Control Theory Tech, Vol. 18, No. 3, pp. 293–306, August 2020

Control Theory and Technology

http://link.springer.com/journal/11768

Robustness analysis and distributed control of a networked system with time-varying delays

Zhike WANG [1,2], Tong ZHOU [1†]

1.*Department of Automation, Tsinghua University, Beijing 100084, China;*

2.*School of Aviation Operations and Services, University of Air Force Aviation, Changchun Jilin 130022, China*

Received 18 June 2019; revised 16 January 2020; accepted 22 May 2020

Abstract

This paper is concerned with the robustness analysis and distributed output feedback control of a networked system with uncertain time-varying communication delays. This system consists of a collection of linear time-invariant subsystems that are spatially interconnected via an arbitrary directed network. Using a dissipation inequality that incorporates dynamic hard IQCs (integral quadratic constraints) for the delay uncertainties, we derive some sufficient robustness conditions in the form of coupled linear matrix inequalities, in which the coupled parts reflect the interconnection structure of the system. We then provide a procedure to construct a distributed controller to ensure the robust stability of the closed-loop system and to achieve a prescribed ℓ_2-gain performance. The effectiveness of the proposed approach is demonstrated by some numerical examples.

Keywords: Networked system, time delay, integral quadratic constraints (IQCs), robust stability, distributed control

DOI *https://doi.org/10.1007/s11768-020-9109-2*

1 Introduction

Recently, the rapid developments in computer networking technology and control engineering have enabled many large-scale networked systems (LSNS), such as unmanned flight formations, automated highway systems, satellite constellations, and smart structures. The common features of these systems are that they usually consist of numerous subsystems and exhibit complex dynamic behavior as a whole by exchanging information among the subsystems through the interconnection networks. When controlling these systems, it is often necessary to adopt a distributed architecture, in which the controller is also composed of several interconnected units, to take advantage of structure information and reduce the computation or communication complexity.

Great effort has been devoted to investigating the stability analysis and distributed controller synthesis problems for special types of systems, including spatially

†Corresponding author.

E-mail: tzhou@mail.tsinghua.edu.cn.

This work was supported by the National Natural Science Foundation of China (Nos. 61573209, 61733008).

invariant systems [1–4], strongly interconnected systems [5], identical dynamically coupled or decoupled systems [6, 7], heterogeneous spatially distributed systems [8, 9]. Most distributed control approaches assume that the communication is ideal. However, due to the spatially distributed nature of LSNS and the limited network capacity, the transfer of information among these subsystems is subjected to communication delays, which are usually uncertain even time-varying. These network-induced delays can lead to serious performance degradation and even destabilization of the system. For such systems, time-delay robustness must be explicitly addressed to ensure that system performance is achieved.

Some references assumed that signals exchanged by the subsystems affected by arbitrarily small delays [9] or one-step delays [10, 11]. By utilizing these assumptions, some delay-independent robust stability criteria were derived. These criteria, however, fail to exploit the gain or phase properties of the delay, and may be overly conservative when the upper and lower bounds of the delay are given. The paper [12] considered an interconnected passive system with time delays in the interconnections, and presented an exact integral quadratic constraint (IQC) based delay-dependent stability condition. However, this condition is only applicable to a special system with a cyclic interconnection structure and is not applicable to distributed controller design.

In this paper, we consider the robustness analysis and distributed output-feedback control of a networked system (NS) with time-varying communication delays. The NS adopted here consists of a collection of linear time-invariant (LTI) subsystems interconnected through an arbitrary directed network. The objective is to derive some delay-dependent and computationally efficient conditions for robustness analysis, and construct a distributed controller with the same architecture as the plant that achieves robust stability and ℓ_2-gain performance against the time-varying communication delays.

To accomplish this, we follow the so-called IQC approach for capturing the input-output properties of time-varying delay uncertainties. The first contribution of this paper is to extend the merging of time-domain IQC descriptions of uncertainties with dissipation theory to networked systems. Inspired by the recent works [8], we construct a dissipation inequality based on the hard factorization of IQCs multipliers for each individual subsystem via a neutral interconnection constraint, in which the time-domain IQCs are interpreted as dynamic supply functions for each subsystem. This paves the way for the distributed controller design by respecting the interconnection structure of the system.

Specifically, by separating the delay operators from the interconnections, we first describe the uncertain NS by a linear fractional transformation (LFT) model. The properties of the time-varying delays are captured by suitable families of dynamic hard IQCs. Using a dissipation inequality that incorporates the IQCs for delay uncertainties, we establish sufficient conditions for robust stability and ℓ_2-gain performance of the NS against delay uncertainties in the form of coupled linear matrix inequalities (LMI). These conditions depend only on parameters of each subsystem, IQC multipliers and interconnection structure information. This property makes them computationally attractive for an LSNS with sparse interconnections.

As the second contribution of this paper, we develop a method for the distributed controller design by adopting the dynamic IQC constraints directly to handle the case when the controller is also affected by communication delays. On the basis of the dualization lemma and elimination lemma in [13] and [14], the conditions for the existence of a distributed output-feedback controller are formulated in terms of several matrix inequalities, which are non-convex. Then, we show how these conditions can be convexified.

The remaining of this paper is organized as follows. Section 2 gives a description of the adopted NS and the problem formulation. Section 3 develops the robust stability and performance conditions of the NS with time-varying communication delays. Section 4 presents the distributed controller design procedure. Some numerical results are reported in Section 5. Section 6 concludes this paper.

Notation Symbols $\mathbb{R}^n, \mathbb{R}^n_{\mathrm{S}}, \mathbb{R}^{m\times n}, \mathbb{R}^+, \mathbb{Z}^+$ denote the sets of n-dimensional real vectors, n-dimensional real symmetric matrices, $m \times n$ real matrices, nonnegative real numbers and nonnegative integers, respectively. ℓ_2^n denotes the set of n-dimensional square summable signals, while $\ell_{2\mathrm{e}}^n$ represents the extended set of n-dimensional locally square summable signals. Notation $\mathbb{R}\mathcal{L}_\infty^{m\times n}$ is used to denote the space of proper rational transfer matrix with no poles on the unit circle. $\mathbb{R}\mathrm{H}_\infty^{m\times n}$

represents the subspace of $\mathbb{R}\mathcal{L}_\infty^{m\times n}$ consisting of functions with no poles outside the open unit disk. The transpose and conjugate transpose of a matrix X are denoted by X^{T} and X^*. The para-Hermitian conjugate of $\Psi \in \mathbb{R}\mathcal{L}_\infty^{m\times n}$, denoted as $\Psi^\sim$, is defined by $\Psi^\sim(\zeta) := \Psi^{\mathrm{T}}(\zeta^{-1})$ where ζ denotes the variable of $\mathcal{Z}$-transformation. The $n\times n$ identity and the $m\times n$ zero matrix are denoted by I_n and $0_{m\times n}$, respectively, or just I and 0 if dimension is clear from context. $\mathrm{diag}\{X_i|_{i=1}^L\}$ denotes a block diagonal matrix, while $\mathrm{col}\{X_i|_{i=1}^L\}$ the vector/matrix stacked by $X_i|_{i=1}^L$. Objects that can be inferred by symmetry are sometimes indicated by $\star$. The Kronecker product is denoted by $\otimes$. The inertia of a symmetric matrix M is defined as $\mathrm{in}(M) = (\mathrm{in}_-(M), \mathrm{in}_0(M), \mathrm{in}_+(M))$ with $\mathrm{in}_-(M)$, $\mathrm{in}_0(M)$, $\mathrm{in}_+(M)$) denoting the number of negative, zero, and positive eigenvalues of M.

Given a time-varying delay $\tau(t) \in \mathbb{Z}^+$. Let $\mathcal{D}_\tau$ denote the time delay operator: $(\mathcal{D}_\tau v)(t) = v(t-\tau(t))$, and $\mathcal{S}_\tau$ denote the "delay-difference" operator $(\mathcal{D}_\tau - I)$; i.e., $\mathcal{S}_\tau(v) := v(t-\tau(t)) - v(t)$.

2 Problem formulation

Consider a networked system $\boldsymbol{\Sigma}$ consisting of N linear time invariant dynamic subsystems. Each subsystem $\boldsymbol{\Sigma}_i$ is described by the following discrete state-space equation:

$$\begin{bmatrix} x(t+1,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{xx,i} & A_{xv,i} & B_{xd,i} \\ A_{zx,i} & A_{zv,i} & B_{zd,i} \\ C_{ex,i} & C_{ev,i} & D_{ed,i} \end{bmatrix} \begin{bmatrix} x(t,i) \\ v(t,i) \\ d(t,i) \end{bmatrix}, \tag{1}$$

where $i = 1, 2, \ldots, N$. t and i stand for the temporal variable and the index number of a subsystem, respectively. $x(t,i) \in \mathbb{R}^{n_{x_i}}$ is the state vector of the ith subsystem $\boldsymbol{\Sigma}_i$ at time t. $z(t,i) \in \mathbb{R}^{n_{z_i}}/v(t,i) \in \mathbb{R}^{n_{v_i}}$ is the output/input vector of the $\boldsymbol{\Sigma}_i$ to/from other subsystems, which is also called internal output/input vector throughout this paper. $e(t,i) \in \mathbb{R}^{n_{e_i}}/d(t,i) \in \mathbb{R}^{n_{d_i}}$ is the performance output/disturbance input of the $\boldsymbol{\Sigma}_i$.

