RESEARCH ARTICLE

Optimal LQG control for discrete time-varying system with multiplicative noise and multiple state delays

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Abstract

This paper is concerned with the optimal linear quadratic Gaussian (LQG) control problem for discrete time-varying system with multiplicative noise and multiple state delays. The main contributions are twofolds. First, in virtue of Pontryagin's maximum principle, we solve the forward and backward stochastic difference equations (FBSDEs) and show the relationship between the state and the costate. Second, based on the solution to the FBSDEs and the coupled difference Riccati equations, the necessary and sufficient condition for the optimal problem is obtained. Meanwhile, an explicit analytical expression is given for the optimal LQG controller. Numerical examples are shown to illustrate the effectiveness of the proposed algorithm.

Keywords LQG control · Discrete time-varying system · Multiplicative noise · Multiple state delays

1 Introduction

The control problem for time-delay systems have received extensively attention since 1950s because of its wide applications in networked control system, intelligent cruise control system, finance, cable-driven manipulators and so on [1–6]. There have been a lot of researches on the optimal control and stabilization problem of time-delay systems in recent years, and many results have been surveyed, which concern with single input/state delay or multiple input/state delays [7–10]. For example, Yue et al. [7] proposed a Lyapunov–Krasovskii functional approach to design the delayed feedback controller of uncertain systems with time-varying input delay, by introducing some relaxation matrices and turning parameters. Lee et al. [9] studied the robust H_∞ control problem for uncertain linear systems with a state-delay. Based on the obtained delay-dependent bounded real lemma, the delay-dependent condition for the existence of a robust controller was presented. Due to the existence of multiple

delays, the optimal controller is related to the past variables, which makes the control problem more challenging.

On the other hand, stochastic uncertainties exist in many control processes, and some results have been shown in [11–14]. Qi et al. [12] presented the optimal estimation and the optimal output feedback controller of the discrete-time multiplicative noise system with intermittent observations by virtue of coupled Riccati equations. The stabilization condition for this system was developed in the mean square sense. Rami et al. [13] considered the discrete-time stochastic LQ problem subject to state and control-dependent noises. A necessary and sufficient condition for the existence of the optimal control was identified in terms of the solution to the proposed difference Riccati equation. As to meet the actual demand in different areas, the control systems with both stochastic uncertainties and time delay(s) have been thoroughly studied [15–19]. Zhang et al. [15] obtained the optimal linear quadratic regulation (LQR) controller for discrete-time system with input delay and multiplicative noise via the Riccati-ZXL difference equation, while the additive noise was not considered in this reference. In [16], Liang et al. took the state- and control-dependent noise, additive noise and input delay into account, and the optimal controller and the suboptimal linear state estimate feedback controller for the linear quadratic Gaussian (LQG) system were both derived, with only single time delay in the input. Besides, when there are multiplicative noise and multiple delays in the input, Li

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et al. [19] presented the optimal controller and the optimal cost under the necessary and sufficient condition. However, additive noise was not considered. It is obviously shown that the system models described in the above literatures are all discrete time-invariant and input-delay(s) systems. Moreover, the optimal problem involving simultaneously multiplicative noise, additive noise and multiple state delays are not mentioned. In addition, when the additive noise is related to the multiplicative noise, the analysis and synthesis for the control problem remain challenging.

Different from the existing research systems, the system considered in this paper contains simultaneously multiplicative noise, multiple state delays and additive noise, which is more complex than before. It should be emphasized that the additive noise and the multiplicative noise are dependent, and the coefficients in this paper are time-varying. The LQG control for our paper is much more sophisticated and unsolved. The main contributions of this paper are as follows: (1) The relations between the state and the costate in terms of the discrete time-varying LQG problem is given by lots of inductive calculations, which is also the solution to the forward and backward stochastic difference equations (FBSDEs). (2) If and only if a sequence of matrices are all positive definite, the optimal controller and the associated cost function will be obtained via the coupled difference Riccati equations, and the explicit expression of the unique controller is presented, which is obviously more complicated than LQR controller in [19]. Our approach is based on the stochastic maximum principle, and the key technique is the solution to the FBSDEs.

The rest of this paper is organized as follows. In Sect. 2, the discrete time-varying stochastic LQG control problem is described. In Sect. 3, the key tool to the solution is presented, and the necessary and sufficient condition for the optimal LQG control problem is shown. The solutions to the general LQG problem are derived. Numerical examples are shown in Sect. 4. Conclusions are provided in Sect. 5. Proofs of the Lemma and the Theorem are described in Appendixes.

Notation $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space. I presents the unit matrix of appropriate dimension. The superscript $'$ denotes the transpose of the matrix. $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k \geqslant 0}\}$ denotes a complete probability space on which random variable v_k and μ_k are defined such that $\{\mathcal{F}_k\}_{k \geqslant 0}$ is the natural filtration generated by v_k and μ_k, i.e., $\mathcal{F}_k = \sigma\{v_0, \ldots, v_k, \mu_0, \ldots, \mu_k\}$, augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. A symmetric $A > 0 \, (\geqslant 0)$ means that it is a positive definite (positive semi-definite) matrix. $\theta_{a,b}$ is the usual Kronecker function, i.e., $\theta_{a,b} = 0$ if $a \neq b$, and $\theta_{a,b} = 1$ if $a = b$. $\text{Tr}(P)$ represents the trace of matrix P.

2 Problem formulation

Consider the discrete time-varying stochastic LQG system with state delays and multiplicative scalar noise:

$$x_{k+1} = \sum_{i=0}^{d} \left[(\mathcal{C}_i(k) + v_k \bar{\mathcal{C}}_i(k)) x_{k-i} \right] + (\mathcal{D}(k) + v_k \bar{\mathcal{D}}(k)) u_k + \mu_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the input control, the positive integer d is the state delay, v_k is the scalar noise with zero mean and variance γ, $\mu_k \in \mathbb{R}^n$ is random variable satisfying $\text{E}[\mu_k | \mathcal{F}_{k-1}] = \bar{\mu}_k$ and $\text{E}[\mu_k \mu_k' | \mathcal{F}_{k-1}] = Q_{\mu_k}$. The coefficient matrices with compatible dimensions $\mathcal{C}_i(k)$, $\bar{\mathcal{C}}_i(k)$, $\mathcal{D}(k)$ and $\bar{\mathcal{D}}(k)$ with $i = 0, \ldots, d$ are time-varying. v_k and μ_k are correlated, satisfying $\text{E}[v_k \mu_k' | \mathcal{F}_{k-1}] = \tau$, $\text{E}[v_k \mu_l' | \mathcal{F}_{k-1}] = 0$, $k \neq l$. The initial states x_i for $i = -d, \ldots, 0$ are deterministic and known.

Consider the associated cost function for system (1):

$$J_N = \text{E}\left\{ \sum_{k=0}^{N} \left[x_k' Q_k x_k + u_k' R_k u_k \right] + x_{N+1}' \mathcal{P}_{N+1}^0 x_{N+1} \right\}, \tag{2}$$

where Q_k, R_k and $\mathcal{P}_{N+1}^0$ are positive semi-definite matrices with appropriate dimensions, and N is the horizon length. In view of the fact that x_k depends on $v_{k-1}, v_{k-2}, \ldots, \mu_{k-1}, \mu_{k-2}, \ldots$, and the controller obeys the causality constraint, u_k must be $\mathcal{F}_{k-1}$-measurable (see [13]). Then, the problem to be addressed is stated as follows.

Problem 1 Find the unique $\mathcal{F}_{k-1}$-measurable state feedback controller u_k, $k = 0, \ldots, N$, for system (1) such that the cost function (2) is minimized.

