problem formulation

2.1 Basic graph theory

A time-varying directed communication graph is defined as $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \boldsymbol{W}(t))$, where $\mathcal{V} = \{1, \dots, n\}$ is a vertex set, $\mathcal{E}(t) \subset \mathcal{V}\times\mathcal{V}$ denotes an edge set and $\boldsymbol{W}(t) = (w_{ij})_{n\times n}$ is a non-negative matrix to represent the weight of adjacent edges. The neighbor set of agent i is defined as $\mathcal{N}_i(t) = \{j \in \mathcal{V}|\ (j, i) \in \mathcal{E}\}$, where agent i can receive information from agent j. For digraph $\mathcal{G}(t)$ and some $B > 0$, $\mathcal{E}_B = \bigcup_{k=tB}^{(t+1)B-1}\mathcal{E}(k)$ denotes the B-edge set. Based on the above conditions, if digraph $\mathcal{G}(t)$ and the edge set $\mathcal{E}_B(t)$ are all strongly connected for any $t \geq 0$. Then, $\mathcal{G}(t)$ is called B-strongly connected graph.

In this paper, the following assumption is made for the communication graph $\mathcal{G}(t)$.

Assumption 1 For any $t \geq 0$, $\mathcal{G}(t)$ is B-strongly connected graph and matrix $\boldsymbol{W}(t)$ is doubly stochastic.

For distributed optimization, $\boldsymbol{W}(t)$ is an essential part in facilitating agents to achieve consensus [25]. For any $t \geq k \geq 0$, we define

$$\begin{cases} \Phi(t,k) = \boldsymbol{W}(t-1)\cdots\boldsymbol{W}(k+1)\boldsymbol{W}(k), & \text{if } t > k; \\ \Phi(t,k) = \mathbf{1}_n, & \text{if } t = k. \end{cases} \tag{1}$$

Lemma 1 [25] *Based on Assumption 1, for any $i, j \in \mathcal{V}$ and $t \geq k$, there exist certain $D > 0$ and $0 < \rho < 1$ satisfying*

$$\left|[\Phi(t,k)]_{ij} - \frac{1}{n}\right| \leq D\rho^{t-k}. \tag{2}$$

2.2 Online distributed optimization

In this paper, we consider a multi-agent system with n agents, where agents exchange local information via graph $\mathcal{G}(t)$. For any $i \in \mathcal{V}$, a sequence of cost functions is given by $\{\psi_i^1, \dots, \psi_i^{\mathrm{T}}\}$, where T is a finite time horizon but is unknown by agents. For any $t \in \lfloor T\rfloor$, $\psi_i^t : \Omega \to \mathbb{R}$ is the cost function of agent i at time t and $\Omega \subset \mathbb{R}^m$. Then, all agents aim to collaboratively solve the following optimization problem with a set constraint, which has the form:

$$\min\ \psi^t(z) = \sum_{i=1}^{n}\psi_i^t(z)\ \text{ s.t. } z \in \Omega. \tag{3}$$

Here, we make some basic assumptions for the problem.

Assumption 2 Constraint set Ω is nonempty, convex, and compact.

Assumption 3 (Strong pseudomonotonicity) For any $m,\ n \in \Omega$, if $\langle \nabla \psi^t(n), m-n\rangle \geq 0$, then $\langle \nabla \psi^t(m), m-n\rangle \geq \beta \|m-n\|^2$ for some $\beta > 0$.

It follows from [27] that if $\nabla \psi^t$ is strong pseudomonotone, then ψ^t is strongly pseudoconvex. The definitions of strongly pseudoconvex functions are given as follows.

Definition 1 For a differentiable function $\psi(\cdot) : \mathbb{R}^m \to \mathbb{R}$ and a convex set $\Omega \in \mathbb{R}^m$, $\psi(\cdot)$ is named a pseudoconvex function if $\langle \nabla \psi(n), m-n\rangle \geq 0$ implies $\psi(m) - \psi(n) \geq 0$ for each pair of different points $m, n \in \Omega$. Moreover, if $\langle \nabla \psi(n), m-n\rangle \geq 0$ implies $\psi(m) - \psi(n) \geq \beta \|m-n\|^2$ for some $\beta > 0$ and each pair of different points $m, n \in \Omega$, then $\psi(\cdot)$ is named a strongly pseudoconvex function on Ω.

Assumption 4 (*Lipschitz continuous gradient*) $\|\nabla \psi_i^t(m) - \nabla \psi_i^t(n)\| \leq L_1\|m-n\|,\ \forall m, n \in \Omega$ for some $L_1 > 0$.

In this paper, we are committed to developing a gradient-free method for solving problem (3). From [28], the smoothed version of ψ^t is defined as

$$\psi_\mu^t(z) = \frac{1}{(2\pi)^{m/2}} \oint_{\mathbb{R}^m} \psi^t(z+\mu\xi) \mathrm{e}^{-\frac{1}{2}\|\xi\|^2} \mathrm{d}\xi, \tag{4}$$

where $\mu > 0$ is the smoothing parameter of function $\psi_\mu^t(z)$. And the multi-point unbiased gradient estimation is defined as

$$\hat{\boldsymbol{g}}(z_i(t)) = \frac{1}{Q_i} \sum_{q_i=1}^{Q_i} \frac{\psi_i^t\big(z_i(t) + \mu_i \xi_{q_i}(t)\big) - \psi_i^t\big(z_i(t)\big)}{\mu_i} \xi_{q_i}(t), \tag{5}$$

where $\hat{\boldsymbol{g}}(z_i(t))$ ia an unbiased estimator of $\nabla \psi_{\mu_i}^t(z_i(t))$, $\mu_i > 0$ is the smoothing parameter and the random sequence $\xi_{q_i}(t)$ is locally generated from an i.i.d. standard Gaussian distribution for any $q \in Q$, where $Q \in \mathbb{N}^+$ is the number of multi-point estimation. Based on results in [29], we know that if $\psi^t(\cdot)$ is differentiable, then $\psi_\mu^t(\cdot)$ is also differentiable, the gradient of $\psi_\mu^t(\cdot)$ is defined as

$$\nabla \psi_\mu^t(z) = \frac{1}{(2\pi)^{m/2}} \oint_{\mathbb{R}^m} \frac{\psi^t\big(z+\mu\xi\big) - \psi^t\big(z\big)}{\mu} \xi \mathrm{e}^{-\frac{1}{2}\|\xi\|^2} \mathrm{d}\xi. \tag{6}$$