Define $\mathcal{N}_{\mathrm{T}}(i)$ as the index set of the subsystems that have internal outputs to subsystem $\boldsymbol{\Sigma}_i$ and $\mathcal{N}_{\mathrm{F}}(i)$ as the index set of the subsystems that have internal inputs from subsystem $\boldsymbol{\Sigma}_i$. Thus, the internal input $v(t,i)$ and internal output $z(t,i)$ can be partitioned as $v(t,i) := \mathrm{col}\{v_p(t,i)|_{p\in\mathcal{N}_{\mathrm{T}}(i)}\}$ and $z(t,i) := \mathrm{col}\{z_q(t,i)|_{q\in\mathcal{N}_{\mathrm{F}}(i)}\}$, respectively.

To a distinct pair of subsystems, indexed by i and j, we assume $j \in \mathcal{N}_{\mathrm{T}}(i)$. The constraint of the interconnection between $\boldsymbol{\Sigma}_i$ and $\boldsymbol{\Sigma}_j$ can be expressed as

$$v_j(t,i) = (\mathcal{D}_{\tau_{ij}} z_i)(t,j), \quad j \in \mathcal{N}_{\mathrm{T}}(i), \tag{2}$$

where $v_j(t,i) \in \mathbb{R}^{n_{v_{ij}}}$ and $z_i(t,j) \in \mathbb{R}^{n_{z_{ji}}}$. It is obvious that $n_{v_{ij}} = n_{z_{ji}}$. $\mathcal{D}_{\tau_{ij}}$ is the delay operator that is defined by $v_j(t,i) = z_i(t-\tau_{ij}(t), j)$, where the delay duration $\tau_{ij}(t)$ is uncertain and time-varying. To simplify the notation, we write $\tau_{ij}(t)$ as τ_{ij}. The upper bound of τ_{ij} is denoted by $\mathcal{T}_{u_{ij}} \in \mathbb{Z}^+$ such that $\tau_{ij} \in [0, \mathcal{T}_{u_{ij}}]$. Hence, the subsystems are connected through

$$v(t) = (\mathcal{D}_\tau \Phi z)(t). \tag{3}$$

Here, $z(t) = \mathrm{col}\{z(t,i)|_{i=1}^N\}$ and $v(t) = \mathrm{col}\{v(t,i)|_{i=1}^N\}$. In addition, $\mathcal{D}_\tau$ is the delay operator generated via $\mathcal{D}_{\tau_{ij}}$,

$$\mathcal{D}_\tau = \mathrm{diag}\{\mathrm{diag}\{\mathcal{D}_{\tau_{ij}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}|_{i=1}^N\}. \tag{4}$$

The interconnected structure is illustrated in Fig. 1 for the case $N = 3$. It is assumed that every row of the subsystem connection matrix (SCM) Φ has only one non-zero element which is equal to one. As argued in [15–17], this assumption explicitly describes the connection between the internal inputs and outputs of different subsystems, and does not introduce any restrictions on the structure of the adopted system. Since each individual subsystem only interact with a small number neighboring subsystems, the SCM Φ usually has a sparse structure. For the case in Fig. 1, the SCM Φ is given by

$$\Phi = \left[\begin{array}{cc|c|c} 0 & 0 & I_{n_{z_{21}}} & 0 \\ \hline I_{n_{z_{12}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{z_{32}}} \\ \hline 0 & I_{n_{z_{13}}} & 0 & 0 \end{array}\right]. \tag{5}$$

Introduce the vector $\bar{v}(t)$ and let $\bar{v}(t) = \Phi z(t)$. We have that $v(t) = (\mathcal{D}_\tau \bar{v})(t)$. For subsystem $\boldsymbol{\Sigma}_i$, the internal input vector $v(t,i)$ can be expressed by $v(t,i) = (\mathcal{D}_{\tau_i}\bar{v})(t,i)$, where $\mathcal{D}_{\tau_i} = \mathrm{diag}\{\mathcal{D}_{\tau_{ij}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$. Based on these relations and equation (1), the state-space description of subsystem $\boldsymbol{\Sigma}_i$ can be rewritten as

$$\begin{bmatrix} x(t+1,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{xx,i} & A_{xv,i} & B_{xd,i} \\ A_{zx,i} & A_{zv,i} & B_{zd,i} \\ C_{ex,i} & C_{ev,i} & D_{ed,i} \end{bmatrix} \begin{bmatrix} x(t,i) \\ (\mathcal{D}_{\tau_i}\bar{v})(t,i) \\ d(t,i) \end{bmatrix}. \tag{6}$$

The interconnections among the subsystems are described by

$$\bar{v}(t) = \Phi z(t). \tag{7}$$

It is obvious that the interconnections described by (7) are turned to be ideal, that is, the flow of information between any two subsystems will be instantaneous. Moreover, the internal input of subsystem $\boldsymbol{\Sigma}_i$ becomes a delayed signal defined by the diagonal delay operators $\mathcal{D}_{\tau_i}$.

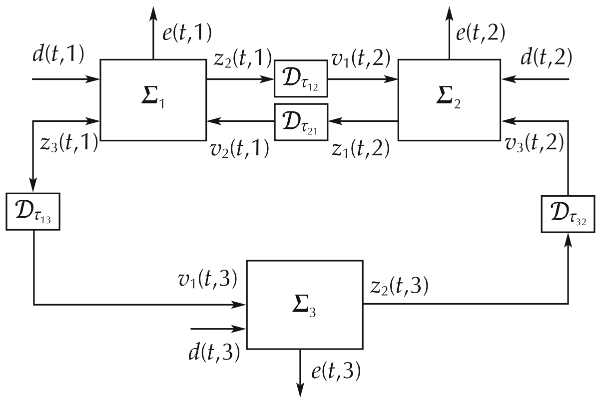

Fig. 1 Networked system with communication delays ($N = 3$).

To facilitate the robustness analysis, a model transformation will be performed on the original model in order to separate the delay uncertainties from the nominal LTI subsystems. Specifically, we introduce two vectors $w(t,i)$ and $q(t,i)$. Let $w(t,i) = \bar{v}(t,i)$ and $q(t,i) = (\mathcal{D}_{\tau_i}\bar{v})(t,i) - \bar{v}(t,i)$. From equation (6), we have the augmented model of the subsystem $\boldsymbol{\Sigma}_i$ described in the following LFT form:

$$\begin{cases} \begin{bmatrix} x(t+1,i) \\ w(t,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{xx,i} & A_{xv,i} & A_{xv,i} & B_{xd,i} \\ 0 & 0 & I & 0 \\ A_{zx,i} & A_{zv,i} & A_{zv,i} & B_{zd,i} \\ C_{ex,i} & C_{ev,i} & C_{ev,i} & D_{ed,i} \end{bmatrix} \begin{bmatrix} x(t,i) \\ q(t,i) \\ \bar{v}(t,i) \\ d(t,i) \end{bmatrix}, \\ q(t,i) = (\mathcal{S}_{\tau_i} w)(t,i), \end{cases} \tag{8}$$

where $\mathcal{S}_{\tau_i}$ is a diagonal delay-difference operator, and it is obvious that $\mathcal{S}_{\tau_i} = \mathrm{diag}\{\mathcal{S}_{\tau_{ij}}|_{j\in\mathcal{N}_\mathrm{T}(i)}\}$. Let $d(t) = \mathrm{diag}\{d(t,i)|_{i=1}^N\}$ and $e(t) = \mathrm{diag}\{e(t,i)|_{i=1}^N\}$. The definition of stability of the NS $\boldsymbol{\Sigma}$ is adapted to the current formulation as follows.

Definition 1 The networked system $\boldsymbol{\Sigma}$ formulated by the feedback interconnection subsystem (8) and the subsystem connection (7) is stable if it is well-posed and if the mapping from $d(t)$ to $e(t)$ has finite ℓ_2 gain.

3 Robust stability and performance analysis

IQCs is a powerful tool in robust stability analysis of uncertain systems. They are extensively applied to specify a constraint on the input/output signals of the uncertainty. More precisely, let Δ denote a bounded, causal operator. Two signals $w \in \ell_2^m$ and $q \in \ell_2^n$ related by $q = \Delta(w)$ satisfy the IQC defined by Π if

$$\int_{-\pi}^{\pi} \begin{bmatrix} \hat{w}(\mathrm{e}^{\mathrm{j}\omega}) \\ \hat{q}(\mathrm{e}^{\mathrm{j}\omega}) \end{bmatrix}^* \Pi(\mathrm{e}^{\mathrm{j}\omega}) \begin{bmatrix} \hat{w}(\mathrm{e}^{\mathrm{j}\omega}) \\ \hat{q}(\mathrm{e}^{\mathrm{j}\omega}) \end{bmatrix} \mathrm{d}\omega \geqslant 0, \tag{9}$$

where $\hat{w}$ and $\hat{q}$ are Fourier transforms of w and q, respectively. Π is a bounded LTI self-adjoint operator on ℓ_2 space, while $\Pi(\mathrm{e}^{\mathrm{j}\omega})$ is its frequency response function. The following definition characterize the IQC in the time domain.

Definition 2 Let $\Pi \in \mathbb{RL}_\infty^{(m+n)\times(m+n)}$ be factorized as $\Psi^\sim M\Psi$ where $M \in \mathbb{R}_\mathrm{S}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times(m+n)}$. Then (Ψ, M) is a hard IQC factorization of Π if for any bounded, causal operator Δ satisfying the IQC defined by Π the following inequality holds:

$$\sum_{t=0}^{T} (z_\psi(t))^\mathrm{T} M z_\psi(t) \geqslant 0 \tag{10}$$

for all $T \geqslant 0$, $v \in \ell_2^n$, $w = \Delta(v)$, and $z = \Psi \begin{bmatrix} v \\ w \end{bmatrix}$.