3 Main results

For simplicity, we make the system (1) to be

$$x_{k+1} = \sum_{i=0}^{d} \mathcal{C}_k^i(k) x_{k-i} + \mathcal{D}_k(k) u_k + \mu_k, \tag{3}$$

where

$$\mathcal{C}_k^i(k) = \mathcal{C}_i(k) + v_k \bar{\mathcal{C}}_i(k), \quad i = 0, \ldots, d,$$
$$\mathcal{D}_k(k) = \mathcal{D}(k) + v_k \bar{\mathcal{D}}(k).$$

Following the similar approach in [19], we apply stochastic Pontryagin's maximum principle [20] to system (3) with the cost function (2) to yield the costate equations:

$$\zeta_N = \mathcal{P}_{N+1}^0 x_{N+1}, \tag{4}$$

$$\zeta_{k-1} = \mathrm{E}\Big[\sum_{m=0}^{d}(\mathcal{C}_{k+m}^{m})'(k+m)\zeta_{k+m}|\mathcal{F}_{k-1}\Big] + Q_k x_k, \tag{5}$$

$$0 = \mathrm{E}\big[\mathcal{D}_k'(k)\zeta_k|\mathcal{F}_{k-1}\big] + R_k u_k, \quad k = 0,\ldots,N, \tag{6}$$

where ζ_k is the costate variable with $\zeta_k = 0$ for $k > N$.

For further study, we define the following Riccati coupled equations and make the backwards recursion for $k = N, N-1, \ldots, 0$:

$$\begin{aligned}
\mathcal{P}_k^j = \sum_{i=0}^{d-j}\Big[& \mathcal{C}_i'(i+k)\mathcal{P}_{i+k+1}^0\mathcal{C}_{i+j}(i+k) \\
& + \gamma\bar{\mathcal{C}}_i'(i+k)\mathcal{P}_{i+k+1}^0 \times \bar{\mathcal{C}}_{i+j}(i+k) \\
& + \mathcal{C}_i'(i+k)\mathcal{P}_{i+k+1}^{j+i+1} + (\mathcal{P}_{i+k+1}^{j+1})'\mathcal{C}_{i+j}(i+k) \\
& - (N_{i+k}^i)'\Omega_{i+k}^{-1}N_{i+k}^{j+i}\Big] + \theta_{j,0}Q_k,
\end{aligned} \tag{7}$$

$$\Omega_k = R_k + \mathcal{D}'(k)\mathcal{P}_{k+1}^0\mathcal{D}(k) + \gamma\bar{\mathcal{D}}'(k)\mathcal{P}_{k+1}^0\bar{\mathcal{D}}(k), \tag{8}$$

$$N_k^j = \mathcal{D}'(k)\mathcal{P}_{k+1}^{j+1} + \mathcal{D}'(k)\mathcal{P}_{k+1}^0\mathcal{C}_j(k) + \gamma\bar{\mathcal{D}}'(k)\mathcal{P}_{k+1}^0\bar{\mathcal{C}}_j(k). \tag{9}$$

The terminal values of the above matrix sequences $\mathcal{P}_k^j$, Ω_k and N_k^j, $j = 0,\ldots,d$ are given by

$$\begin{cases}
\mathcal{P}_{N+1}^j = 0, \quad j = 1,\ldots,d+1, \\
\mathcal{P}_{N+i}^j = 0, \quad i = 2,\ldots,d+1, j = 0,\ldots,d+1, \\
\Omega_{N+i} = I, \quad N_{N+i}^j = 0, \quad i = 1,\ldots,d, j = 0,\ldots,d.
\end{cases} \tag{10}$$

It should be emphasized that the recursion will stop unless assuming that Ω_k is invertible. To give the main results of Problem 1, we need to obtain the solution to the FBSDEs (3) and (4)–(6), and then the following lemma is proposed.

Lemma 1 *Assuming that Ω_k are positive definite, i.e., $\Omega_k > 0$, for $k = 0,\ldots,N$, then the following equation*

$$\zeta_{k-1} = \sum_{j=0}^{d}\mathcal{P}_k^j x_{k-j} + \Phi_k \tag{11}$$

is the solution to FBSDEs (3) and (4)–(6), where

$$\begin{aligned}
\Phi_k = \sum_{i=0}^{d}\Big[& \big(\mathcal{C}_i'(k+i) - (N_{k+i}^i)'\Omega_{k+i}^{-1}\mathcal{D}'(k+i)\big) \\
& \times (\Phi_{k+i+1} + \mathcal{P}_{k+i+1}^0\bar{\mu}_{k+i}) + \big(\bar{\mathcal{C}}_i'(k+i) \\
& - (N_{k+i}^i)'\Omega_{k+i}^{-1}\bar{\mathcal{D}}'(k+i)\big)\mathcal{P}_{k+i+1}^0\tau \\
& + (\mathcal{P}_{k+i+1}^{i+1})'\bar{\mu}_{k+i}\Big]
\end{aligned} \tag{12}$$

with the terminal value $\Phi_{N+1} = 0$, and $\mathcal{P}_k^j$, N_k^j, Ω_k satisfy the coupled equations (7)–(10).

Proof The proof of Lemma 1 is in Appendix A. □

Remark 1 It is noted that the system model in this paper is discrete time-varying, and contains simultaneously multiplicative noise, additive noise and multiple state delays. Meanwhile, the multiplicative noise is related with the additive noise. Thus, the problem of optimal LQG control is particularly difficult.

Remark 2 We have defined $\mathcal{P}_k^j$, N_k^j with $j \in [0,d]$ by the equation (10). As using the notations $\mathcal{P}_k^j$, N_k^j for $j > d$, we extend the definition $\mathcal{P}_k^j = 0$, $N_k^j = 0$ for $j > d$. Besides, the coefficient matrices $\mathcal{C}_j(k)$, $\bar{\mathcal{C}}_j(k)$ are set to be 0 for $j > d$.

Now, we are in the position to present the solution to Problem 1. The results are stated in the following theorem.

Theorem 1 *Problem 1 has a unique $\mathcal{F}_{k-1}$-measurable u_k if and only if Ω_k, for $k = 0,\ldots,N$, are positive definite. In this context, the optimal controller u_k is calculated by*

$$u_k = -\Omega_k^{-1}\sum_{j=0}^{d}N_k^j x_{k-j} - \Omega_k^{-1}\Sigma_k. \tag{13}$$

The minimum performance index is as

$$\begin{aligned}
J_N^* = {} & x_0'\mathcal{P}_0^0 x_0 + 2x_0'\sum_{j=1}^{d}\mathcal{P}_0^j x_{-j} + \sum_{j=1}^{d}\sum_{i=1}^{d}\sum_{l=0}^{d-1}x_{-j}'\Big[\mathcal{C}_{j+l}'(l) \\
& \times \mathcal{P}_{l+1}^0\mathcal{C}_{i+l}(l) + \gamma\bar{\mathcal{C}}_{j+l}'(l)\mathcal{P}_{l+1}^0\bar{\mathcal{C}}_{i+l}(l) + (\mathcal{P}_{l+1}^{j+l+1})' \\
& \times \mathcal{C}_{i+l}(l) + \mathcal{C}_{j+l}'(l)\mathcal{P}_{l+1}^{i+l+1} - (N_l^{j+l})'\Omega_l^{-1}N_l^{i+l}\Big]x_{-i} \\
& + 2x_0'\Phi_0 - \sum_{k=0}^{N}\Sigma_k'\Omega_k^{-1}\Sigma_k + 2\sum_{k=0}^{N}\bar{\mu}_k'\Phi_{k+1} \\
& + \sum_{k=0}^{N}\mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}],
\end{aligned} \tag{14}$$

where

$$\Sigma_k = \mathcal{D}'(k)(\Phi_{k+1} + \mathcal{P}_{k+1}^0\bar{\mu}_k) + \bar{\mathcal{D}}'(k)\mathcal{P}_{k+1}^0\tau, \tag{15}$$

while $\mathcal{P}_k^j$, N_k^j, Ω_k satisfy the coupled equations (7)–(10).

Proof The proof of Theorem 1 is in Appendix B. □

Remark 3 Different from the existing work [19], the difficulties caused by the additive noise are mainly as follows.