Any online algorithm should mimic the performance of its offline counterpart, and the gap between them is called regret [25]. The regret with most stringent offline benchmark is the dynamic regret, whose offline benchmark is to minimize $\psi^t(z)$ at each time. The definition of dynamic regrets is given by

$$\mathcal{R}_i^d(T) = \sum_{t=0}^{T} \psi^t(z_i(t)) - \sum_{t=0}^{T} \psi^t(z^*(t)),\ i \in \mathcal{V}, \tag{7}$$

where $z^*(t) = \mathrm{argmin}_{z\in\Omega} \psi^t(z)$ for any $t \in \lfloor T \rfloor$. An online optimization algorithm is announced "good" if regret (7) increases sublinearly, i.e., $\lim_{T\to\infty} \mathcal{R}_i^d(T)/T = 0$. Unfortunately, using dynamic regret will cause the problem to become insolvable when the minimizer of the cost function changes rapidly. Inspired by [25], the difficulty is described by the deviation of the minimization sequence $\{z^*(t)\}_{t=0}^T$:

$$\Theta_T = \sum_{t=0}^{T} \|z^*(t+1) - z^*(t)\|. \tag{8}$$

2.3 Online distributed gradient-free algorithm

Now, we consider an offline and centralized optimization problem, defined as

$$\min\ \psi(z),\quad \text{s.t. } z \in \Omega, \tag{9}$$

where objective function ψ is strongly pseudoconvex, and constraint set Ω satisfies Assumption 2. The KKT condition of pseudoconvex optimization in terms of variational inequality is given in the following lemma.

Lemma 2 [25] *Suppose function $\psi : \mathbb{R}^m \to \mathbb{R}$ is pseudoconvex and differentiable and constraint set Ω is convex. Then, z^* is a minimum point of ψ on Ω if it can satisfy the following variational inequality*

$$\langle \nabla \psi(z^*), z - z^*\rangle \geq 0,\quad \forall\, z \in \Omega. \tag{10}$$

Based on Lemma 2, we can know problem (9) is solved if there exists a point $z \in \Omega$ satisfying

$$\langle \nabla \psi(z), u - z\rangle \geq 0,\quad \forall\, u \in \Omega. \tag{11}$$

Combining with multi-point unbiased gradient estimation, we construct an auxiliary problem, defined as

$$\min\ z^{\mathrm{T}} P z + \langle \alpha \hat{\boldsymbol{g}}(z_0) - 2Pz_0, z\rangle,\quad s.t.\ z \in \Omega, \tag{12}$$

where $z_0 \in \Omega$, $\alpha > 0$ and matrix $P \in \mathbb{R}^{m\times m}$ is positive definite and symmetric. Based on KKT condition, $z^* \in \Omega$ is the solution to (12) if and only if

$$\langle 2Pz^* + \alpha \hat{\boldsymbol{g}}(z_0) - 2Pz_0, u - z^*\rangle,\quad \forall u \in \Omega. \tag{13}$$

By comparing (11) and (13), we can achieve that z^* is also the solution to (9) when $z^* = z_0$. By replacing z_0 with $z(t)$

and z^* *with* $z(t+1)$*, we propose the following auxiliary optimization strategy*:

$$z(t+1) = \arg\min_{z\in\Omega}\left\{z^{\mathrm{T}}Pz + \langle \alpha(t)\hat{\boldsymbol{g}}(z(t)) - 2Pz(t), z\rangle\right\}. \tag{14}$$

Note that if $z(t+1) = z(t)$ *in* (14)*, then problem* (9) *is solved, which means that the solution of problem* (9) *is the equilibrium point of problem* (14)*. Let matrix* $P = \boldsymbol{I}_m$*, if* ψ_i^t *is a convex function, then we have* $\arg\min_{z\in\Omega}\{z^{\mathrm{T}}Pz + \langle \alpha(t)\hat{\boldsymbol{g}}(z(t)) - 2Pz(t), z\rangle\} = z(t) - \frac{\alpha(t)}{2}\hat{\boldsymbol{g}}(z(t))$*. Thus,* (14) *can be regarded as an extension of the gradient descent algorithm. The detailed proofs of the convergence for algorithm* (14) *can be found in* [30, 31].

To solve problem (3), a random gradient-free online distributed algorithm is proposed

$$\begin{cases} z_i(t+1) \\ = \arg\min_{z\in\Omega}\left\{z^{\mathrm{T}}Pz + \langle \alpha(t)\hat{\boldsymbol{g}}(z_i(t)) - 2Pv_i(t), z\rangle\right\}, \\ v_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}z_j(t), \\ \hat{\boldsymbol{g}}(z_i(t)) \\ = \frac{1}{Q_i}\sum_{q_i=1}^{Q_i} \frac{\psi_i^t(z_i(t)+\mu_i\xi_{q_i}(t)) - \psi_i^t(z_i(t))}{\mu_i}\xi_{q_i}(t), \end{cases} \tag{15}$$

where $z_i(t)$ is the state of agent i at time t with $z_i(0) \in \Omega$ and $\alpha(t) > 0$ is a diminishing learning rate with initial value $\alpha(0) = \alpha_0$. Motivated by the multi-point gradient estimation method, consensus algorithm, and the auxiliary optimization strategy, we propose algorithm (15). When running algorithm (15), each agent updates status at each time only using the information received from its neighbors and its own gradient estimation at past time. Therefore, (15) is an online and distributed algorithm.