A time domain IQC as in Definition 1 is referred to as a hard IQC in [18]. In contrast, factorizations for which the time domain constraint holds only for $T = \infty$ are called soft IQCs. It should be noted that the hard/soft property depends on the factorization (Ψ, M) but not be inherent to the multiplier Π [19]. The distinction is important because the dissipation inequality is valid only for hard IQCs. The next lemma provides a sufficient condition that Π has a hard factorization.

Lemma 1 Let $\Pi \in \mathbb{RL}_\infty^{(m+n)\times(m+n)}$ if $\Pi = \Psi^\sim M\Psi$ and partition as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_\infty^{m\times m}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{n\times n}$. Assume $\Pi_{11} > 0$ and $\Pi_{22} < 0$, then Π has a hard factorization (Ψ, M).

Assume that the delay-difference operator $\mathcal{S}_{\tau_i}$ satisfies a collection of IQC multipliers $\{\Pi_k, k = 1, 2, \ldots, N_\tau\}$, which can be partitioned as $\begin{bmatrix} \Pi_{11,k} & \Pi_{12,k} \\ \Pi_{12,k}^\sim & \Pi_{22,k} \end{bmatrix}$ with $\Pi_{11,k} > 0$ and $\Pi_{22,k} < 0$. According to Lemma 1, every Π_k

has a $J_{n_{v_i},n_{v_i}}$-spectral factorization $(\Psi_{k,i}, M_{k,i})$ using the methods of [19]. We further assume the factorization $(\Psi_{k,i}, M_{k,i})$ has the form of $\Psi_{k,i} = \begin{bmatrix}\psi_{11,ki} & \psi_{12,ki}\\ 0 & I_{n_{v_i}}\end{bmatrix}$ and $M_{k,i} = \begin{bmatrix}I_{n_{v_i}} & 0\\ 0 & -I_{n_{v_i}}\end{bmatrix}$. As argued in [20], this assumption is without loss any generality.

Then, the state-space realization of the associated system $\Psi_{k,i}$ can be described as

$$\begin{bmatrix}x_{\psi_k}(t+1,i)\\ z_{\psi_k}(t,i)\end{bmatrix} = \begin{bmatrix}A_{\psi_k,i} & B_{\psi_k q,i} & B_{\psi_k w,i}\\ \bar{C}_{\psi_k,i} & \bar{D}_{\psi_k q,i} & \bar{D}_{\psi_k w,i}\end{bmatrix}\begin{bmatrix}x_{\psi_k}(t,i)\\ q(t,i)\\ w(t,i)\end{bmatrix}, \tag{11}$$

where $x_{\psi_k}(t,i) \in \mathbb{R}^{n_{\psi_{k,i}}}$ denotes the state vector of the associated system $\Psi_{k,i}$ with $x_{\psi_k}(0,i) = 0$, and $z_{\psi_k}(t,i) \in \mathbb{R}^{2n_{v_i}}$ is the output of $\Psi_{k,i}$. The output matrices have the following structure for all $k = 1, 2, \ldots, N_\tau$,

$$\bar{C}_{\psi_k,i} = \begin{bmatrix}C_{\psi_k,i}\\ 0\end{bmatrix},\ \bar{D}_{\psi_k q,i} = \begin{bmatrix}D_{\psi_k q,i}\\ I_{n_{v_i}}\end{bmatrix},\ \bar{D}_{\psi_k w,i} = \begin{bmatrix}D_{\psi_k w,i}\\ 0\end{bmatrix}.$$

All $\{\Psi_{k,i}, k = 1, 2, \ldots, N_\tau\}$ are aggregated into a single system Ψ_i with the following state-space realization:

$$\begin{bmatrix}x_{\psi}(t+1,i)\\ z_{\psi}(t,i)\end{bmatrix} = \begin{bmatrix}A_{\psi,i} & B_{\psi q,i} & B_{\psi w,i}\\ \begin{bmatrix}C_{\psi,i} & D_{\psi q,i} & D_{\psi w,i}\\ 0 & I_{n_{v_i}} & 0\end{bmatrix}\end{bmatrix}\begin{bmatrix}x_{\psi}(t,i)\\ q(t,i)\\ w(t,i)\end{bmatrix}, \tag{12}$$

where $x_\psi(t,i) = \operatorname{col}\{x_{\psi_k}(t,i)|_{k=1}^{N_\tau}\}$ with $x_\psi(t,i) \in \mathbb{R}^{n_{\psi_i}}$. The matrix parameters are partitioned as

$$\begin{aligned}
&A_{\psi,i} = \operatorname{diag}\{A_{\psi_k,i}|_{k=1}^{N_\tau}\},\ B_{\psi q,i} = \operatorname{col}\{B_{\Psi_k q,i}|_{k=1}^{N_\tau}\},\\
&B_{\psi w,i} = \operatorname{col}\{B_{\Psi_k w,i}|_{k=1}^{N_\tau}\},\ C_{\Psi,i} = \operatorname{diag}\{C_{\Psi_k,i}|_{k=1}^{N_\tau}\},\\
&D_{\psi q,i} = \operatorname{col}\{D_{\Psi_k q,i}|_{k=1}^{N_\tau}\},\ D_{\psi w,i} = \operatorname{col}\{D_{\Psi_k w,i}|_{k=1}^{N_\tau}\}.
\end{aligned}$$

Now, we combine the IQC-induced system (12) to the augmented system (8). The resulting extended system can be described in the following form for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N_\tau$,

$$\begin{bmatrix}\tilde{x}(t+1,i)\\ z_{\psi_k}(t,i)\\ z(t,i)\\ e(t,i)\end{bmatrix} = \begin{bmatrix}A_{t,i} & B_{t1,i} & B_{t2,i} & B_{t3,i}\\ C_{k,i} & D_{k1,i} & D_{k2,i} & 0\\ \bar{A}_{zx,i} & A_{zv,i} & A_{zv,i} & B_{zd,i}\\ \bar{C}_{ex,i} & C_{ev,i} & C_{ev,i} & D_{ed,i}\end{bmatrix}\begin{bmatrix}\tilde{x}(t,i)\\ q(t,i)\\ \bar{v}(t,i)\\ d(t,i)\end{bmatrix}, \tag{13}$$

where $\tilde{x}(t,i) = [x_\psi^{\mathrm{T}}(t,i)\ x^{\mathrm{T}}(t,i)]^{\mathrm{T}} \in \mathbb{R}^{n_{\tilde{x}_i}}$ with $n_{\tilde{x}_i} = n_{x_i} + n_{\psi_i}$ and the system matrices given by

$$\begin{cases}
A_{t,i} = \begin{bmatrix}A_{\psi,i} & 0\\ 0 & A_{xx,i}\end{bmatrix},\ B_{t1,i} = \begin{bmatrix}B_{\psi q,i}\\ A_{xv,i}\end{bmatrix},\\
B_{t2,i} = \begin{bmatrix}B_{\psi w,i}\\ A_{xv,i}\end{bmatrix},\ B_{t3,i} = \begin{bmatrix}0\\ B_{xd,i}\end{bmatrix},\\
C_{k,i} = \begin{bmatrix}[0_{n_{v_i}\times n_\alpha}\ C_{\psi_k,i}\ 0_{n_{v_i}\times n_\beta}] & 0_{n_{v_i}\times n_{x_i}}\\ 0 & 0\end{bmatrix},\\
D_{k1,i} = \begin{bmatrix}0\\ I_{n_{v_i}}\end{bmatrix},\ D_{k2,i} = \begin{bmatrix}D_{\psi_k w,i}\\ 0\end{bmatrix},\\
\bar{A}_{zx,i} = [0_{n_{z_i}\times n_{\psi_i}}\ A_{zx,i}],\ \bar{C}_{ex,i} = [0_{n_{e_i}\times n_{\psi_i}}\ C_{ex,i}],
\end{cases} \tag{14}$$

where $n_\alpha = \sum_{\varsigma=1}^{k-1} n_{\psi_{\varsigma,i}}$ and $n_\beta = \sum_{\varsigma=k+1}^{N_\tau} n_{\psi_{\varsigma,i}}$.

In order to establish the stability conditions of the NS against time-varying delays, we will establish the connection between time-domain IQCs and the dissipation theory. We first introduce the following definition of dissipativity [21].

Definition 3 (Dissipativity) A discrete time system Σ_G is (Q, S, R)-dissipative with respect to the energy supply rate $s(u, y) := y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$, if there exists a non-negative storage function $V(x) : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$V(x(j)) \leqslant V(x(i)) + \sum_{t=i}^{j-1}\begin{bmatrix}y(t)\\ u(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q & S\\ S^{\mathrm{T}} & R\end{bmatrix}\begin{bmatrix}y(t)\\ u(t)\end{bmatrix} \tag{15}$$

holds for all $x \in \mathbb{R}^n$, all times $j > i \geqslant 0, i, j \in \mathbb{Z}^+$, and all input $u \in \ell_{2\mathrm{e}}^m$.

The energy supply rate function $s(u, y)$ can be defined according to the quadratic performance criteria of the system. In this paper, the NS $\boldsymbol{\Sigma}$ is required to be finite-gain ℓ_2 stable, which is equivalent to that there exists $\gamma > 0$ such that it is $(-I, 0, \gamma^2 I)$-dissipative.