First, this paper considers the optimal LQG control problem with both multiplicative noise and correlated additive noise, which is more challenging than [19]. Second, due to the existence of the additive noise, the key technique to this optimal control problem, i.e., the solution to the FBSDEs (11) is quite more difficult than that of [19]. Besides, the optimal controller u_k satisfying (13) and the associated optimal cost (14) are more difficult to obtain, and the expression of u_k and J_N^* are more complicated than [19].

Remark 4 For a stochastic discrete-time system with no state delays, i.e., $d = 0$ in system (3), it is obviously obtained that the coupled Riccati difference equations:

$$\mathcal{P}_k^0 = \mathcal{C}_0'(k)\mathcal{P}_{k+1}^0\mathcal{C}_0(k) + \gamma\bar{\mathcal{C}}_0'(k)\mathcal{P}_{k+1}^0\bar{\mathcal{C}}_0(k) \\ - (N_k^0)'\Omega_k^{-1}N_k^0 + Q_k,$$

where

$$\Omega_k = R_k + \mathcal{D}'(k)\mathcal{P}_{k+1}^0\mathcal{D}(k) + \gamma\bar{\mathcal{D}}'(k)\mathcal{P}_{k+1}^0\bar{\mathcal{D}}(k),$$
$$N_k^0 = \mathcal{D}'(k)\mathcal{P}_{k+1}^0\mathcal{C}_0(k) + \gamma\bar{\mathcal{D}}'(k)\mathcal{P}_{k+1}^0\bar{\mathcal{C}}_0(k).$$

The optimal controller reduces to

$$u_k = -\Omega_k^{-1}N_k^0 x_k - \Omega_k^{-1}\Sigma_k,$$

where Σ_k as (15). In this context, the solution to the FBSDEs is as

$$\zeta_{k-1} = \mathcal{P}_k^0 x_k + \Phi_k$$

with

$$\Phi_k = \big(\mathcal{C}_0'(k) - (N_k^0)'\Omega_k^{-1}\mathcal{D}'(k)\big)(\Phi_{k+1} + \mathcal{P}_{k+1}^0\bar{\mu}_k) \\ + \big(\bar{\mathcal{C}}_0'(k) - (N_k^0)'\Omega_k^{-1}\bar{\mathcal{D}}'(k)\big)\mathcal{P}_{k+1}^0\rho + (\mathcal{P}_{k+1}^1)'\bar{\mu}_k.$$

Remark 5 From Theorem 1, when the disturbance term μ_k and the multiplicative noise v_k are independent, i.e., $\tau = 0$, and when the additive noise is Gaussian white noise, i.e., $\bar{\mu}_k = 0$, the Riccati difference equations are as (7)–(10), and the matrices can be rewritten as

$$\Phi_k = \sum_{i=0}^{d} \big(\mathcal{C}_i'(k+i) - (N_{k+i}^i)'\Omega_{k+i}^{-1}\mathcal{D}'(k+i)\big)\Phi_{k+i+1}, \quad (16)$$

$$\Sigma_k = \mathcal{D}'(k)\Phi_{k+1}. \quad (17)$$

As the terminal value $\Phi_{N+1} = 0$, it is obviously obtained that Σ_k and Φ_k in (16), (17) always equal to be zero for $k = 0, \ldots, N$. Then, the optimal controller reduces to

$$u_k = -\Omega_k^{-1}\sum_{j=0}^{d} N_k^j x_{k-j}.$$

and the solution to the FBSDEs is as

$$\zeta_{k-1} = \sum_{j=0}^{d} \mathcal{P}_k^j x_{k-j}.$$

In this case, the associated optimal cost is given by

$$J_N^* = x_0'\mathcal{P}_0^0 x_0 + 2x_0'\sum_{j=1}^{d}\mathcal{P}_0^j x_{-j} + \sum_{j=1}^{d}\sum_{i=1}^{d}\sum_{l=0}^{d-1} x_{-j}'\big[\mathcal{C}_{j+l}'(l) \\ \times \mathcal{P}_{l+1}^0\mathcal{C}_{i+l}(l) + \gamma\bar{\mathcal{C}}_{j+l}'(l)\mathcal{P}_{l+1}^0\bar{\mathcal{C}}_{i+l}(l) + (\mathcal{P}_{l+1}^{j+l+1})' \\ \times \mathcal{C}_{i+l}(l) + \mathcal{C}_{j+l}'(l)\mathcal{P}_{l+1}^{i+l+1} - (N_l^{j+l})'\Omega_l^{-1}N_l^{i+l}\big]x_{-i} \\ + \sum_{k=0}^{N}\text{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}].$$

In view of obtaining the scalar case of optimal LQG control system (3), we derive the results to the general system with multiple delays and multiplicative noise.

Consider the following case of discrete time-varying system:

$$x_{k+1} = \sum_{i=0}^{d}\Big[\mathcal{C}_i(k) + \sum_{m=1}^{f} v_k(m)\bar{\mathcal{C}}_{i,m}(k)\Big]x_{k-i} \\ + \Big[\mathcal{D}(k) + \sum_{m=1}^{f} v_k(m)\bar{\mathcal{D}}_m(k)\Big]u_k + \mu_k, \quad (18)$$

where $\mathcal{V}_k = (v_k(1)\ldots v_k(f))'$ is a f-dimensional white noise defined on a complete probability $\{\Omega, \mathcal{P}, \mathcal{F}\}$. $\mathcal{V}_k$ satisfies the variance γ, i.e.,

$$\text{E}[\mathcal{V}_k\mathcal{V}_k'] = \gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1f} \\ \vdots & & \vdots \\ \gamma_{f1} & \cdots & \gamma_{ff} \end{bmatrix} \in \mathbb{R}^{f\times f}, \quad \gamma \geqslant 0.$$

Here $\mathcal{F}_k$ is the natural filtration generated by $\mathcal{V}_k$ and μ_k, i.e., $\mathcal{F}_k$ is the σ-algebra generated by $\{\mathcal{V}_0, \ldots, \mathcal{V}_k, \mu_0, \ldots, \mu_k\}$. Then, the general case of discrete time-varying LQG control problem is stated as follows.

Problem 2 Find the unique $\mathcal{F}_{k-1}$-measurable state feedback controller u_k, $k = 0, \ldots, N$, for system (18) such that the cost function (2) is minimized.

To solve Problem 2, we derive the definition as

$$\mathcal{C}_k^i(k) = \mathcal{C}_i(k) + \sum_{m=1}^{f} v_k(m)\bar{\mathcal{C}}_{i,m}(k),$$

$$\mathcal{D}_k(k) = \mathcal{D}(k) + \sum_{m=1}^{f} v_k(m)\bar{\mathcal{D}}_m(k),$$

and the coupled difference Riccati equations (7)–(9) extend to

$$
\begin{aligned}
\mathcal{P}_k^j =& \sum_{i=0}^{d-j} \Bigg[\mathcal{C}_i'(i+k) \mathcal{P}_{i+k+1}^0 \mathcal{C}_{i+j}(i+k) + \sum_{a=1}^f \sum_{b=1}^f \gamma_{ab} \\
& \times \bar{\mathcal{C}}_{i,a}'(i+k) \mathcal{P}_{i+k+1}^0 \bar{\mathcal{C}}_{i+j,b}(i+k) + \mathcal{C}_i'(i+k) \\
& \times \mathcal{P}_{i+k+1}^{j+i+1} + (\mathcal{P}_{i+k+1}^{i+1})' \mathcal{C}_{i+j}(i+k) - (N_{i+k}^i)' \\
& \times \Omega_{i+k}^{-1} N_{i+k}^{j+i} \Bigg] + \theta_{j,0} Q_k,
\end{aligned} \tag{19}
$$

$$
\Omega_k = R_k + \mathcal{D}'(k) \mathcal{P}_{k+1}^0 \mathcal{D}(k) + \sum_{a=1}^f \sum_{b=1}^f \gamma_{ab} \bar{\mathcal{D}}_a'(k) \mathcal{P}_{k+1}^0 \bar{\mathcal{D}}_b(k), \tag{20}
$$

$$
\begin{aligned}
N_k^j =& \mathcal{D}'(k) \mathcal{P}_{k+1}^{j+1} + \mathcal{D}'(k) \mathcal{P}_{k+1}^0 \mathcal{C}_j(k) + \sum_{a=1}^l \sum_{b=1}^l \gamma_{ab} \\
& \times \bar{\mathcal{D}}_a(k)' \mathcal{P}_{k+1}^0 \bar{\mathcal{C}}_{j,b}(k), \quad k = 0, \dots, N.
\end{aligned} \tag{21}
$$

Based on the above definitions, the solution to Problem 2 is derived in the following theorem.