3 Main results

In this part, we will elaborate the main results of this paper and give their concrete proof.

Theorem 1 *Based on Assumptions* 1–4*, if the learning rate* $\alpha(t) = \frac{c}{\sqrt{t+1}}$ *for some* $c > 0$*, then for any* $i \in \mathcal{V}$ *and* $T \in \mathbb{N}$*, the following inequality holds*

$$\mathbb{E}\{\mathcal{R}_i^d(T)\} \le nL_0\sqrt{\Gamma + \frac{F_1T}{\upsilon\ln(T+1)}\sum_{i=1}^{n}\mu_i + \frac{16hM\Theta_T}{c\mu\ln 2}} \cdot \left((T+1)^{\frac{3}{4}}\sqrt{\ln(T+1)}\right), \tag{16}$$

where $\Gamma = \frac{(4\hat{\rho}_1/(c\upsilon)+2\hat{\mathcal{K}}_1))+3c^2(4\hat{\rho}_2/(c\upsilon)+2\hat{\mathcal{K}}_2)}{\rho(1-\rho)\ln 2} + \frac{6ncL_0^2}{\upsilon\varepsilon} + \frac{4d}{c\upsilon\ln 2}$, $\hat{\mathcal{K}}_1 = \mathcal{C}^2 + \frac{\mathcal{C}L_0Dn\sqrt{m}}{\varepsilon(1-\rho)}(m+4)L_0$, $\hat{\rho}_2 = \frac{n(5M\mathcal{K}_2+\sqrt{m+4}L_0(kL_1+L_0)Dn\sqrt{m})}{\varepsilon}$, $\hat{\mathcal{K}}_2 = \frac{mn^2H^2}{4\varepsilon^2(1-\rho)}(m+4)^2L_0^2$, $\hat{\rho}_1 = n(5M\hat{\mathcal{K}}_1 + (kL_1+L_0)c\mathcal{C})$, $F_1 = L_1kc(m+3)^{\frac{3}{2}}$, $L_0 = \sup_{t\in\lfloor T\rfloor, i\in\mathcal{V}, z\in\Omega}\|\nabla\psi_i^t(z)\|$, $d = Mk^2$, $\varepsilon = \lambda_{\min}(P)$, $\mathcal{C} = D\sqrt{m}\sum_{i=1}^{n}\|z_i(0)\|_1$, $\beta_1 = \sup_{t\in\lfloor T\rfloor, i\in\mathcal{V}, z\in\Omega}\|\hat{\boldsymbol{g}}(z_i(t))\|$, $M = n\lambda_{\max}(P)$, $L_1 = \sup_{t\in\lfloor T\rfloor, i\in\mathcal{V}, z\in\Omega}\|\nabla^2 f_i^t(z)\|$*, and* $h = \sup_{z\in\Omega}\|z\|$.

By Theorem 1, we know that both Θ_T and μ_i play important roles in the bound of dynamic regret expectation. Note that if $\mu_i = T^{-\frac{1}{2}-\varrho}$ for any $\varrho > 0$ and Θ_T grows sublinearly with $\frac{\sqrt{T+1}}{\ln(T+1)}$, then $\mathbb{E}\{\mathcal{R}_i^d(T)\}$ increase sublinearly with T, which implies the performance of online distributed algorithm (15) is "good". Therefore, algorithm (15) can be employed to solve some strongly pseudoconvex optimization problems, where the specific gradient information is unavailable or expensive to obtain. If the fluctuation of minimizer sequence $\{z^*(t)\}_{t=0}^T$ is dramatical, Θ_T might become linear with $\frac{\sqrt{T+1}}{\ln(T+1)}$, then the problem becomes insolvable. It is a natural phenomenon, even in online convex optimization.

Before proving Theorem 1, some useful lemmas are need to be presented. First, we use $\mathcal{F}_t$ to denote the σ-field generated by the entire history of the random variable as

$$\mathcal{F}_t = \{\mathrm{col}(z_i(s))_{i\in\mathcal{V}}, \mathrm{col}(\varsigma_i(s))_{i\in\mathcal{V}}, s = 0,\ldots,t\}. \tag{17}$$

Based on the above definition, the following lemma can be achieved.

Lemma 3 [19, 28] *If* $\nabla\psi_i^t$ *is* L_1*-Lipschitz continuous on* Ω*, then*

$$\begin{aligned} &\text{(a)}\ \mathbb{E}\{\|\hat{\boldsymbol{g}}(z_i(t))\|^2|\mathcal{F}_t\} \le (m+4)^2L_0^2. \\ &\text{(b)}\ \|\nabla\psi_{\mu_i}^t(z_i(t)) - \nabla\psi_i^t(z_i(t))\| \le \frac{\mu}{2}L_1(m+3)^{\frac{3}{2}}. \\ &\text{(c)}\ \mathbb{E}\{\|\hat{\boldsymbol{g}}(z_i(t)) - \nabla\psi_{\mu_i}^t(z_i(t))\||\mathcal{F}_t\} = 0. \end{aligned} \tag{18}$$

To prove Theorem 1, first, we present the following lemma, which gives the upper bound of the discrepancy between each agent's state and their average state at each update.

Lemma 4 *Based on Assumptions* 1–4*, for any* $i \in \mathcal{V}$*,*

$$\|z_i(t) - \bar{z}(t)\| \le \mathcal{C}\rho^t + \frac{nD\sqrt{m}\beta_1}{\varepsilon}\sum_{\gamma=0}^{t}\rho^{t-\gamma}\alpha(\gamma) \tag{19}$$

and

$$\|z_i(t) - \bar{z}(t)\|^2 \le \mathcal{K}_1\rho^t + \mathcal{K}_2\sum_{\gamma=0}^{t}\rho^{t-\gamma}(\alpha(\gamma))^2, \tag{20}$$

where $\bar{z}(t) = \frac{1}{n}\sum_{i=1}^{n} z_i(t)$, $\mathcal{K}_1 = \mathcal{C}^2 + \frac{\mathcal{C}Dn\sqrt{m}\beta_1}{\varepsilon(1-\rho)}$, $\mathcal{K}_2 = \frac{m(nD\beta_1)^2}{4\varepsilon^2(1-\rho)}$.