Furthermore, utilizing the neutral interconnection constraint, we introduce the following quadratic form as a special energy supply rate for each subsystem

$$\begin{aligned}
Q_i(z(t,i), \bar{v}(t,i)) &= \begin{bmatrix}z(t,i)\\ \bar{v}(t,i)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}U_i & 0\\ 0 & V_i\end{bmatrix}\begin{bmatrix}z(t,i)\\ \bar{v}(t,i)\end{bmatrix}\\
&= \sum_{j=1}^{N}\begin{bmatrix}z_j(t,i)\\ \bar{v}_j(t,i)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}X_{ij} & 0\\ 0 & Y_{ij}\end{bmatrix}\begin{bmatrix}z_j(t,i)\\ \bar{v}_j(t,i)\end{bmatrix},
\end{aligned} \tag{16}$$

where $U_i = \operatorname{diag}\{X_{ij}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\}$ and $V_i = \operatorname{diag}\{Y_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$.

Let the scaling matrix $X_{ij} = -Y_{ji}$, then

$$Q(z(t), \bar{v}(t)) = \sum_{i=1}^{N} Q_i(z(t,i), \bar{v}(t,i)) = 0. \tag{17}$$

The following theorem provides sufficient conditions for robustness analysis of an NS with time-varying communication delays.

Theorem 1 Consider the networked system $\boldsymbol{\Sigma}$. Given the upper bound vector $\mathcal{T}_{u_i} = \mathrm{col}\{\mathcal{T}_{u_{ij}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ of the time-varying delay $\tau_i = \mathrm{col}\{\tau_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ for all $i = 1, 2, \dots, N$ with τ_i and $\mathcal{T}_{u_{ij}} \in \mathbb{Z}^+$, if there exist positive-definite matrices $P_i \in \mathbb{R}_{\mathrm{S}}^{n_{x_i}+n_{\psi_i}}$ and $X_{ij} \in \mathbb{R}_{\mathrm{S}}^{n_{z_{ij}}}$, and positive-definite diagonal matrices $X_{k,i} \in \mathbb{R}_{\mathrm{S}}^{n_{v_i}}$ for all $i = 1, 2, \dots, N$, $j \in \mathcal{N}_{\mathrm{T}}(i)$ and $k = 1, 2, \dots, N_\tau$ such that

$$\begin{aligned}&\begin{bmatrix} A_{t,i}^{\mathrm{T}}P_iA_{t,i} - P_i & \star \\ B_{t,i}^{\mathrm{T}}P_iA_{t,i} & B_{t,i}^{\mathrm{T}}P_iB_{t,i}\end{bmatrix} + [\star]^{\mathrm{T}}\begin{bmatrix} U_i & 0 \\ 0 & V_i\end{bmatrix}[\,C_{\phi,i}\;\; D_{\phi,i}\,]\\ &+\sum_{k=1}^{N_\tau}(I_2\otimes X_{k,i})[\star]^{\mathrm{T}}M_{k,i}[\,C_{k,i}\;\; D_{k,i}\,]\\ &+[\star]^{\mathrm{T}}\begin{bmatrix} I & 0\\ 0 & -\gamma^2 I\end{bmatrix}[\,C_{p,i}\;\; D_{p,i}\,] < 0\end{aligned} \tag{18}$$

is satisfied for all $i = 1, 2, \dots, N$ with

$$\begin{cases} U_i = \mathrm{diag}\{X_{ij}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\},\;\; V_i = \mathrm{diag}\{Y_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\},\\ X_{ij} = -Y_{ji},\;\; B_{t,i} = [B_{t1,i}\;\; B_{t2,i}\;\; B_{t3,i}],\\ C_{\phi,i} = \begin{bmatrix} 0_{n_{z_i}\times n_{\psi_i}} & A_{zx,i}\\ 0 & 0\end{bmatrix},\;\; C_{p,i} = \begin{bmatrix} 0_{n_{e_i}\times n_{\psi_i}} & C_{ex,i}\\ 0 & 0\end{bmatrix},\\ D_{\phi,i} = \begin{bmatrix} A_{zv,i} & A_{zv,i} & B_{zd,i}\\ 0 & I & 0\end{bmatrix},\;\; D_{p,i} = \begin{bmatrix} C_{ev,i} & C_{ev,i} & D_{ed,i}\\ 0 & 0 & I\end{bmatrix},\\ D_{k,i} = [D_{k1,i}\;\; D_{k2,i}].\end{cases} \tag{19}$$

Then, system $\boldsymbol{\Sigma}$ is robustly stable for all the time-varying communication delay $\tau_{ij} \in [0, \mathcal{T}_{u_{ij}}]$, and the induced ℓ_2 gain from d to e is no more than a priori given $\gamma > 0$.

Proof Assume that all the IQC multipliers $\{\Pi_k, k = 1, 2, \dots, N_\tau\}$ for $\mathcal{S}_{\tau_i}$ depend on the upper bound vector $\mathcal{T}_{u_i}$ of the time-varying delay τ_i. When the value of $\mathcal{T}_{u_i}$ is given, we can construct the $J_{n_{v_i},n_{v_i}}$-spectral factorization (Ψ_k, M_k) of Π_k. From the $(1,1)$ block of the left-hand side of (18), we have

$$\begin{aligned}&A_{t,i}^{\mathrm{T}}P_iA_{t,i} - P_i + \sum_{k=1}^{N_\tau} X_{k,i}C_{k,i}^{\mathrm{T}}C_{k,i} + C_{\phi,i}^{\mathrm{T}}U_iC_{\phi,i}\\ &+ C_{p,i}^{\mathrm{T}}C_{p,i} < 0,\end{aligned} \tag{20}$$

where P_i and U_i are positive definite according to hypothesis. This implies that

$$A_{t,i}^{\mathrm{T}}P_iA_{t,i} - P_i < 0. \tag{21}$$

Then, each individual subsystem is stable.

Define the storage function $V : \mathbb{R}^{n_{x_i}+n_\psi} \to \mathbb{R}^+$ as $V(\tilde{x}(t,i)) = \tilde{x}^{\mathrm{T}}(t,i)P_i\tilde{x}(t,i)$. Multiplying the left and right sides of (18) by $[\tilde{x}^{\mathrm{T}}(t,i)\;\; q^{\mathrm{T}}(t,i)\;\; \bar{v}^{\mathrm{T}}(t,i)\;\; d^{\mathrm{T}}(t,i)]$ and its transpose and summing over $i = 1, \dots, N$ yield

$$\begin{aligned}&\tilde{x}^{\mathrm{T}}(t+1)P\tilde{x}(t+1) - \tilde{x}^{\mathrm{T}}(t)P\tilde{x}(t) + Q(z(t), \bar{v}(t))\\ &+\sum_{k=1}^{N_\tau} X_k z_{\psi_k}^{\mathrm{T}}(t)M_k z_{\psi_k}(t) + e^{\mathrm{T}}(t)e(t) - \gamma^2 d^{\mathrm{T}}(t)d(t) < 0,\end{aligned} \tag{22}$$

in which

$$\begin{aligned}&\tilde{x}(t) = \mathrm{col}\{\tilde{x}(t,i)|_{i=1}^{N}\},\;\; P = \mathrm{diag}\{P_i|_{i=1}^{N}\},\\ &z_{\psi_k}(t) = \mathrm{col}\{z_{\psi_k}(t,i)|_{i=1}^{N}\},\;\; X_k = \mathrm{diag}\{X_{k,i}|_{i=1}^{N}\},\\ &M_k = \mathrm{diag}\{M_{k,i}|_{i=1}^{N}\}.\end{aligned}$$

Furthermore, using the fact that $Q(z(t), \bar{v}(t)) = 0$ and summing both sides of the above inequality from $t = 0$ to $t = T$ with zero initial conditions, we obtain

$$\begin{aligned}&\tilde{x}^{\mathrm{T}}(T+1)P\tilde{x}(T+1) + \sum_{k=1}^{N_\tau} X_k\sum_{t=0}^{T} z_{\psi_k}^{\mathrm{T}}(t)M_k z_{\psi_k}(t)\\ &+\sum_{t=0}^{T} e^{\mathrm{T}}(t)e(t) - \gamma^2\sum_{t=0}^{T} d^{\mathrm{T}}(t)d(t) < 0.\end{aligned} \tag{23}$$

Applying the hard IQC condition (10) and nonnegativity of the storage function V, we obtain

$$\sum_{t=0}^{T} e^{\mathrm{T}}(t)e(t) < \gamma^2\sum_{t=0}^{T} d^{\mathrm{T}}(t)d(t). \tag{24}$$

Now, we conclude that the NS is robustly stable for all the communication time-varying delay $\tau_{ij} \in [0, \mathcal{T}_{u_{ij}}]$, and the worst-case ℓ_2 gain from d to e is no more than γ. □

Theorem 1 involves parameter dependent LMI conditions. Note that inequality (22) requires the time-domain IQC to hold over finite time horizons. Hence the existence of a hard factorization of Π_k permits the merging of IQC descriptions of delay uncertainties with dissipation theory. As was shown in [18,22,23], a broad class of multipliers has a hard factorization. Thus, for the given $\mathcal{T}_{u_{ij}}$ and γ, Theorem 1 provides convex conditions on P_i, X_{ij} and $X_{k,i}$ that are sufficient to upper bound the ℓ_2 gain from d to e.