Theorem 2 *Problem 2 has a unique $\mathcal{F}_{k-1}$- measurable u_k if and only if Ω_k, for $k = 0, \dots, N$, are positive definite. In this case, the optimal controller u_k is given by*

$$
u_k = -\Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} - \Omega_k^{-1} \Sigma_k,
$$

and the associated optimal cost function is as

$$
\begin{aligned}
J_N^* =& x_0' \mathcal{P}_0^0 x_0 + 2 x_0' \sum_{j=1}^d \mathcal{P}_0^j x_{-j} + \sum_{j=1}^d \sum_{i=1}^d \sum_{l=0}^{d-1} x_{-j}' \Bigg[\mathcal{C}_{j+l}'(l) \\
& \times \mathcal{P}_{l+1}^0 \mathcal{C}_{i+l}(l) + \sum_{a=1}^f \sum_{b=1}^f \gamma_{ab} \bar{\mathcal{C}}_{j+l,a}'(l) \mathcal{P}_{l+1}^0 \bar{\mathcal{C}}_{i+l,b}(l) \\
& + (\mathcal{P}_{l+1}^{j+l+1})' \mathcal{C}_{i+l}(l) + \mathcal{C}_{j+l}'(l) \mathcal{P}_{l+1}^{i+l+1} - (N_l^{j+l})' \\
& \times \Omega_l^{-1} N_l^{i+l} \Bigg] x_{-i} + 2 x_0' \Phi_0 - \sum_{k=0}^N \Sigma_k' \Omega_k^{-1} \Sigma_k \\
& + 2 \sum_{k=0}^N \bar{\mu}_k' \Phi_{k+1} + \sum_{k=0}^N \mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}],
\end{aligned}
$$

where $\mathcal{P}_k^j, N_k^j, \Omega_k$ satisfy the coupled equations (19)–(21).

Remark 6 It is obviously that the multiplicative noise $\mathcal{V}_k$ in the general LQG control system (18) is expanded by multiple dimensions of white noises. The existence of multidimensional white noise has no essential influence on the optimal control problem, and we can treat it as a whole. Then, the approach of Theorem 1 is also applied to the

general situation. Thus, combining the mathematical characteristics of $\mathcal{V}_k$, Theorem 2 is derived as the above.

4 Numerical examples

Example 1 Consider the scalar case of time-varying LQG control system (3) in Theorem 1, as the additive noise μ_k correlated with ν_k. Let the associated parameters be as

$$
\begin{aligned}
& \mathcal{C}_0(0) = 1, \ \mathcal{C}_1(0) = -1, \ \mathcal{C}_2(0) = 1, \\
& \bar{\mathcal{C}}_0(0) = -4, \ \bar{\mathcal{C}}_1(0) = 3, \ \bar{\mathcal{C}}_2(0) = 2, \\
& \mathcal{C}_0(1) = 2, \ \mathcal{C}_1(1) = 3, \ \mathcal{C}_2(1) = -2, \\
& \bar{\mathcal{C}}_0(1) = 2, \ \bar{\mathcal{C}}_1(1) = -2, \ \bar{\mathcal{C}}_2(1) = 1, \\
& \mathcal{D}(0) = 4, \ \bar{\mathcal{D}}(0) = 1, \ \mathcal{D}(1) = -1, \ \bar{\mathcal{D}}(1) = -1, \\
& \gamma = 1, \ \tau = 1, \ Q_{\nu_k} = 1, \ Q_{\mu_k} = 1, \ \bar{\mu}_k = 0.2,
\end{aligned}
$$

and the cost function (2) with

$$
d = 2, \ N = 1, \ \mathcal{P}_{N+1}^0 = 1, \ Q_k = 1, \ R_k = 1.
$$

By direct calculation , it yields

$$
\begin{aligned}
& \mathcal{P}_0^0 = 64.55, \ \mathcal{P}_1^0 = -48.73, \ \mathcal{P}_2^0 = -25.93, \\
& N_0^0 = 2.67, \ N_1^0 = -6.33, \ N_2^0 = 22, \\
& \mathcal{P}_0^1 = 3.67, \ \mathcal{P}_1^1 = 0.67, \ \mathcal{P}_2^1 = -0.67, \\
& N_0^1 = -4, \ N_1^1 = -1, \ N_2^1 = 1 \\
& \Sigma_0 = 6.6, \ \Sigma_1 = -1.2, \ \Omega_0 = 63.33, \ \Omega_1 = 3.
\end{aligned}
$$

It is obviously known that Ω_k is positive definite for $k = 0, 1$. Thus, from Theorem 1, there exists a unique u_k, which is given by

$$
\begin{aligned}
u_0^* =& -0.0421 \times x_0 + 0.1 \times x_{-1} - 0.3474 \times x_{-2} - 0.1042; \\
u_1^* =& 0.0632 \times x_1 + 0.0158 \times x_0 - 0.0158 \times x_{-1} + 0.0189.
\end{aligned}
$$

Second, we shall illustrate that u_k^* can minimize the cost function (2). Let the controller be arbitrary. For example,

$$
\begin{aligned}
\hat{u}_0 =& 0.34 \times x_0 + 2 \times x_{-1} + 0.8 \times x_{-2} - 0.2; \\
\hat{u}_1 =& 2 \times x_1 + 1 \times x_0 - 0.2 \times x_{-1} - 1.
\end{aligned}
$$

Compare the cost function under u_k^* and $\hat{u}_k$ with different initial values as follows:

1) $x_0 = 1, \ x_{-1} = 1, \ x_{-2} = 2, \ J^* = 3.56, \ \hat{J} = 21.88$;
2) $x_0 = -1, \ x_{-1} = 2, \ x_{-2} = 2, \ J^* = 61.65, \ \hat{J} = 141.16$;
3) $x_0 = 0, \ x_{-1} = -1, \ x_{-2} = -3, \ J^* = 22.99, \ \hat{J} = 36.79$;
4) $x_0 = 2, \ x_{-1} = 1, \ x_{-2} = -2, \ J^* = 23.16, \ \hat{J} = 42.09$.

Hence, u_k^* and J^* are optimal. This demonstrates the correctness of our results.