Proof See Appendix 1. □

Next, we give the expectation for the upper bound of the cumulative square error between agents' optimal state and their average state at each iteration time.

Lemma 5 *Under Assumptions 1–4, we have*

$$\mathbb{E}\Big\{\sum_{t=0}^{T}\|\bar{z}(t)-z^*(t)\|^2\Big\} \le \frac{2d}{u\alpha(T)} + \frac{8hM\Theta_T}{u\alpha(T)} + \frac{2\hat{\rho}_1}{u\alpha(T)}\sum_{t=0}^{T}\rho^t + \frac{2\hat{\rho}_2}{u\alpha(T)}\sum_{t=0}^{T}\sum_{\gamma=0}^{t+1}\rho^{t-\gamma}(\alpha(\gamma))^2 + \frac{F_1T\sum_{i=1}^{n}\mu_i}{u\alpha(T)} + \frac{nL_0}{\varepsilon u\alpha(T)}\sum_{t=0}^{T}(\alpha(t))^2. \quad (21)$$

Proof See Appendix 1. □

Proof of Theorem 1 Based on Lemmas 4–5, we can achieve

$$\begin{aligned}\mathbb{E}\Big\{\sum_{t=0}^{T}\|z(t)-z^*(t)\|^2\Big\} &\le 2\mathbb{E}\Big\{\sum_{t=0}^{T}\|\bar{z}(t)-z^*(t)\|^2\Big\} + 2\mathbb{E}\Big\{\sum_{t=0}^{T}\|z_i(t)-\bar{z}(t)\|^2\Big\}\\ &\le \Big(\frac{4\hat{\rho}_1}{u\alpha(T)} + 2\hat{\mathcal{K}}_1\Big)\sum_{t=0}^{T}\rho^t + \frac{4d}{u\alpha(T)} + \frac{16hM\Theta_T}{u\alpha(T)}\\ &\quad + \Big(\frac{4\hat{\rho}_2}{u\alpha(T)} + 2\hat{\mathcal{K}}_2\Big)\sum_{t=0}^{T}\sum_{\gamma=0}^{t+1}\rho^{t-\gamma}(\alpha(\gamma))^2\\ &\quad + \frac{2F_1T\sum_{i=1}^{n}\mu_i}{u\alpha(T)} + \frac{2nL_0}{\varepsilon u\alpha(T)}\sum_{t=0}^{T}(\alpha(t))^2. \end{aligned}\quad (22)$$

Let $\alpha(t) = \frac{c}{\sqrt{t+1}}$, there holds

$$\begin{aligned}\sum_{t=0}^{T}\sum_{\gamma=0}^{t+1}\rho^{t-\gamma}(\alpha(\gamma))^2 &= \sum_{\gamma=1}^{T+1}\sum_{t=\gamma-1}^{T}\rho^{t-\gamma}(\alpha(\gamma))^2 + \sum_{t=0}^{T}\rho^t(\alpha(0))^2\\ &\le \Big(\sum_{\gamma=1}^{T+1}(\alpha(\gamma))\Big)^2\Big(\sum_{t=0}^{T}\rho^t\Big)^2 + \sum_{t=0}^{T}\rho^t(\alpha(0))^2\\ &\le \frac{c^2(1+\rho+\ln(T+1))}{(1-\rho)\rho} \le \frac{3c^2\ln(T+1)}{(1-\rho)\rho\ln 2}. \end{aligned}\quad (23)$$

Based on Jensen's inequality, we have

$$\Big(\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|\Big)^2 \le (T+1)\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|^2. \quad (24)$$

Now, by taking expectation on both sides of inequality (24), we can obtain

$$\mathbb{E}\Big\{\Big(\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|\Big)^2\Big\} \le (T+1)\mathbb{E}\Big\{\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|^2\Big\}. \quad (25)$$

Based on (22), (23), and (25), we have

$$\begin{aligned}\mathbb{E}\Big\{\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|\Big\} &\le \sqrt{(T+1)\mathbb{E}\Big\{\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|^2\Big\}}\\ &\le \sqrt{\Gamma + \frac{2(T+1)F_1\sum_{i=1}^{n}\mu_i}{\upsilon c\ln(T+1)} + \frac{16hM\Theta_T}{c\mu\ln 2}}\\ &\quad\cdot (T+1)^{\frac{3}{4}}\sqrt{\ln(T+1)}. \end{aligned}\quad (26)$$

Note that $\nabla\psi_i^t$ is bounded by L_0 for any $i \in \mathcal{V}$ and $t \in \lfloor T\rfloor$. Thus,

$$\begin{aligned}\mathbb{E}\Big\{\mathcal{R}_i^d(T)\Big\} &= \mathbb{E}\Big\{\sum_{t=0}^{T}(\psi^t(z_i(t)) - \psi^t(z^*(t))\Big\}\\ &\le \mathbb{E}\Big\{\sum_{t=0}^{T}\sum_{j=1}^{n}\|\psi_j^t(z_i(t)) - \psi_j^t(z^*(t)\|\Big\}\\ &\le \mathbb{E}\Big\{nL_0\sum_{t=0}^{T}\|z_i(t)-z^*(t)\|\Big\}. \end{aligned}\quad (27)$$