Note also that the conditions (18) are coupled LMIs through the scaling matrices X_{ij} for all $i = 1, 2, \ldots, N$ and $j \in \mathcal{N}_{\mathrm{T}}(i)$. This is based on the constraint (16) that expresses the neutral property of the interconnections in terms of energy supply. The number of coupled independent scaling matrices in (18) is equal to the number of non-zero blocks of the SCM Φ. Hence, if Φ has a sparse structure, the sparsity of the interconnection structure can also be reflected by the number of coupled scaling matrices.

The classical approach to analyze the robust stability and performance of the NS is to eliminate the interconnection constraint (8) to describe the entire system as a lumped system. Then, a robustness condition based on the lumped formulation of the system can be derived using IQCs and a standard dissipation argument as in [19, Theorem 3]. Compared with the conditions based on the lumped formulation of the IQC analysis problem (LF-IQC), the conditions in Theorem 1 are completely determined by the parameters of subsystem $\boldsymbol{\Sigma}_i$, the IQC multipliers and the interconnection structure information of the system.

It should be noted that the conditions in Theorem 1 are more conservative than those in LF-IQC although both of them are based on the IQC framework, because we have restricted the quadratic supply rate to a particular type as in equation (16) when utilizing the neutral interconnection property of dissipative systems. However, by considering these supply rates as free parameters, we take advantage of the interconnection information of the directed network and give less conservative analysis conditions than those with fixed supply rates. Another source of conservatism in Theorem 1 is derived from the selected IQC multipliers modeling the delay uncertainties, which can be reduced by exploring more compact IQCs to bound the delay-difference operators.

4 Robust distributed control design

We extend the subsystem $\boldsymbol{\Sigma}_i$ with a control input $u(t,i) \in \mathbb{R}^{n_{u_i}}$ and a measurement output $y(t,i) \in \mathbb{R}^{n_{y_i}}$, which leads to the following state-space description

$$\begin{bmatrix} x(t+1,i) \\ z(t,i) \\ e(t,i) \\ y(t,i) \end{bmatrix} = \begin{bmatrix} A_{xx,i} & A_{xv,i} & B_{xd,i} & B_{xu,i} \\ A_{zx,i} & A_{zv,i} & B_{zd,i} & B_{zu,i} \\ C_{ex,i} & C_{ev,i} & D_{ed,i} & D_{eu,i} \\ C_{yx,i} & C_{yv,i} & D_{yd,i} & 0 \end{bmatrix} \begin{bmatrix} x(t,i) \\ v(t,i) \\ d(t,i) \\ u(t,i) \end{bmatrix}. \tag{25}$$

We assume that the to-be-designed controller K is another interconnected system that consists of N controller subsystems. The controller subsystem K_i associated with (25) is described as

$$\begin{bmatrix} x^{\mathrm{K}}(t+1,i) \\ z^{\mathrm{K}}(t,i) \\ u(t,i) \end{bmatrix} = \underbrace{\begin{bmatrix} A^{\mathrm{K}}_{xx}(i) & A^{\mathrm{K}}_{xv,i} & B^{\mathrm{K}}_{xu,i} \\ A^{\mathrm{K}}_{zx,i} & A^{\mathrm{K}}_{zv,i} & B^{\mathrm{K}}_{zu,i} \\ C^{\mathrm{K}}_{x,i} & C^{\mathrm{K}}_{v,i} & D^{\mathrm{K}}_{u,i} \end{bmatrix}}_{\mathcal{K}_i} \begin{bmatrix} x^{\mathrm{K}}(t,i) \\ v^{\mathrm{K}}(t,i) \\ y(t,i) \end{bmatrix}, \tag{26}$$

where $x^{\mathrm{K}}(t,i) \in \mathbb{R}^{n_{x_i^{\mathrm{K}}}}$, $z^{\mathrm{K}}(t,i) \in \mathbb{R}^{n_{z_i^{\mathrm{K}}}}$ and $v^{\mathrm{K}}(t,i) \in \mathbb{R}^{n_{v_i^{\mathrm{K}}}}$ with $i = 1, \ldots, N$. Notice that controller subsystems can communicate via the signals $v^{\mathrm{K}}(t,i)$ and $z^{\mathrm{K}}(t,i)$. Assume that the distributed controller shares the same interconnection topology with that of the plant. In this way, the $v^{\mathrm{K}}(t,i)$ and $z^{\mathrm{K}}(t,i)$ can be partitioned as $v^{\mathrm{K}}(t,i) := \mathrm{col}\{v^{\mathrm{K}}_p(t,i)|_{p\in\mathcal{N}_{\mathrm{T}}(i)}\}$ and $z^{\mathrm{K}}(t,i) := \mathrm{col}\{z^{\mathrm{K}}_q(t,i)|_{q\in\mathcal{N}_{\mathrm{F}}(i)}\}$, respectively. If we assume that $j \in \mathcal{N}_{\mathrm{T}}(i)$, the constraint of the interconnection between K_i and K_j is described by

$$v^{\mathrm{K}}_j(t,i) = (\mathcal{D}_{\tau^{\mathrm{K}}_{ij}} z^{\mathrm{K}}_i)(t,j), \quad j \in \mathcal{N}_{\mathrm{T}}(i), \tag{27}$$

where $v^{\mathrm{K}}_j(t,i) \in \mathbb{R}^{n_{v^{\mathrm{K}}_{ij}}}$ and $z^{\mathrm{K}}_i(t,j) \in \mathbb{R}^{n_{z^{\mathrm{K}}_{ji}}}$ with $n_{v^{\mathrm{K}}_{ij}} = n_{z^{\mathrm{K}}_{ji}}$. $\mathcal{D}_{\tau^{\mathrm{K}}_{ij}}$ is the delay operator, where τ^{K}_{ij} is time-varying delay duration. We further assume that τ_{ij} and τ^{K}_{ij} do not have to be equal but share the same upper bound $\mathcal{T}_{u_{ij}} \in \mathbb{Z}^+$.

The objective here is to design an interconnected output-feedback controller K such that the closed-loop system is robustly stable and has finite ℓ_2 gain in the face of communication delays. The distributed control system is depicted in Fig. 2.

A similar model transformation will be performed on (26) to separate the delay uncertainties from the LTI part of the controller subsystem. Then, we have the augmented model of the K_i described in the following LFT form:

$$\begin{cases} \begin{bmatrix} x^{\mathrm{K}}(t+1,i) \\ w^{\mathrm{K}}(t,i) \\ z^{\mathrm{K}}(t,i) \\ y(t,i) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{K}}_{xx,i} & A^{\mathrm{K}}_{xv,i} & A^{\mathrm{K}}_{xv,i} & B^{\mathrm{K}}_{xu,i} \\ 0 & 0 & I & 0 \\ A^{\mathrm{K}}_{zx,i} & A^{\mathrm{K}}_{zv,i} & A^{\mathrm{K}}_{zv,i} & B^{\mathrm{K}}_{zd,i} \\ C^{\mathrm{K}}_{x,i} & C^{\mathrm{K}}_{v,i} & C^{\mathrm{K}}_{v,i} & D^{\mathrm{K}}_{u,i} \end{bmatrix} \begin{bmatrix} x^{\mathrm{K}}(t,i) \\ q^{\mathrm{K}}(t,i) \\ \bar{v}^{\mathrm{K}}(t,i) \\ u(t,i) \end{bmatrix}, \\ q^{\mathrm{K}}(t,i) = (\mathcal{S}_{\tau^{\mathrm{K}}_i} w^{\mathrm{K}})(t,i). \end{cases} \tag{28}$$

Define the delay vector $\tau^{\mathrm{C}}_{ij} := \mathrm{col}\{\tau_{ij}, \tau^{\mathrm{K}}_{ij}\}$ and the delay-difference operator $\mathcal{S}_{\tau^{\mathrm{C}}_{ij}} := \mathrm{diag}\{\mathcal{S}_{\tau_{ij}}, \mathcal{S}_{\tau^{\mathrm{K}}_{ij}}\}$. Let $\mathcal{S}_{\tau^{\mathrm{C}}_i} = \mathrm{diag}\{\mathcal{S}_{\tau^{\mathrm{C}}_{ij}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ denote a diagonal delay-difference operator associated with the closed-loop subsystem consisting of K_i and $\boldsymbol{\Sigma}_i$. We can employ a set of dynamic IQC multipliers $\{\Pi_k, k = 1, 2, \ldots, N_\tau\}$ to characterize the delay-difference operator $\mathcal{S}_{\tau^{\mathrm{C}}_i}$. Denote a state-space realization of the IQC-induced system Ψ^{C}_i

by $(A^{\mathrm{C}}_{\psi,i}, [B^{\mathrm{C}}_{\psi q,i}\ B^{\mathrm{C}}_{\psi w,i}], C^{\mathrm{C}}_{\psi,i}, [D^{\mathrm{C}}_{\psi q,i}\ D^{\mathrm{C}}_{\psi w,i}])$ with the same structure as equation (12). This system essentially replaces the original relation $q^{\mathrm{C}}(t,i) = (\mathcal{S}_{\tau_i^{\mathrm{C}}} w^{\mathrm{C}})(t,i)$, which is shown in Fig. 3.