Example 2 Consider LQG control system (3) with $x_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}^2$, $d = 1$, $\gamma = 1$, $\tau = [1, 1]'$, and the coefficient matrixes are time-invariant with

$$\mathcal{C}_0 = \begin{bmatrix} -1 & 1.2 \\ 0.8 & 1 \end{bmatrix}, \ \bar{\mathcal{C}}_0 = \begin{bmatrix} -0.2 & 0 \\ 1 & -0.5 \end{bmatrix}, \ \mathcal{C}_1 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\bar{\mathcal{C}}_1 = \begin{bmatrix} 0.2 & -2 \\ 1 & 1 \end{bmatrix}, \ \mathcal{C}_2 = \begin{bmatrix} -1 & 2 \\ -2 & -0.4 \end{bmatrix}, \ \bar{\mathcal{C}}_2 = \begin{bmatrix} 0.6 & -2 \\ 1 & 0 \end{bmatrix},$$

$$\mathcal{C}_3 = \begin{bmatrix} 0 & 1.5 \\ -0.7 & 1 \end{bmatrix}, \ \bar{\mathcal{C}}_3 = \begin{bmatrix} 0.8 & -1.3 \\ 1 & 0.4 \end{bmatrix}, \ \mathcal{D} = \begin{bmatrix} 0.3 & 0 \\ -1 & 0.2 \end{bmatrix},$$

$$\bar{\mathcal{D}} = \begin{bmatrix} 1 & 1.2 \\ 0.3 & -1 \end{bmatrix}, \ Q_v = I, \ Q_\mu = I,$$

and the cost function (2) with

$$N = 3, \ Q = I, \ R = I, \ \mathcal{P}_{N+1}^0 = 0.$$

Then by direct calculation, the solution to coupled difference Riccati equations is yielded as

$$\mathcal{P}_0^0 = \begin{bmatrix} -6.28 & -11.33 \\ -0.41 & -37.19 \end{bmatrix}, \ \mathcal{P}_1^0 = \begin{bmatrix} 1.43 & -1.29 \\ 1.73 & -2.84 \end{bmatrix},$$

$$\mathcal{P}_2^0 = \begin{bmatrix} 2.46 & -0.02 \\ -0.72 & 2.78 \end{bmatrix}, \ \mathcal{P}_3^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathcal{P}_0^1 = \begin{bmatrix} -2.11 & 5.09 \\ 4.08 & -6.45 \end{bmatrix}, \ \mathcal{P}_1^1 = \begin{bmatrix} 1.72 & -5.66 \\ 0.15 & -3.70 \end{bmatrix},$$

$$\mathcal{P}_2^1 = \begin{bmatrix} 0.81 & -1.77 \\ -0.84 & 0.29 \end{bmatrix}, \ \mathcal{P}_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$N_0^0 = \begin{bmatrix} 1.10 & 3.96 \\ 0.52 & -1.53 \end{bmatrix}, \ N_1^0 = \begin{bmatrix} -2.24 & -2.27 \\ -3.12 & 1.85 \end{bmatrix},$$

$$N_2^0 = \begin{bmatrix} -1 & -0.79 \\ -1.08 & 0.7 \end{bmatrix}, \ N_3^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$N_0^1 = \begin{bmatrix} -0.45 & -6.19 \\ 0.94 & 1.46 \end{bmatrix}, \ N_1^1 = \begin{bmatrix} -0.19 & -3.54 \\ -1.93 & -9.9 \end{bmatrix},$$

$$N_2^1 = \begin{bmatrix} 0.2 & -2.1 \\ -0.76 & -3.2 \end{bmatrix}, \ N_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Omega_0 = \begin{bmatrix} 4.11 & 0.99 \\ 0.99 & 4.53 \end{bmatrix}, \ \Omega_1 = \begin{bmatrix} 6.71 & 1.32 \\ 2.23 & 8.33 \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} 3.18 & 0.7 \\ 0.7 & 3.48 \end{bmatrix}, \ \Omega_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Obviously, $\Omega_k > 0$ for $k = 0, \ldots, N$, therefore, there exist a optimal controller for Problem 1. In addition, when the initial values are

$$x_0 = \begin{bmatrix} -0.5 \\ 0.8 \end{bmatrix}, \ x_{-1} = \begin{bmatrix} 0.3 \\ -0.7 \end{bmatrix},$$

the optimal controller can be calculated as

$$u_0^* = \begin{bmatrix} 2.52 \\ -1.36 \end{bmatrix}, \ u_1^* = \begin{bmatrix} -3.68 \\ 1.58 \end{bmatrix}, \ u_2^* = \begin{bmatrix} -3.89 \\ -4.6 \end{bmatrix}, \ u_3^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Accordingly, the associated cost function is $J^* = 148.8$.

5 Conclusions

In this paper, the discrete time-varying LQG control problem with both multiplicative noise and multiple state delays has been studied. We obtain the solution to the FBSDEs for the discrete time-varying systems. A necessary and sufficient condition for the existence of a unique optimal controller is proposed. The basis of this approach is the stochastic maximum principle and the key is the relationship between the state and costate. In the future works, we expect that the results in this paper shall pave new ways for networked control system with both state delays and packet dropout.

Appendix A

With the stochastic maximum principle (4)–(6) to LQG control system (3) involving multiple state delays and multiplicative noise, we can obtain for $k = N$,

$$\begin{aligned} 0 = &\sum_{i=0}^d \left(\mathcal{D}'(N)\mathcal{P}_{N+1}^0 \mathcal{C}_i(N) + \gamma \bar{\mathcal{D}}'(N)\mathcal{P}_{N+1}^0 \bar{\mathcal{C}}_i(N) \right) x_{N-i} \\ &+ \left(\mathcal{D}'(N)\mathcal{P}_{N+1}^0 D(N) + \gamma \bar{\mathcal{D}}'(N)\mathcal{P}_{N+1}^0 \bar{\mathcal{D}}(N) + R_N \right) \\ &\times u_N + \mathcal{D}'(N)\mathcal{P}_{N+1}^0 \bar{\mu}_N + \bar{\mathcal{D}}'(N)\mathcal{P}_{N+1}^0 \tau. \end{aligned}$$

Using Eqs. (8) and (9), the optimal controller u_N is as

$$u_N = -\Omega_N^{-1} \sum_{i=0}^d N_N^i x_{N-i} - \Omega_N^{-1} \Sigma_N,$$

where $\Sigma_N = \mathcal{D}'(N)\mathcal{P}_{N+1}^0 \bar{\mu}_N + \bar{\mathcal{D}}'(N)\mathcal{P}_{N+1}^0 \tau$. From (4) and (5), we also have

$$\zeta_{N-1} = \mathrm{E}\Big[(\mathcal{C}_N^0)'(N)\mathcal{P}_{N+1}^0 x_{N+1}|\mathcal{F}_{N-1}\Big] + Q_N x_N$$

$$= \mathrm{E}\Big[\Big(\sum_{i=0}^d (\mathcal{C}_N^0)'(N)\mathcal{P}_{N+1}^0 \mathcal{C}_N^i(N) - (N_N^0)'\Omega_N^{-1}N_N^i\Big)$$

$$\times x_{N-i} - (N_N^0)'\Omega_N^{-1}\mathcal{D}_N'(N)\mathcal{P}_{N+1}^0 \mu_N + (\mathcal{C}_N^0)'(N)$$

$$\times \mathcal{P}_{N+1}^0 \mu_N |\mathcal{F}_{N-1}\Big] + Q_N x_N.$$

Substituting (7), ζ_{N-1} yields

$$\zeta_{N-1} = \sum_{i=1}^d \mathcal{P}_N^i x_{N-i} + \mathcal{P}_N^0 x_N + (\mathcal{C}_0'(N)\mathcal{P}_{N+1}^0 \bar\mu_N$$

$$+ \bar{\mathcal{C}}_0'(N)\mathcal{P}_{N+1}^0 \tau - (N_N^0)'\Omega_N^{-1}\mathcal{D}'(N)\mathcal{P}_{N+1}^0$$

$$\times \bar\mu_N - (N_N^0)'\Omega_N^{-1}\bar{\mathcal{D}}'(N)\mathcal{P}_{N+1}^0 \tau)$$

$$= \sum_{j=0}^d \mathcal{P}_N^j x_{N-j} + \Phi_N,$$

where Φ_N satisfied (12) with the terminal values being zero.