Substituting (26) into (27) yields (16). Thus, Theorem 1 is proved. □

4 A simulation example

In this part, we illustrate the validity of proposed algorithm by using a numerical example. Assume a multi-agent system with six agents, labeled as $\mathcal{V} = \{1, 2, \dots, 6\}$. Each agent exchange information with its neighbors via a time-varying digraph as shown in Fig. 1. For any $i \in \mathcal{V}$, cost function of agent i is given by

$$\psi_i(z) = p_i z^3 + q_i(t)z,$$

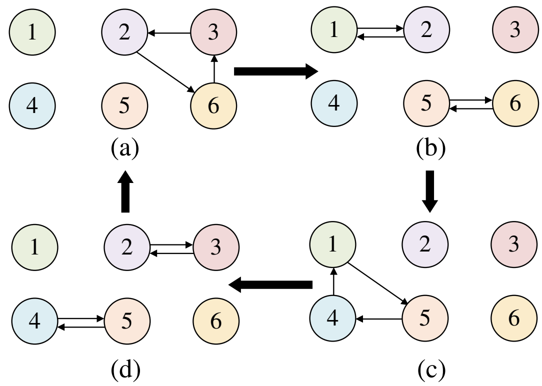

Fig. 1 The time-varying directed graph sequence

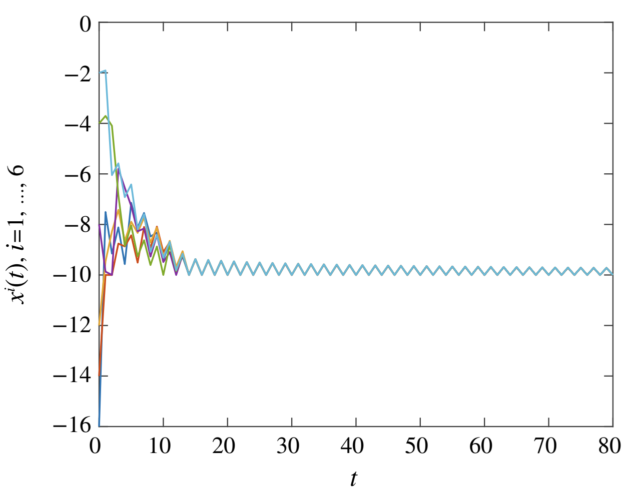

Fig. 2 The state trajectories of all agents under algorithm (15)

where $z \in \Omega$. In this simulation, parameters are selected as $p_1 = 0.6$, $p_2 = 0.8$, $p_3 = p_6 = 1$, $p_4 = 0.5$, $p_5 = 0.1$, $q_i(t)$ is randomly selected from $[-i, i]$ and subject to a uniform distribution. Moreover, $\Omega = \{z| -10 \leq z \leq -1\}$ is the constraint set of the objective function. Initial states of agents are selected as $z_1(0) = 0.3$, $z_2(0) = -0.5$, $z_3(0) = -0.5$, $z_4(0) = 0.4$, $z_5(0) = -0.1$ and $z_6(0) = -0.2$. Algorithm (15) is applied to solving this problem. Let $\alpha(t) = 1/\sqrt{200t + 500}$, $\mu_i = 0.05$ and $Q = 1$. The trajectories of $x_i(t), i = 1, \ldots, 6$ are represented in Fig. 2 and Fig. 3 displays the trajectories of the average regrets of all agents. From Fig. 2, it can be clearly seen that all agents' states gradually approach to the optimal solution $z^*(t)$. Furthermore, by observing Fig. 3, one can find that average regret of each agent gradually diminishes to zero. These observations indicate the correctness of achieved theoretical results.

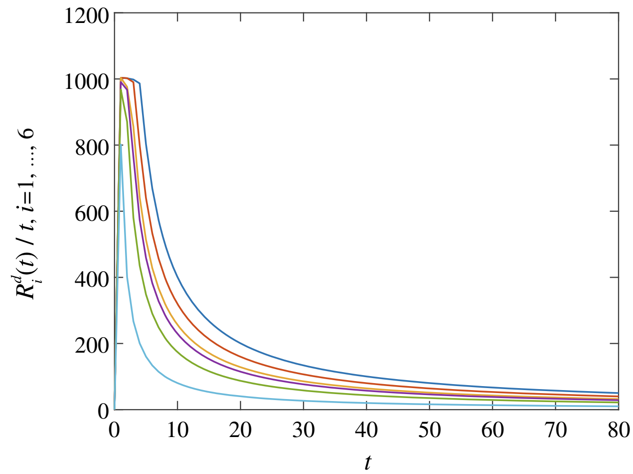

Fig. 3 The average regret trajectories of all agents under algorithm (15)

5 Conclusions

This paper deals with the zeroth-order online distributed optimization problem where the sum of local cost functions is strongly pseudoconvex. To solve this problem, a random gradient-free algorithm based on multi-point gradient estimation method and auxiliary optimization strategy is proposed. Under this algorithm, each agent updates state at each time only using the information received from its neighbors at the current moment and the information of its own gradient estimation at previous time.The proposed algorithm is measured by the dynamic regret. The results indicate that if the communication graph is B-strongly connected, and the deviation of the minimizer sequence grows with a certain rate, then the expectation of dynamic regret increases with some sublinear bound. The simulation example in the previous section has verified its validity. The cases of communication delays and packet losses will be our future work, which means that there will be new difficulty for the online distributed pseudoconvex optimization problems.