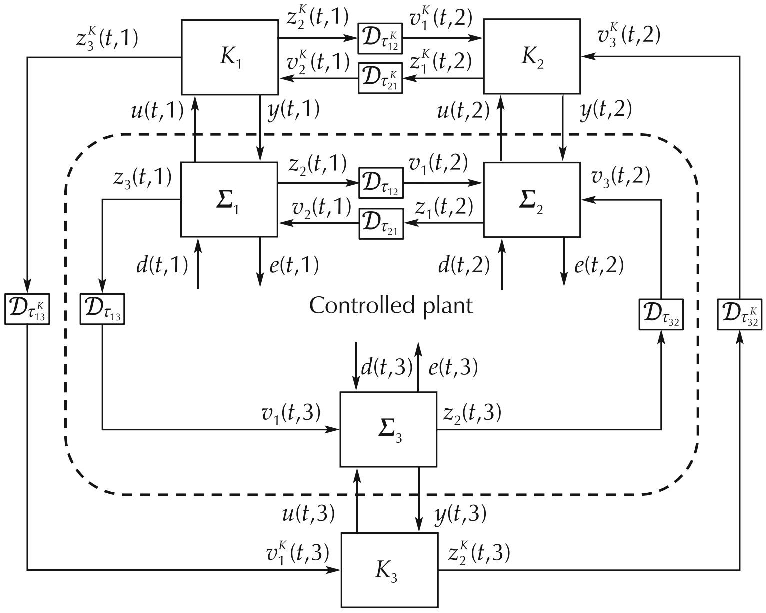

Fig. 2 Closed-loop networked system with $N = 3$.

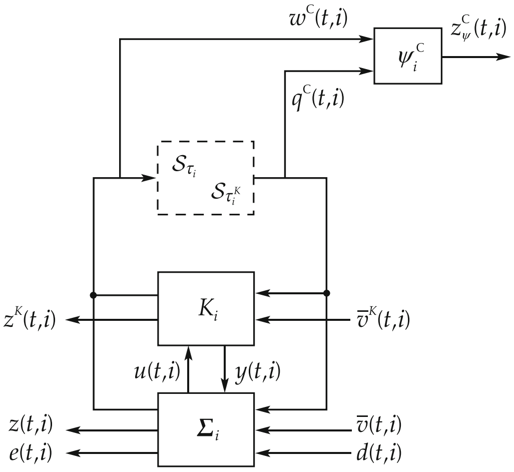

Fig. 3 Robust synthesis interconnection.

By connecting each LFT sub-controller (28) to the subsystem (8) and absorbing the IQC-induced system (12), the extended closed-loop subsystem are given by

$$\begin{bmatrix} x^{\mathrm{C}}(t+1,i) \\ z^{\mathrm{C}}_{\psi}(t,i) \\ z^{\mathrm{C}}(t,i) \\ e(t,i) \end{bmatrix} = \underbrace{\begin{bmatrix} A^{\mathrm{C}}_{i} & B^{\mathrm{C}}_{1,i} & B^{\mathrm{C}}_{2,i} & B^{\mathrm{C}}_{3,i} \\ \bar{C}^{\mathrm{C}}_{\psi,i} & \bar{D}^{\mathrm{C}}_{\psi q,i} & \bar{D}^{\mathrm{C}}_{\psi w,i} & 0 \\ A^{\mathrm{C}}_{zx,i} & A^{\mathrm{C}}_{zv,i} & A^{\mathrm{C}}_{zv,i} & B^{\mathrm{C}}_{z,i} \\ C^{\mathrm{C}}_{ex,i} & C^{\mathrm{C}}_{ev,i} & C^{\mathrm{C}}_{ev,i} & D^{\mathrm{C}}_{ed,i} \end{bmatrix}}_{\mathcal{M}_i} \begin{bmatrix} x^{\mathrm{C}}(t,i) \\ q^{\mathrm{C}}(t,i) \\ \bar{v}^{\mathrm{C}}(t,i) \\ d(t,i) \end{bmatrix}, \quad (29)$$

where $x^{\mathrm{C}}(t,i) \in \mathbb{R}^{n_{\psi_i^{\mathrm{C}}}+n_{x_i^{\mathrm{C}}}}$, $z^{\mathrm{C}}(t,i) \in \mathbb{R}^{n_{z_i^{\mathrm{C}}}}$, $\bar{v}^{\mathrm{C}}(t,i) \in \mathbb{R}^{n_{v_i^{\mathrm{C}}}}$, $q^{\mathrm{C}}(t,i) \in \mathbb{R}^{n_{v_i^{\mathrm{C}}}}$ and $z^{\mathrm{C}}_{\psi}(t,i) \in \mathbb{R}^{n_{z^{\mathrm{C}}_{\psi i}}}$ with $n_{x_i^{\mathrm{C}}} = n_{x_i} + n_{x_i^{\mathrm{K}}}$, $n_{z_i^{\mathrm{C}}} = n_{z_i} + n_{z_i^{\mathrm{K}}}$, $n_{v_i^{\mathrm{C}}} = n_{v_i} + n_{v_i^{\mathrm{K}}}$ and $n_{z^{\mathrm{C}}_{\psi i}} = n_{(N_\tau+1)n_{v_i^{\mathrm{C}}}}$. Accordingly, we can partition the interconnection vectors $v^{\mathrm{C}}(t,i)$ and $z^{\mathrm{C}}(t,i)$ as $v^{\mathrm{C}}(t,i) = \mathrm{col}\{v^{\mathrm{C}}_j(t,i)|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ and $z^{\mathrm{C}}(t,i) = \mathrm{col}\{z^{\mathrm{C}}_j(t,i)|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\}$, respectively, in which $v^{\mathrm{C}}_j(t,i) \in \mathbb{R}^{n_{v^{\mathrm{C}}_{ij}}}$ and $z^{\mathrm{C}}_j(t,i) \in \mathbb{R}^{n_{z^{\mathrm{C}}_{ij}}}$.

The state-space matrices of (29) can be described as

$$\mathcal{M}_i = \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{xu,i} \\ 0 \\ \mathcal{B}_{zu,i} \\ \mathcal{D}_{eu,i} \end{bmatrix} \mathcal{K}_i [\mathcal{C}_{yx,i}\ \mathcal{C}_{yv,i}\ \mathcal{C}_{yv,i}\ \mathcal{D}_{yd,i}], \quad (30)$$

where

$$\mathcal{A}_i = \left[\begin{array}{cccc|cc|cc} A_{\psi,i} & 0 & 0 & 0 & B_{\psi q,i} & 0 & B_{\psi w,i} & 0 \\ 0 & A^{\mathrm{K}}_{\psi,i} & 0 & 0 & 0 & B^{\mathrm{K}}_{\psi q,i} & 0 & B^{\mathrm{K}}_{\psi w,i} \\ 0 & 0 & A_{xx,i} & 0 & A_{xv,i} & 0 & A_{xv,i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline C_{\psi,i} & 0 & 0 & 0 & 0 & 0 & D_{\psi w,i} & 0 \\ 0 & C^{\mathrm{K}}_{\psi,i} & 0 & 0 & 0 & 0 & 0 & D^{\mathrm{K}}_{\psi w,i} \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & A_{zx,i} & 0 & A_{zv,i} & 0 & A_{zv,i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c|c} A_i & 0 & B_{1,i} & B_{2,i} \\ 0 & 0 & 0 & 0 \\ \hline \tilde{C}^{\mathrm{C}}_{\psi,i} & 0 & 0 & \tilde{D}^{\mathrm{C}}_{\psi w,i} \\ 0 & 0 & I & 0 \\ \hline \tilde{A}_{zx,i} & 0 & \tilde{A}_{zv,i} & \tilde{A}_{zv,i} \end{array}\right], \tag{31}$$

$$\begin{aligned} \mathcal{B}_i &= [0\ \ 0\ \ B^{\mathrm{T}}_{xd,i}\ \ 0|\ 0\ \ 0\ \ 0\ \ 0|\ B^{\mathrm{T}}_{zd,i}\ \ 0]^{\mathrm{T}} \\ &= [B^{\mathrm{T}}_{3,i}\ \ 0|\ 0\ \ 0|\ \tilde{B}^{\mathrm{T}}_{zd,i}]^{\mathrm{T}}, \end{aligned} \tag{32}$$

$$\begin{aligned} \mathcal{C}_i &= [0\ \ 0\ \ C_{ex,i}\ \ 0|\ C_{ev,i}\ \ 0|\ C_{ev,i}\ \ 0] \\ &= [\tilde{C}_{ex,i}\ \ 0|\ \tilde{C}_{ev,i}\ |\ \tilde{C}_{ev,i}], \end{aligned} \tag{33}$$

$$\mathcal{D}_i = D_{ed,i}, \tag{34}$$

$$\mathcal{B}_{xu,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{xu,i} \\ I & 0 & 0 \end{bmatrix}, \tag{35}$$

$$\mathcal{B}_{zu,i} = \begin{bmatrix} 0 & 0 & B_{zu,i} \\ 0 & I & 0 \end{bmatrix}, \tag{36}$$

$$\mathcal{C}_{yx,i} = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_{yx,i} & 0 \end{bmatrix}, \tag{37}$$

$$\mathcal{C}_{yv,i} = \begin{bmatrix} 0 & 0 \\ 0 & I \\ C_{yv,i} & 0 \end{bmatrix}, \tag{38}$$

$$\mathcal{D}_{yd,i} = \begin{bmatrix} 0 \\ 0 \\ D_{yd,i} \end{bmatrix}, \tag{39}$$

$$\mathcal{D}_{eu,i} = [0\ \ 0\ \ D_{eu,i}]. \tag{40}$$

Now, we show the conditions for the existence of a distributed output-feedback controller in the following theorem.