Now, we have verified (11) for $k = N$. Supposing that ζ_{k-1} are as (11) for all $k \geqslant n+1$, we will show that (11) also holds for $k = n$. For $k = n+1$, with (3) and (11), ζ_n can be calculated as

$$\zeta_n = \sum_{j=0}^d \mathcal{P}_{n+1}^j x_{n+1-j} + \Phi_{n+1}$$

$$= \sum_{j=1}^d \mathcal{P}_{n+1}^j x_{n+1-j} + \mathcal{P}_{n+1}^0 \sum_{i=0}^d \big(\mathcal{C}_n^i(n) x_{n-i} + \mathcal{D}(n) u_n \tag{22}$$

$$+ \mu_n\big) + \Phi_{n+1}.$$

Inserting ζ_n to (6), (6) will become

$$0 = \mathrm{E}\Big[\sum_{j=0}^d \big(\mathcal{D}_n'(n)\mathcal{P}_{n+1}^{j+1} + \mathcal{D}_n'(n)\mathcal{P}_{n+1}^0 \mathcal{C}_n^j(n)\big) x_{n-j} + \mathcal{D}_n'(n)$$

$$\times \mathcal{P}_{n+1}^0 \mathcal{D}_n(n) u_n + \mathcal{D}_n'(n)\mathcal{P}_{n+1}^0 \mu_n + \mathcal{D}_n'(n)\Phi_{n+1}|\mathcal{F}_{n-1}\Big] + R_n u_n$$

$$= \sum_{j=0}^d N_n^j x_{n-j} + \Omega_n u_n + \mathcal{D}'(n)\big(\mathcal{P}_{n+1}^0 \bar\mu_n + \Phi_{n+1}\big)$$

$$+ \bar{\mathcal{D}}'(n)\mathcal{P}_{n+1}^0 \tau.$$

Thus, the optimal controller is given by

$$u_n = -\Omega_n^{-1}\sum_{j=0}^d N_n^j x_{n-j} - \Omega_n^{-1}\Phi_n. \tag{23}$$

In virtue of equations (3), (5) and (23), ζ_{n-1} yields that

ζ_{n-1}

$$= \mathrm{E}\Big[\sum_{m=0}^{d-1}(\mathcal{C}_{n+m}^m)'(n+m)\zeta_{n+m} + (\mathcal{C}_{n+d}^d)'(n+d)$$

$$\times \Big(\sum_{j=1}^d \mathcal{P}_{n+d+1}^j x_{n+d+1-j} + \mathcal{P}_{n+d+1}^0\Big(\sum_{i=0}^d \mathcal{C}_{n+d}^i(n+d)x_{n+d-i}$$

$$+ \mu_{n+d} + \mathcal{D}_{n+d}(n+d)u_{n+d}\Big) + \Phi_{n+d+1}\Big)|\mathcal{F}_{n-1}\Big] + Q_n x_n$$

$$= \mathrm{E}\Big[\sum_{m=0}^{d-1}(\mathcal{C}_{n+m}^m)'(n+m)\zeta_{n+m} + \sum_{j=0}^d \big((\mathcal{C}_{n+d}^d)'(n+d)\mathcal{P}_{n+d+1}^{j+1}$$

$$+ (\mathcal{C}_{n+d}^d)'(n+d)\mathcal{P}_{n+d+1}^0 \mathcal{C}_{n+d}^j(n+d) - (N_{n+d}^d)'\Omega_{n+d}^{-1}$$

$$\times N_{n+d}^j\big)x_{n+d-j} - (N_{n+d}^d)'\Omega_{n+d}^{-1}\Sigma_{n+d} + (\mathcal{C}_{n+d}^d)'(n+d)$$

$$\times (\mathcal{P}_{n+d+1}^0 \mu_{n+d} + \Phi_{n+d+1})|\mathcal{F}_{n-1}\Big] + Q_n x_n$$

$$= \mathrm{E}\Big[\sum_{m=0}^{d-2}(\mathcal{C}_{n+m}^m)'(n+m)\zeta_{n+m} + \sum_{j=0}^d \big((\mathcal{C}_{n+d-1}^{d-1})'(n+d-1)$$

$$\times \mathcal{P}_{n+d}^{j+1} + (\mathcal{C}_{n+d-1}^{d-1})'(n+d-1)\mathcal{P}_{n+d}^0 \mathcal{C}_{n+d-1}^j(n+d-1)$$

$$+ (\mathcal{C}_{n+d}^d)'(n+d)\mathcal{P}_{n+d+1}^{j+2} + (\mathcal{C}_{n+d}^d)'(n+d)\mathcal{P}_{n+d+1}^0$$

$$\times \mathcal{C}_{n+d}^{j+1}(n+d) - (N_{n+d}^d)'\Omega_{n+d}N_{n+d}^{j+1} + (\mathcal{P}_{n+d}^d)'$$

$$\times \mathcal{C}_{n+d-1}^j(n+d-1)\big)x_{n+d-1-j} + (N_{n+d-1}^{d-1})'u_{n+d-1}$$

$$- (N_{n+d}^d)'\Omega_{n+d}^{-1}\Sigma_{n+d} + (\mathcal{C}_{n+d}^d)'(n+d)(\mathcal{P}_{n+d+1}^0 \mu_{n+d}$$

$$+ \Phi_{n+d+1}) + (\mathcal{C}_{n+d-1}^{d-1})'(n+d-1)(\mathcal{P}_{n+d}^0 g_{n+d-1} + \Phi_{n+d})$$

$$+ (\mathcal{P}_{n+d}^d)'\mu_{n+d-1}|\mathcal{F}_{n-1}\Big] + Q_n x_n$$

$$= \mathrm{E}\Big[\sum_{m=0}^{d-3}(\mathcal{C}_{n+m}^m)'(n+m)\zeta_{n+m} + \sum_{j=0}^d \sum_{i=d-2}^d \big((\mathcal{C}_{n+i}^i)'(n+i)$$

$$\times \mathcal{P}_{n+i+1}^0 \mathcal{C}_{n+i}^{i+j-d+2}(n+i) + (\mathcal{C}_{n+i}^i)'(n+i)\mathcal{P}_{n+i+1}^{j+j-d+3}$$

$$+ (\mathcal{P}_{n+i+1}^{i+1})'\mathcal{C}_{n+i}^{i+1}(n+i) - (N_{n+i}^i)'\Omega_{n+i}^{-1}N_{n+i}^{i+j-d+2}\big)$$

$$\times x_{n+d-2-j} - \sum_{i=d-1}^d \big((N_{n+i}^i)'\Omega_{n+i}^{-1}\Sigma_{n+i} + (\mathcal{C}_{n+i}^i)'(n+i)$$

$$\times (\mathcal{P}_{n+i+1}^0 \mu_{n+i} + \Phi_{n+i+1}) + (\mathcal{P}_{n+i+1}^{i+1})'\mu_{n+i})|\mathcal{F}_{n-1}\Big]$$

$$+ Q_n x_n.$$

Plugging (3) and (23) into the above equation for times d, we can calculate ζ_{n-1} as follows:

ζ_{n-1}

$$= E\Bigg[\sum_{j=0}^{d} \sum_{i=0}^{d} \Big((\mathcal{C}_{n+i}^{i})'(n+i)\mathcal{P}_{n+i+1}^{0}\mathcal{C}_{n+i}^{i+j}(n+i)$$

$$+ (\mathcal{C}_{n+i}^{i})'(n+i)\mathcal{P}_{n+i+1}^{i+j} + (\mathcal{P}_{n+i+1}^{i+1})'\mathcal{C}_{n+i}^{j+1}(n+i)$$

$$- (N_{n+i}^{i})'\Omega_{n+i}^{-1}N_{n+i}^{i+j} \Big) x_{n+1-j} + \sum_{i=0}^{d} \Big((-N_{n+i}^{i})'\Omega_{n+i}^{-1}\Sigma_{n+i}$$

$$+ (\mathcal{C}_{n+i}^{i})'(n+i)(\mathcal{P}_{n+i+1}^{0}\mu_{n+i} + \Phi_{n+i+1})$$

$$+ (\mathcal{P}_{n+i+1}^{i+1})'\mu_{n+i} \Big)|\mathcal{F}_{n-1}\Bigg] + Q_n x_n,$$

where $\mathcal{P}_{n+i+1}^{i+j} = 0$ for $i+j > d$ from Remark 1, and then

ζ_{n-1}

$$= \sum_{j=0}^{d} \sum_{i=0}^{d-j} \Big(\mathcal{C}_i'(n+i)\mathcal{P}_{n+i+1}^{0}\mathcal{C}_{i+j}(n+i)$$