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Appendix

Proof of Lemma 4

Note that $z_i(t+1)$ generated by algorithm (15) is the solution to the following optimization:

$$\min_{z\in\Omega} z^{\mathrm{T}} P z + \langle \alpha(t)\hat{\boldsymbol{g}}(z_i(t)) - 2P\nu_i(t), z\rangle. \tag{28}$$

Then according to the KKT condition, we can have

$$\langle z_i(t+1) - \nu_i(t), z_i(t+1) - z\rangle_P \le \frac{\alpha(t)}{2}\langle \hat{\boldsymbol{g}}(z_i(t)), z_i(t+1) - z\rangle \tag{29}$$

for any $z \in \Omega$. Note that $\nu_i(t) \in \Omega$. Let $z = \nu_i(t)$, based on $2\varepsilon\|z_i(t+1) - \nu_i(t)\|^2 \le \|z_i(t+1) - \nu_i(t)\|_P^2$ and $\|\hat{\boldsymbol{g}}(z_i(t))\| \le \beta_1$, the following inequality holds

$$2\varepsilon\|z_i(t+1) - \nu_i(t)\|^2 \le \frac{\beta_1\alpha(t)}{2}\|z_i(t+1) - \nu_i(t)\|. \tag{30}$$

For any $i \in \mathcal{V}$, $r_i(t)$ is defined as $r_i(t) = z_i(t+1) - \nu_i(t)$. From inequality (30), one can obtain $\|r_i(t)\| \le \beta_1\alpha(t)/(2\varepsilon)$. Moreover,

$$z_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} w_{ij} z_j(t) + r_i(t). \tag{31}$$

For any $i \in \mathcal{V}$, vector $\tilde{z}_s(t) \in \mathbb{R}^n$ and $\tilde{r}_s(t) \in \mathbb{R}^n$ are defined as a superposition of the sth entry of $z_i(t)$ and $r_i(t)$, respectively. Then, it follows that

$$\tilde{z}_s(t+1) = \boldsymbol{W}(t)\tilde{z}_s(t) + \tilde{r}_s(t), \tag{32}$$

which means the following equation holds

$$\tilde{z}_s(t) = \Phi(t,0)\tilde{z}_s(0) + \sum_{\gamma=1}^{t} \Phi(t,\gamma)\tilde{r}_s(\gamma - 1), \tag{33}$$

where the definition of $\Phi(t,k)$ can be found in (1). Since matrix $\Phi(t,k)$ is doubly stochastic, Eq. (33) can be further expressed

$$\mathbf{1}^{\mathrm{T}}\tilde{z}_s(t) = \mathbf{1}^{\mathrm{T}}\tilde{z}_s(0) + \sum_{\gamma=1}^{t} \mathbf{1}^{\mathrm{T}}\tilde{r}_s(\gamma - 1). \tag{34}$$

Combining (33) and (34), one can obtain

$$\begin{aligned}
&\left|[\tilde{z}_s(t)]_i - \frac{1}{n}\mathbf{1}^{\mathrm{T}}\tilde{z}_s(t)\right| \\
&\le \left|\left([\Phi(t,0)]_{i\cdot} - \frac{1}{n}\mathbf{1}^{\mathrm{T}}\right)\tilde{z}_s(0)\right| \\
&\quad + \sum_{\gamma=1}^{t}\left|\left([\Phi(t,\gamma)]_{i\cdot} - \frac{1}{n}\mathbf{1}^{\mathrm{T}}\right)\tilde{r}_s(\gamma-1)\right| \\
&\le \max_{1\le j\le n}\left|[\Phi(t,0)]_{ij} - \frac{1}{n}\right|\|\tilde{z}_s(0)\|_1 \frac{n\beta_1}{2\varepsilon} \\
&\quad \cdot \sum_{\gamma=1}^{t}\alpha(\gamma-1)\max_{1\le j\le n}\left|[\Phi(t,\gamma)]_{ij} - \frac{1}{n}\right|
\end{aligned} \tag{35}$$

for any $i \in \mathcal{V}$. Based on inequality (2), one has

$$\begin{aligned}
&\left|[\tilde{z}_s(t)]_i - \frac{1}{n}\mathbf{1}^{\mathrm{T}}\tilde{z}_s(t)\right| \\
&\le D\rho^t\|\tilde{z}_s(0)\|_1 + \frac{nD\beta_1}{2\varepsilon}\sum_{\gamma=1}^{t}\rho^{t-\gamma}\alpha(\gamma-1) \\
&\le D\rho^t\|\tilde{z}_s(0)\|_1 + \frac{nD\beta_1}{2\varepsilon}\sum_{\gamma=0}^{t}\rho^{t-\gamma}\alpha(\gamma).
\end{aligned} \tag{36}$$

This inequality directly proves the validity of inequality (19). □

Furthermore, we can have

$$\begin{aligned}
&E\|z_i(t) - \bar{z}(t)\| \\
&\le \mathcal{C}\rho^t + \frac{nD\sqrt{m}}{2\varepsilon}(m+4)L_0\sum_{\gamma=1}^{t}\rho^{t-\gamma}\alpha(\gamma).
\end{aligned} \tag{37}$$

Note that $0 < \rho < 1$ and learning rate $\alpha(t)$ is non-increasing, we can obtain

$$\begin{aligned}
&\|z_i(t+1) - \bar{z}(t)\|^2 \\
&\le \left(\mathcal{C}^2 + \frac{\mathcal{C}nD\sqrt{m}\alpha_0\beta_1}{\varepsilon(1-\rho)}\right)\rho^t \\
&\quad + \frac{m(n\beta_1 D)^2}{4\varepsilon^2}\left(\sum_{\gamma=0}^{t}\rho^{t-\gamma}\alpha(\gamma)\right)^2.
\end{aligned} \tag{38}$$

By Cauchy–Schwarz inequality, we can obtain

$$\begin{aligned}
&\left(\sum_{\gamma=0}^{t}\rho^{t-\gamma}\alpha(\gamma-1)\right)^2 \\
&\le \left(\sum_{\gamma=0}^{t}\rho^{t-\gamma}\right)\left(\sum_{\gamma=0}^{t}\rho^{t-\gamma}(\alpha(\gamma))^2\right) \\
&\le \frac{1}{1-\rho}\sum_{\gamma=0}^{t}\rho^{t-\gamma}(\alpha(\gamma))^2.
\end{aligned} \tag{39}$$

Combining (38) and (39), we can obtain the validity of inequality (20). □

Furthermore, we can achieve

$$\begin{aligned}
&\mathbb{E}\|z_i(t) - \bar{z}(t)\|^2 \\
&\le \mathcal{K}_1\rho^t + \mathcal{K}_2(m+4)^2L_0^2\sum_{\gamma=1}^{t}\rho^{t-\gamma}\alpha(\gamma)^2.
\end{aligned} \tag{40}$$

Proof of Lemma 5

To prove Lemma 5, an auxiliary lemma is given as follows.