Theorem 2 Consider the networked system $\boldsymbol{\Sigma}$. Given the upper bound vector $\mathcal{T}_{u_i} = \mathrm{col}\{\mathcal{T}_{u_{ij}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ of the time-varying delay $\tau^{\mathrm{C}}_i = \mathrm{col}\{\tau^{\mathrm{C}}_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\}$ for all $i = 1,2,\ldots,N$ with τ^{C}_i and $\mathcal{T}_{u_{ij}} \in \mathbb{Z}^+$, if there exist positive-definite matrices P^{G}_i, $\hat{P}^{\mathrm{G}}_i \in \mathbb{R}^{n_{x_i}+n_{\psi^{\mathrm{C}}_i}}_{\mathrm{S}}$ and X^{C}_{ij}, $\hat{X}^{\mathrm{C}}_{ij} \in \mathbb{R}^{n_{z^{\mathrm{C}}_{ij}}}_{\mathrm{S}}$, and diagonal positive-definite matrices $X^{\mathrm{C}}_{k,i}, \hat{X}^{\mathrm{C}}_{k,i} \in \mathbb{R}^{n_{v^{\mathrm{C}}_i}}$ for all $i = 1,2,\ldots,N$, $j \in \mathcal{N}_{\mathrm{F}}(i)$ and $k = 1,2,\ldots,N_\tau$ such that (41a)–(41f) are satisfied for all $i = 1,2,\ldots,N$. Then, there exists a distributed controller with the same structure as $\boldsymbol{\Sigma}$ such that the closed-loop system (29) is robustly stable for all the communication time-varying delay $\tau^{\mathrm{C}}_{ij} \in [0,\mathcal{T}_{u_{ij}}]$, and the induced ℓ_2 gain from d to e is no more than a priori given $\gamma > 0$.

$$[\star]^{\mathrm{T}} \left[\begin{array}{c|cc} \Xi_{1,i} & \star & \star \\ \hline \Upsilon_{1,i} & -I & \star \\ \Omega_{1,i} & 0 & -\hat{\Lambda}_i \end{array}\right] \left[\begin{array}{c|c} \Gamma_{\mathrm{X},i} & 0 \\ \hline 0 & I \end{array}\right] < 0, \tag{41a}$$

$$[\star]^{\mathrm{T}} \left[\begin{array}{c|cc} \Xi_{2,i} & \star & \star \\ \hline \Upsilon_{2,i} & \gamma^2 I & \star \\ \Omega_{2,i} & 0 & \sum\limits_{k=1}^{N_\tau} X^{\mathrm{C}}_{k,i} \end{array}\right] \left[\begin{array}{c|c} \Gamma_{\mathrm{Y},i} & 0 \\ \hline 0 & I \end{array}\right] > 0, \tag{41b}$$

$$\begin{bmatrix} P^{\mathrm{G}}_i & I \\ I & \hat{P}^{\mathrm{G}}_i \end{bmatrix} \geqslant 0, \tag{41c}$$

$$\begin{bmatrix} X^{\mathrm{C}}_{k,i} & I \\ I & \hat{X}^{\mathrm{C}}_{k,i} \end{bmatrix} \geqslant 0, \tag{41d}$$

$$X^{\mathrm{C}}_{ij}\hat{X}^{\mathrm{C}}_{ij} = I_{n_{z^{\mathrm{C}}_{ij}}}, \tag{41e}$$

$$X^{\mathrm{C}}_{k,i}\hat{X}^{\mathrm{C}}_{k,i} = I_{n_{v^{\mathrm{C}}_{ij}}}, \tag{41f}$$

where

$$\Gamma_{\mathrm{X},i} = \ker\left[\begin{array}{cc|cc|cc|c} 0_{n_{v_i}\times n_{\psi^{\mathrm{C}}_i}} & 0 & 0 & I_{n_{v_i}} & 0 & I_{n_{v_i}} & 0 \\ 0 & C_{yx,i} & C_{yv,i} & 0 & C_{yv,i} & 0 & D_{yd,i} \end{array}\right], \tag{42}$$

$$\Gamma_{\mathrm{Y},i} = \ker\left[\begin{array}{cc|c|cc|c} 0_{n_{v_i}\times n_{\psi^{\mathrm{C}}_i}} & 0 & 0_{n_{v_i}\times n_{z^{\mathrm{C}}_\psi}} & 0 & I_{n_{v_i}} & 0 \\ 0 & B^{\mathrm{T}}_{xu,i} & 0 & B^{\mathrm{T}}_{zu,i} & 0 & D^{\mathrm{T}}_{eu,i} \end{array}\right], \tag{43}$$

$$\begin{aligned} \Xi_{1,i} = {} & [\star]^{\mathrm{T}} \begin{bmatrix} -P^{\mathrm{G}}_i & 0 \\ 0 & P^{\mathrm{G}}_i \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ A_i & B_{1,i} & B_{2,i} & B_{3,i} \end{bmatrix} \\ & + [\star]^{\mathrm{T}} \begin{bmatrix} U^{\mathrm{C}}_i & 0 \\ 0 & V^{\mathrm{C}}_i \end{bmatrix} \begin{bmatrix} \tilde{A}_{zx,i} & \tilde{A}_{zv,i} & \tilde{A}_{zv,i} & \tilde{B}_{zd,i} \\ 0 & 0 & I & 0 \end{bmatrix} \\ & + [\star]^{\mathrm{T}} \begin{bmatrix} -\sum\limits_{k=1}^{N_\tau} X_{k,i} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \end{aligned} \tag{44}$$

$$\begin{aligned} \Xi_{2,i} = {} & [\star]^{\mathrm{T}} \begin{bmatrix} -\hat{P}^{\mathrm{G}}_i & 0 \\ 0 & \hat{P}^{\mathrm{G}}_i \end{bmatrix} \begin{bmatrix} A^{\mathrm{T}}_i & \tilde{C}^{\mathrm{T}}_{\psi,i} & \tilde{A}^{\mathrm{T}}_{zx,i} & \tilde{C}^{\mathrm{T}}_{ex,i} \\ -I & 0 & 0 & 0 \end{bmatrix} \\ & + [\star]^{\mathrm{T}} \begin{bmatrix} \hat{U}^{\mathrm{C}}_i & 0 \\ 0 & \hat{V}^{\mathrm{C}}_i \end{bmatrix} \begin{bmatrix} 0 & 0 & -I & 0 \\ B^{\mathrm{T}}_{2,i} & D^{\mathrm{T}}_{\psi w,i} & A^{\mathrm{T}}_{zv,i} & C^{\mathrm{T}}_{ev,i} \end{bmatrix} \\ & + [\star]^{\mathrm{T}} \begin{bmatrix} \hat{\Lambda}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}, \end{aligned} \tag{45}$$

$$\Upsilon_{1,i} = [\tilde{C}_{ex,i}\ \ \tilde{C}_{ev,i}\ \ \tilde{C}_{ev,i}\ \ D_{ed,i}], \tag{46}$$

$$\Upsilon_{2,i} = [B^{\mathrm{T}}_{3,i}\ \ 0\ \ B^{\mathrm{T}}_{zd,i}\ \ D^{\mathrm{T}}_{ed,i}], \tag{47}$$

$$\Omega_{1,i} = [\tilde{C}_{\psi,i}\ \ 0\ \ \tilde{D}_{\psi w,i}\ \ 0], \tag{48}$$

$$\Omega_{2,i} = [B_{1,i}^{\mathrm{T}} \ \ 0 \ \ \tilde{A}_{zv,i}^{\mathrm{T}} \ \ \tilde{C}_{ev,i}^{\mathrm{T}}], \tag{49}$$

$$\Lambda_i = \mathrm{diag}\{X_{k,i}^{\mathrm{C}}|_{i=1}^{N_\tau}\}, \tag{50}$$

$$\hat{\Lambda}_i = \mathrm{diag}\{\hat{X}_{k,i}^{\mathrm{C}}|_{i=1}^{N_\tau}\}, \tag{51}$$

$$U_i^{\mathrm{C}} = \begin{bmatrix} \mathrm{diag}\{X_{ij}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} & \mathrm{diag}\{X_{ij}^{\mathrm{H}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} \\ \mathrm{diag}\{(X_{ij}^{\mathrm{H}})^{\mathrm{T}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} & \mathrm{diag}\{X_{ij}^{\mathrm{K}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} \end{bmatrix}, \tag{52}$$

$$V_i^{\mathrm{C}} = \begin{bmatrix} \mathrm{diag}\{Y_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} & \mathrm{diag}\{Y_{ij}^{\mathrm{H}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} \\ \mathrm{diag}\{(Y_{ij}^{\mathrm{H}})^{\mathrm{T}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} & \mathrm{diag}\{Y_{ij}^{\mathrm{K}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} \end{bmatrix}, \tag{53}$$

$$X_{ij}^{\mathrm{C}} = \begin{bmatrix} X_{ij} & X_{ij}^{\mathrm{H}} \\ (X_{ij}^{\mathrm{H}})^{\mathrm{T}} & X_{ij}^{\mathrm{K}} \end{bmatrix}, \tag{54}$$

$$Y_{ij}^{\mathrm{C}} = \begin{bmatrix} Y_{ij} & Y_{ij}^{\mathrm{H}} \\ (Y_{ij}^{\mathrm{H}})^{\mathrm{T}} & Y_{ij}^{\mathrm{K}} \end{bmatrix}, \tag{55}$$

$$X_{ij}^{\mathrm{C}} = -Y_{ji}^{\mathrm{C}}, \tag{56}$$

$$\hat{U}_i^{\mathrm{C}} = \begin{bmatrix} \mathrm{diag}\{\hat{X}_{ij}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} & \mathrm{diag}\{\hat{X}_{ij}^{\mathrm{H}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} \\ \mathrm{diag}\{(\hat{X}_{ij}^{\mathrm{H}})^{\mathrm{T}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} & \mathrm{diag}\{\hat{X}_{ij}^{\mathrm{K}}|_{j\in\mathcal{N}_{\mathrm{F}}(i)}\} \end{bmatrix}, \tag{57}$$