$$+ \gamma\bar{\mathcal{C}}_i'(n+i)\mathcal{P}_{n+i+1}^{0}\bar{\mathcal{C}}_{i+j}(n+i) + \mathcal{C}_i'(n+i)\mathcal{P}_{n+i+1}^{i+j+1}$$

$$+ (\mathcal{P}_{n+i+1}^{i+1})'\mathcal{C}_{i+j}(n+i) - (N_{n+i}^{i})'\Omega_{n+i}^{-1}N_{n+i}^{i+j} \Big) x_{n-j}$$

$$+ \sum_{i=0}^{d} \Big((-N_{n+i}^{i})'\Omega_{n+i}^{-1}\Sigma_{n+i} + \mathcal{C}_i'(n+i)(\mathcal{P}_{n+i+1}^{0}\bar{\mu}_{n+i}$$

$$+ \Phi_{n+i+1}) + \bar{\mathcal{C}}_i'(n+i)(\mathcal{P}_{n+i+1}^{0}\tau + (\mathcal{P}_{n+i+1}^{i+1})'\bar{\mu}_{n+i}) + Q_n x_n.$$

After inserting (7), we can summarize that

$$\zeta_{n-1} = \sum_{j=0}^{d} \mathcal{P}_n^j x_{n-j} + \Phi_n.$$

This completes the proof of the lemma.

Appendix B

(Necessity) Suppose that there exists the unique $\mathcal{F}_{k-1}$-measurable u_k to make the cost function (2) minimized. We will show that $\Omega_k, k = 0, \ldots, N$ are positive definite by induction and the optimal controller can be designed as (13). Define

$$J(k) = E\Bigg[\sum_{i=k}^{N} x_i'Q_i x_i + u_i'R_i u_i + x_{N+1}'\mathcal{P}_{N+1}^{0}x_{N+1} \Bigg].$$

When $k = N$, $J(N)$ is presented as

$$J(N) = u_N'\Omega_N u_N + 2u_N(\mathcal{D}(N)\mathcal{P}_{N+1}^{0}\bar{\mu}_N + \bar{\mathcal{D}}(N)\mathcal{P}_{N+1}^{0}\tau)$$
$$+ \mathrm{Tr}[\mathcal{P}_{N+1}^{0}Q_{\mu_N}],$$

where $x_N = 0$ and $x_{N-j} = 0$ for $j = 0, \ldots, d$ as the uniqueness of the optimal controller is unrelated with x_k.

As $J(N)$ can be expressed as a quadratic function of u_N, and the performance index must be positive, it can be obviously

know that $\Omega_N > 0$, i.e., Ω_k is positive definite for $k = N$. Assuming $\Omega_k > 0$ for all $k \geqslant n + 1$, we will prove that $\Omega_n > 0$. With (3), (5) and (6), for $k \geqslant n + 1$, we construct that

$$E\Big[x_k'\zeta_{k-1} - x_{k+1}'\zeta_k \Big]$$

$$= E\Bigg[x_k'E[\sum_{m=0}^{d} \mathcal{C}_{k+m}^{m}(k+m)\zeta_{k+m}|\mathcal{F}_{k-1}] + x_k'Q_k x_k$$

$$- \Big(\sum_{i=0}^{d} \mathcal{C}_k^i(k)x_{k-i} + \mathcal{D}_k(k)u_k + \mu_k \Big)'\zeta_k \Bigg] \quad (24)$$

$$= E\Big[x_k'Q_k x_k - u_k'E[\mathcal{D}'(k)\zeta_k|\mathcal{F}_{k-1}] - \mu_k'\zeta_k \Big]$$

$$= E\Big[x_k'Q_k x_k + u_k'R_k u_k - \mu_k'\zeta_k \Big].$$

To obtain the form of $J(N)$, we add both sides of (24) from $k = n + 1$ to $k = N$, we have

$$E[x_{n+1}'\zeta_n - x_{N+1}'\zeta_N] = \sum_{k=n+1}^{N} E\Big[x_k'Q_k x_k + u_k'R_k u_k - \mu_k'\zeta_k \Big].$$

Then,

$$E\Bigg[\sum_{k=n+1}^{N} \Big(x_k'Q_k x_k + u_k'R_k u_k \Big) + x_{N+1}'\mathcal{P}_{N+1}^{0}x_{N+1} \Bigg]$$

$$= E\Big[x_{n+1}'\zeta_n - \sum_{k=n}^{N} \mu_k'\zeta_k \Big].$$

Compared with (2), it yields that

$$J(n) = \big[x_n'Q_n x_n + u_n'R_n u_n \big] + E\Big[x_{n+1}'\zeta_n + \sum_{k=n+1}^{N} \mu_k'\zeta_k \Big]. \quad (25)$$

Setting $x_n = 0$ and $x_{n-i} = 0$, for $i = 0, \ldots, d$ as the same as the condition $k = N$, and plugging (11) into (25), we obtain

$$J(n) = E\Bigg[u_n'R_n u_n + u_n'\mathcal{D}_n'(n)\zeta_n + \sum_{k=n}^{N} \mu_k'\zeta_k \Bigg]$$

$$= E\Big[u_n'R_n u_n + u_n'\mathcal{D}_n'(n)(\mathcal{P}_{n+1}^{0}\mathcal{D}_n(n)u_n + \mu_n)$$

$$+ u_n'\mathcal{D}_n'(n)\Phi_{n+1} + \sum_{k=n}^{N} \mu_k'\zeta_k \Big]$$

$$= u_n'\Omega_n u_n + u_n'\big(\mathcal{D}'(n)(\mathcal{P}_{n+1}^{0}\bar{\mu}_n + \Phi_{n+1})$$

$$+ \bar{\mathcal{D}}'(n)\mathcal{P}_{n+1}^{0}\tau \big) + \sum_{k=n+1}^{N} \mu_k'\zeta_k.$$

Similarly to the case $\Omega_N > 0$ above, we obviously get $\Omega_n > 0$ for all $k = 0, \ldots, N$. This completes the proof of necessity.

(Sufficiency) Suppose that $\Omega_k > 0$ for $k = 0, \ldots, N$ is ture, we will show the existence of the unique $\mathcal{F}_{k-1}$-measurable u_k to minimize (2). Make the definition:

$V(x_k)$

$$
\begin{aligned}
= \mathrm{E}\Big[& x_k' \mathcal{P}_k^0 x_k + 2x_k' \sum_{j=1}^d \mathcal{P}_k^j x_{k-j} \\
& + \sum_{j=1}^d \sum_{i=1}^d \sum_{l=0}^{d-1} x_{k-j}' \Big[\mathcal{C}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^0 \mathcal{C}_{i+l}(k+l) \\
& + \gamma \bar{\mathcal{C}}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^0 \bar{\mathcal{C}}_{i+l}(k+l) + \mathcal{C}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^{i+l+1} \\
& + (\mathcal{P}_{k+l+1}^{j+l+1})' \mathcal{C}_{i+l}(k+l) - (N_{k+l}^{j+l})' \Omega_{k+l}^{-1} N_{k+l}^{i+l} \Big] x_{k-i} \\
& + 2x_k' \Phi_k \Big].
\end{aligned}
$$

First, as $k = k+1$, using the equivalent substitution $l = l+1$, $j = j-1$, and $i = i-1$ in turn, the $V(x_{k+1})$ can be calculated as