Lemma 6 *Based on Assumptions 1–4, for $t \in \lfloor T\rfloor$ and any $u \in \mathbb{R}^m$,*

$$\sum_{i=1}^{n}\langle z_i(t) - \nu_i(t), u - z_i(t+1)\rangle_P$$

$$\leq \frac{5nL}{2}\mathcal{K}_1\rho^t + \frac{5nL}{2}\mathcal{K}_2\sum_{\gamma=0}^{t+1}\rho^{t-\gamma}(\alpha(\gamma))^2. \quad (41)$$

Proof of Lemma 6 Note that

$$\begin{aligned}
&\sum_{i=1}^{n}\langle z_i(t)-v_i(t), u-z_i(t+1)\rangle_P \\
&= \sum_{i=1}^{n}\langle z_i(t)-v_i(t), u-\bar{z}(t+1)\rangle_P \\
&\quad + \sum_{i=1}^{n}\langle z_i(t)-v_i(t), \bar{z}(t+1)-z_i(t+1)\rangle_P \\
&\leq \left\langle \sum_{i=1}^{n}(z_i(t)-v_i(t)), u-\bar{z}(t+1)\right\rangle_P \\
&\quad + M\sum_{i=1}^{n}\|z_i(t)-v_i(t)\|\|\bar{z}(t+1)-z_i(t+1)\|. \quad (42)
\end{aligned}$$

It is not hard to prove that $\sum_{i=1}^{n} P v_i(t) = \sum_{i=1}^{n} P z_i(t)$, Then, $\left\langle \sum_{i=1}^{n}\left(z_i(t+1)-v_i(t)\right), u-\bar{z}(t+1)\right\rangle_P = 0$. Using Young's inequality and the fact that $\sum_{i=1}^{n}\|z_i(t)-v_i(t)\|^2 \leq 4\sum_{i=1}^{n}\|z_i(t)-\bar{z}(t)\|^2$, we can have

$$\begin{aligned}
&\sum_{i=1}^{n}\langle z_i(t+1)-v_i(t), u-z_i(t+1)\rangle_P \\
&\leq 2M\sum_{i=1}^{n}\|z_i(t)-\bar{z}(t)\|^2 \\
&\quad + \frac{M}{2}\sum_{i=1}^{n}\|\bar{z}(t+1)-z_i(t+1)\|^2. \quad (43)
\end{aligned}$$

Combining with (20) in Lemma 4, it immediately implies (41). □

Proof of Lemma 5 Note that

$$\begin{aligned}
&\frac{1}{2}\|z^*(t+1)-z_i(t+1)\|_P^2 - \frac{1}{2}\|z^*(t)-z_i(t)\|_P^2 \\
&= \langle z_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\rangle_P - \frac{1}{2}\|z_i(t) \\
&\quad - z_i(t+1)\|_P^2 + \sum_{i=1}^{n}\langle \frac{1}{2}\left(z^*(t+1)+z^*(t)\right) \\
&\quad - z_i(t+1), z^*(t+1)+z^*(t)\rangle_P. \quad (44)
\end{aligned}$$

Now, we denote $\mathcal{O}(t) = \frac{1}{2}\sum_{i=1}^{n}\|z_i(t)^*-z_i(t)\|_P^2$. Then,

$$\begin{aligned}
&\nabla\mathcal{O}(t) \\
&= \mathcal{O}(t+1)-\mathcal{O}(t) \\
&= \sum_{i=1}^{n}\left\langle z_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\right\rangle_P \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\|z_i(t)-z_i(t+1)\|_P^2 + \sum_{i=1}^{n}\left\langle \frac{1}{2}(z^*(t+1)\right. \\
&\quad \left. + z^*(t)) - z_i(t+1), z^*(t+1)-z^*(t)\right\rangle_P \\
&\leq \sum_{i=1}^{n}\left\langle z_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\right\rangle_P \\
&\quad - \sum_{i=1}^{n}\frac{\varepsilon}{2}\|z_i(t)-z_i(t+1)\|^2 \\
&\quad + 2hM\|z^*(t+1)-z^*(t)\|^2. \quad (45)
\end{aligned}$$

Based on KKT condition, one can obtain

$$\begin{aligned}
&\langle v_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\rangle_P \\
&\leq \frac{\alpha(t)}{2}\langle \hat{\boldsymbol{g}}(z_i(t)), z^*(t)-z_i(t+1)\rangle. \quad (46)
\end{aligned}$$

Using (41) in Lemma 6, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\langle z_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\rangle_P \\
&= \sum_{i=1}^{n}\langle v_i(t)-z_i(t+1), z^*(t)-z_i(t+1)\rangle_P \\
&\quad + \sum_{i=1}^{n}\langle z_i(t)-v_i(t), z^*(t)-z_i(t+1)\rangle_P \\
&\leq \sum_{i=1}^{n}\frac{\alpha(t)}{2}\langle \hat{\boldsymbol{g}}(z_i(t)), z^*(t)-z_i(t+1)\rangle_P \\
&\quad + \frac{5Mn}{2}\mathcal{K}_1\rho^t + \frac{5Mn}{2}\mathcal{K}_2\sum_{\gamma=0}^{t+1}\rho^{t-\gamma}\alpha((\gamma))^2. \quad (47)
\end{aligned}$$

Based on Lemma 2 and Assumption 3, the following inequality can be achieved

$$\left\langle \sum_{i=1}^{n}\nabla\psi_i^t(\bar{z}(t)), \bar{z}(t)-z^*(t)\right\rangle \geq \frac{\upsilon}{2}\|\bar{z}(t)-z^*(t)\|^2. \quad (48)$$