$V(x_{k+1})$

$$
\begin{aligned}
= \mathrm{E}\Big[& x_k' \big((\mathcal{C}_k^0)'(k) \mathcal{P}_{k+1}^0 \mathcal{C}_k^0(k) + (\mathcal{C}_k^0)'(k) \mathcal{P}_{k+1}^1 \\
& + (\mathcal{P}_{k+1}^1)' \mathcal{C}_k^0(k) \big) x_k + 2x_k' \sum_{j=1}^d \big((\mathcal{C}_k^0)'(k) \mathcal{P}_{k+1}^0 \mathcal{C}_k^j(k) \\
& + (\mathcal{C}_k^0)'(k) \mathcal{P}_{k+1}^{j+1} + (\mathcal{P}_{k+1}^1)' \mathcal{C}_k^j(k) \big) x_{k-j} \\
& + \sum_{j=1}^d \sum_{i=1}^d x_{k-i}' \big((\mathcal{C}_k^i)'(k) \mathcal{P}_{k+1}^0 \mathcal{C}_k^j(k) + (\mathcal{C}_k^i)'(k) \mathcal{P}_{k+1}^{j+1} \\
& + (\mathcal{P}_{k+1}^{i+1})' \mathcal{C}_k^j(k) \big) x_{k-j} + 2u_k' \sum_{j=0}^d N_k^j x_{k-j} + u_k'(\Omega_k - R) u_k \\
& + \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} \sum_{l=1}^d x_{k-j}'[\mathcal{C}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^0 \mathcal{C}_{i+l}(k+l) \\
& + \gamma \bar{\mathcal{C}}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^0 \bar{\mathcal{C}}_{i+l}(k+l) + \mathcal{C}_{j+l}'(k+l) \mathcal{P}_{k+l+1}^{i+l+1} \\
& + (\mathcal{P}_{k+l+1}^{j+l+1})' \mathcal{C}_{i+l}(k+l) - (N_{k+l}^{j+l})' \Omega_{k+l}^{-1} N_{k+l}^{i+l}] x_{k-i} \\
& + 2\mu_k' \sum_{j=0}^d \mathcal{P}_{k+1}^0 \mathcal{C}_k^j(k) x_{k-j} + 2\mu_k' \mathcal{P}_{k+1}^0 \mathcal{D}_k(k) u_k + \mu_k' \mathcal{P}_{k+1}^0 \mu_k \\
& + \sum_{j=0}^d \mu_k' \mathcal{P}_{k+1}^{j+1} x_{k-j} + 2x_{k+1}' \Phi_{k+1} \Big].
\end{aligned}
$$

Constructing the form $V(x_k) - V(x_{k+1})$, we have

$V(x_k) - V(x_{k+1})$

$$
\begin{aligned}
= \mathrm{E}\Big[& x_k' Q_k x_k + u_k' R_k u_k - \Big(u_k + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big)' \Omega_k \Big(u_k \\
& + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big) - 2u_k' \Sigma_k + 2x_k \Phi_k - 2 \sum_{j=0}^d x_{k-j}' \\
& \times \Big((\mathcal{C}_k^j)'(k) \Big(\mathcal{P}_{k+1}^0 \mu_k + \Phi_{k+1} \Big) + \mathcal{P}_{k+1}^{j+1} \mu_k \Big) - 2\mu_k' \Phi_{k+1} \\
& - \mu_k' \mathcal{P}_{k+1}^0 \mu_k \Big].
\end{aligned}
$$

Denote

$$
\begin{aligned}
\phi_k^i = & \big(\mathcal{C}_i'(k) - (N_k^i)' \Omega_k^{-1} \mathcal{D}'(k) \big) (\mathcal{P}_{k+1}^0 \bar{\mu}_k + \Phi_{k+1}) \\
& + \big(\bar{\mathcal{C}}_i'(k) - (N_k^i)' \Omega_k^{-1} \bar{\mathcal{D}}'(k) \big) \mathcal{P}_{k+1}^0 \tau + (\mathcal{P}_{k+1}^{i+1})' \bar{\mu}_k.
\end{aligned}
$$

We can obviously know that $\Phi_k = \sum_{i=0}^d \phi_{k+i}^i$. Then,

$$
\begin{aligned}
\mathrm{E}\Big[& 2x_k \Phi_k - 2 \sum_{j=0}^d x_{k-j}' \big((\mathcal{C}_k^j)'(k)(\mathcal{P}_{k+1}^0 \mu_k + \Phi_{k+1}) + \mathcal{P}_{k+1}^{j+1} \mu_k \big) \Big] \\
& = 2 \sum_{j=0}^d x_{k-j}' \phi_k^j - 2 \sum_{j=0}^d \big(\phi_k^j + (N_k^j)' \Omega_k^{-1} \Sigma_k \big) \\
& = -2 \sum_{j=0}^d x_{k-j}' (N_k^j)' \Omega_k^{-1} \Sigma_k.
\end{aligned}
$$

(26)

By virtue of (26), the following equation becomes

$V(x_k) - V(x_{k+1})$

$$
\begin{aligned}
= \mathrm{E}\Big[& x_k' Q_k x_k + u_k' R_k u_k - \Big(u_k + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big)' \Omega_k \Big(u_k \\
& + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big) - 2u_k' \Sigma_k - 2 \sum_{j=0}^d x_{k-j}' (N_k^j)' \Omega_k^{-1} \Sigma_k \\
& - 2\mu_k' \Phi_{k+1} - \mu_k' \mathcal{P}_{k+1}^0 \mu_k \Big] \\
= \mathrm{E}\Big[& x_k' Q_k x_k + u_k' R_k u_k - \Big(u_k + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big)' \Omega_k \Big(u_k \\
& + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} \Big) - 2u_k' \Sigma_k - 2 \sum_{j=0}^d x_{k-j}' (N_k^j)' \Omega_k^{-1} \Sigma_k \\
& - \Sigma_k' \Omega_k^{-1} \Sigma_k + \Sigma_k' \Omega_k^{-1} \Sigma_k - 2\mu_k' \Phi_{k+1} - \mu_k' \mathcal{P}_{k+1}^0 \mu_k \Big] \\
= & x_k' Q_k x_k + u_k' R_k u_k - \Big(u_k + \Omega_k^{-1} \sum_{j=0}^d N_k^j x_{k-j} + \Omega_k^{-1} \Sigma_k \Big)'
\end{aligned}
$$

.

$$\times \Omega_k \Big(u_k + \Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j} + \Omega_k^{-1} \Sigma_k \Big) + \Sigma_k' \Omega_k^{-1} \Sigma_k$$
$$- 2\bar{\mu}_k' \Phi_{k+1} - \mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}].$$

Adding from $k = 0$ to $k = N$, the following equation is obtained:

$$V(x_0) - V(x_{N+1})$$
$$= \sum_{k=0}^{N} \Bigg[x_k' Q_k x_k + u_k' R_k u_k - \Big(u_k + \Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j}$$
$$+ \Omega_k^{-1} \Sigma_k \Big)' \Omega_k \Big(u_k + \Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j} + \Omega_k^{-1} \Sigma_k \Big)$$
$$+ \Sigma_k' \Omega_k^{-1} \Sigma_k - 2\bar{\mu}_k' \Phi_{k+1} - \mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}] \Bigg].$$

Then, the cost function (2) becomes

$$J_N = V(x_0) + \sum_{k=0}^{N} \Bigg[\Big(u_k + \Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j} + \Omega_k^{-1} \Sigma_k \Big)'$$
$$\times \Omega_k \Big(u_k + \Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j} + \Omega_k^{-1} \Sigma_k \Big)$$
$$- \Sigma_k' \Omega_k^{-1} \Sigma_k + 2\bar{\mu}_k' \Phi_{k+1} + \mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{\mu_k}] \Bigg].$$

As $\Omega_k > 0$, the unique optimal controller is

$$u_k^* = -\Omega_k^{-1} \sum_{j=0}^{d} N_k^j x_{k-j} - \Omega_k^{-1} \Sigma_k,$$

which minimized the cost function (2), and the optimal cost is

$$J_N^* = V(x_0) + \sum_{k=0}^{N} \Big(2\bar{\mu}_k' \Phi_{k+1} - \Sigma_k' \Omega_k^{-1} \Sigma_k + \mathrm{Tr}[\mathcal{P}_{k+1}^0 Q_{g_k} 1] \Big).$$

Now, the proof of Theorem 1 is completed.

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