Note that $\|\nabla^2\psi_i^t\| \leq L_1$ for any $z \in \Omega$, it implies $\|\nabla\psi_i^t(z_i(t))-\nabla\psi_i^t(\bar{z}(t))\| \leq L_1\|z_i(t)-\bar{z}(t)\|$ for any $i \in \mathcal{V}$. By combining the boundedness of Ω in Assumption 2, one can obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\langle \nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t)\rangle \\
&= \sum_{i=1}^{n}\langle \nabla\psi_i^t(z_i(t))-\nabla\psi_i^t(\bar{z}(t)), z^*(t)-z_i(t)\rangle \\
&\quad + \sum_{i=1}^{n}\langle \nabla\psi_i^t(\bar{z}(t)), \bar{z}(t)-z_i(t)\rangle \\
&\quad - \sum_{i=1}^{n}\langle \nabla\psi_i^t(\bar{z}(t)), \bar{z}(t)-z^*(t)\rangle \\
&\leq \sum_{i=1}^{n}(\kappa L_1+L_0)\|z_i(t)-\bar{z}(t)\| - \frac{\upsilon}{2}\|\bar{z}(t)-z^*(t)\|^2.
\end{aligned}$$
(49)

Thus, the following inequality can be achieved

$$
\begin{aligned}
&\alpha(t)\sum_{i=1}^{n}\langle\nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&=\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t)\rangle\\
&\quad+\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_i^t(z_i(t)), z_i(t)-z_i(t+1)\rangle\\
&\le-\frac{\upsilon\alpha(t)}{2}\|\bar{z}(t)-z^*(t)\|^2\\
&\quad+\sum_{i=1}^{n}L_0\alpha(t)\|z_i(t)-z_i(t+1)\|\\
&\quad+\sum_{i=1}^{n}(\kappa L_1+L_0)\alpha(t)\|z_i(t)-\bar{z}(t)\|\\
&\le-\frac{\upsilon\alpha(t)}{2}\|\bar{z}(t)-z^*(t)\|^2\\
&\quad+\sum_{i=1}^{n}(\kappa L_1+L_0)\alpha(t)\|z_i(t)-\bar{z}(t)\|\\
&\quad+\sum_{i=1}^{n}\frac{\varepsilon}{2}\|z_i(t)-z_i(t+1)\|^2+\frac{n(L_0\alpha(t))^2}{2\varepsilon}.
\end{aligned}\tag{50}
$$

Moreover, we can also achieve that

$$
\begin{aligned}
&\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_{\mu^i}^t(z_i(t))-\nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&\le\sum_{i=1}^{n}\alpha(t)k\|\nabla\psi_{\mu^i}^t(z_i(t))-\nabla\psi_i^t(z_i(t))\|\\
&\le\frac{L_1ck(m+3)^{\frac{3}{2}}\sum_{i=1}^{n}\mu_i}{2}.
\end{aligned}\tag{51}
$$

Note that

$$
\begin{aligned}
&\sum_{i=1}^{n}\alpha(t)\langle\hat{\boldsymbol{g}}(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&=\sum_{i=1}^{n}\alpha(t)\langle\hat{\boldsymbol{g}}(z_i(t))-\nabla\psi_{\mu^i}^t(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&\quad+\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&\quad+\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_{\mu^i}^t(z_i(t))-\nabla\psi_i^t(z_i(t)), z^*(t)\\
&\quad-z_i(t+1)\rangle.
\end{aligned}\tag{52}
$$

Substituting (50) and (51) into (52), using (c) in Lemma 3, then taking the total expectation for the achieved inequality after taking the conditional expectation over $\mathcal{F}_t$, so we can obtain

$$
\begin{aligned}
&\mathbb{E}\{\sum_{i=1}^{n}\alpha(t)\langle\hat{\boldsymbol{g}}(z_i(t)), z^*(t)-z_i(t+1)\rangle\}\\
&=\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_i^t(z_i(t)), z^*(t)-z_i(t+1)\rangle\\
&\quad+\sum_{i=1}^{n}\alpha(t)\langle\nabla\psi_{\mu^i}^t(z_i(t))-\nabla\psi_i^t(z_i(t)), z^*(t)\\
&\quad-z_i(t+1)\rangle\\
&\le\frac{L_1kc(m+3)^{\frac{3}{2}}\sum_{i=1}^{n}\mu_i}{2}-\frac{\upsilon\alpha(t)}{2}\|\bar{z}(t)-z^*(t)\|^2\\
&\quad+\sum_{i=1}^{n}(\kappa L_1+L_0)\alpha(t)\|z_i(t)-\bar{z}(t+1)\|\\
&\quad+\sum_{i=1}^{n}\frac{\varepsilon}{2}\|z_i(t)-z_i(t+1)\|^2+\frac{n(L_0\alpha(t))^2}{2\varepsilon}.
\end{aligned}\tag{53}
$$

By (45), (47), and (53), using (25), we have

$$
\begin{aligned}
&\mathbb{E}\{\nabla\mathcal{O}(t)\}\\
&\le-\frac{\upsilon\alpha(t)}{4}\mathbb{E}\{\|\bar{z}(t)-z^*(t)\|^2\}+\frac{F_1\sum_{i=1}^{n}\mu_i}{4}\\
&\quad+\frac{\hat{\rho}_1}{2}\rho^t+\frac{n(L_0\alpha(t))^2}{2\varepsilon}+\frac{\hat{\rho}_2}{2}\sum_{\gamma=1}^{t+1}\rho^{t-\gamma}(\alpha(\gamma))^2\\
&\quad+2hL\|z^*(t+1)-z^*(t)\|.
\end{aligned}\tag{54}
$$

Due to $\mathcal{O}(t)\ge 0$ for any $t\in\lfloor T\rfloor$, we can have $-\sum_{t=0}^{T}\nabla\mathcal{O}(t)=\mathcal{O}(0)-\mathcal{O}(T)\le\mathcal{O}(0)\le d/2$. By summing from $t=0$ to T on both sides of inequality (54), the validity of Lemma 5 is verified. □

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