$$\hat{V}_i^{\mathrm{C}} = \begin{bmatrix} \mathrm{diag}\{\hat{Y}_{ij}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} & \mathrm{diag}\{\hat{Y}_{ij}^{\mathrm{H}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} \\ \mathrm{diag}\{(\hat{Y}_{ij}^{\mathrm{H}})^{\mathrm{T}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} & \mathrm{diag}\{\hat{Y}_{ij}^{\mathrm{K}}|_{j\in\mathcal{N}_{\mathrm{T}}(i)}\} \end{bmatrix}, \tag{58}$$

$$\hat{X}_{ij}^{\mathrm{C}} = \begin{bmatrix} \hat{X}_{ij} & \hat{X}_{ij}^{\mathrm{H}} \\ (\hat{X}_{ij}^{\mathrm{H}})^{\mathrm{T}} & \hat{X}_{ij}^{\mathrm{K}} \end{bmatrix}, \tag{59}$$

$$\hat{Y}_{ij}^{\mathrm{C}} = \begin{bmatrix} \hat{Y}_{ij} & \hat{Y}_{ij}^{\mathrm{H}} \\ (\hat{Y}_{ij}^{\mathrm{H}})^{\mathrm{T}} & \hat{Y}_{ij}^{\mathrm{K}} \end{bmatrix}, \tag{60}$$

$$\hat{X}_{ij}^{\mathrm{C}} = -\hat{Y}_{ji}^{\mathrm{C}}. \tag{61}$$

Proof Applying our robustness analysis results to the extended closed-loop subsystems (29), we derive the following LMI condition for all $i = 1, 2, \dots, N$,

$$W_i^{\mathrm{T}} \Theta_i W_i < 0, \tag{62}$$

where

$$W_i = \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ A_i^{\mathrm{C}} & B_{1,i}^{\mathrm{C}} & B_{2,i}^{\mathrm{C}} & B_{3,i}^{\mathrm{C}} \\ \hline \bar{C}_{\psi,i}^{\mathrm{C}} & \bar{D}_{\psi 1,i}^{\mathrm{C}} & \bar{D}_{\psi 2,i}^{\mathrm{C}} & 0 \\ 0 & I & 0 & 0 \\ \hline A_{zx,i}^{\mathrm{C}} & A_{zv,i}^{\mathrm{C}} & A_{zv,i}^{\mathrm{C}} & B_{z,i}^{\mathrm{C}} \\ 0 & 0 & I & 0 \\ \hline C_{\mathrm{ex},i}^{\mathrm{C}} & C_{\mathrm{ev},i}^{\mathrm{C}} & C_{\mathrm{ev},i}^{\mathrm{C}} & D_{ed,i}^{\mathrm{C}} \\ 0 & 0 & 0 & I \end{array}\right], \tag{63}$$

$$\Theta_i = \left[\begin{array}{cc|cc|cc|cc} -P_i^{\mathrm{C}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i^{\mathrm{C}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \Lambda_i^{\mathrm{C}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sum_{k=1}^{N_\tau} X_{k,i}^{\mathrm{C}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & U_i^{\mathrm{C}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_i^{\mathrm{C}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{array}\right]. \tag{64}$$

Notice that the state-space matrices of the extended closed-loop subsystem depends affinely on $\mathcal{K}_i$. Thus, we would like to derive equivalent conditions that do not involve the controller's data $\mathcal{K}_i$, $i = 1, 2, \dots, N$.

We first discuss the inertia of the matrix Θ_i. Because every $(\Psi_{k,i}, M_{k,i})$ is the $J_{n_{v_i}, n_{v_i}}$-spectral factorization of $\Pi_{k,i}$ and $X_{k,i}^{\mathrm{C}} > 0$, thus

$$\mathrm{in}\Big(\begin{bmatrix} \Lambda_i & 0 \\ 0 & -\sum_{k=1}^{N_\tau} X_{k,i} \end{bmatrix}\Big) = (n_{v_i} + n_{v_i^{\mathrm{K}}}, 0, N_\tau \times (n_{v_i} + n_{v_i^{\mathrm{K}}})). \tag{65}$$

Since $X_{ij}^{\mathrm{C}} > 0$ and $X_{ij}^{\mathrm{C}} = -Y_{ji}^{\mathrm{C}}$, we have $\mathrm{in}_{\pm}(X_{ij}^{\mathrm{C}}) = \mathrm{in}_{\mp}(Y_{ji}^{\mathrm{C}})$. Hence

$$\mathrm{in}\Big(\begin{bmatrix} U_i^{\mathrm{C}} & 0 \\ 0 & V_i^{\mathrm{C}} \end{bmatrix}\Big) = (n_{v_i} + n_{v_i^{\mathrm{K}}}, 0, n_{z_i} + n_{z_i^{\mathrm{K}}}). \tag{66}$$

Since $P_i^{\mathrm{C}} \in \mathbb{R}_{\mathrm{S}}^{n_{x_i^{\mathrm{C}}} + n_{\psi_i^{\mathrm{C}}}}$, thus

$$\mathrm{in}\Big(\begin{bmatrix} -P_i^{\mathrm{C}} & 0 \\ 0 & P_i^{\mathrm{C}} \end{bmatrix}\Big) = (n_{x_i^{\mathrm{C}}} + n_{\psi_i^{\mathrm{C}}}, 0, n_{x_i^{\mathrm{C}}} + n_{\psi_i^{\mathrm{C}}}). \tag{67}$$

If we take P_i^{G}, $\hat{P}_i^{\mathrm{G}}$ to be the top-left blocks of P_i^{C}, $(P_i^{\mathrm{C}})^{-1}$, respectively. By applying the elimination lemma from [13] to LMIs (62) , we obtain (41c)–(41f) and the following LMIs

$$\Gamma_{X,i}^{\mathrm{T}}[\Xi_{1,i} + \Upsilon_{1,i}^{\mathrm{T}}\Upsilon_{1,i} + \Omega_{1,i}^{\mathrm{T}}\Lambda_i\Omega_{1,i}]\Gamma_{X,i} < 0, \tag{68}$$

$$\Gamma_{Y,i}^{\mathrm{T}}[\Xi_{2,i} - \gamma^{-2}\Upsilon_{2,i}^{\mathrm{T}}\Upsilon_{2,i} - \Omega_{2,i}^{\mathrm{T}}(\sum_{k=1}^{N_\tau} X_{k,i})^{-1}\Omega_{2,i}]\Gamma_{Y,i} > 0, \tag{69}$$

which lead to (41a) and (41b) by taking the Shur-complement.

If the conditions (41a)–(41f) are feasible, we can follow the similar techniques as in [8] to construct the extended scales P_i^{C} by taking $n_{x_i^{\mathrm{K}}} = n_{x_i} + n_{\psi_i^{\mathrm{C}}}$ such that LMI (62) is satisfied for all $i = 1, 2, \dots, N$. □

Note that the conditions (41a)–(41f) are LMIs, but the conditions (41e) and (41f) are non-convex. To address this problem, we transform the original feasibility problem into a so-called cone complementarity problem (CCP)

$$\begin{aligned}&\min\ \sum_{i=1}^{N}\Big(\sum_{j\in N_{\mathrm{F}}(i)}\mathrm{Tr}(X_{ij}^{\mathrm{C}}\hat{X}_{ij}^{\mathrm{C}}) + \sum_{k=1}^{N_\tau}\mathrm{Tr}(X_{k,i}^{\mathrm{C}}\hat{X}_{k,i}^{\mathrm{C}})\Big)\\ &\ \text{s.t.}\ \ (41\text{a})\text{–}(41\text{d}).\end{aligned} \tag{70}$$

The global minimum of optimization problem (70) is required to be $n_{z^{\mathrm{C}}} + N_\tau \times n_{v^{\mathrm{C}}}$. A linearization method proposed in [24] can be employed to solve such a problem. Given a set of feasible solutions $(X_{ij}^{\mathrm{C}})_0$, $(\hat{X}_{ij}^{\mathrm{C}})_0$, $(X_{k,i}^{\mathrm{C}})_0$, $(\hat{X}_{k,i}^{\mathrm{C}})_0$ that solve the LMIs (41a)–(41d), a linear approximation of (70) has the form of

$$\begin{aligned}&\min\ \sum_{i=1}^{N}\Big(\sum_{j\in N_{\mathrm{F}}(i)}\mathrm{Tr}((X_{ij}^{\mathrm{C}})_0\hat{X}_{ij}^{\mathrm{C}} + (\hat{X}_{ij}^{\mathrm{C}})_0X_{ij}^{\mathrm{C}})\\ &\qquad + \sum_{k=1}^{N_\tau}\mathrm{Tr}((X_{k,i}^{\mathrm{C}})_0\hat{X}_{k,i}^{\mathrm{C}} + (\hat{X}_{k,i}^{\mathrm{C}})_0X_{k,i}^{\mathrm{C}})\Big)\\ &\ \text{s.t.}\ \ (41\text{a})\text{–}(41\text{d}).\end{aligned} \tag{71}$$

Hence the non-convex conditions (41a)–(41f) in Theorem 1 can be verified by solving a standard semi-definite program. For more details of the linearization algorithm we refer the reader to [24] and the references therein.

Once the optimization problem (71) has global optimal solutions, it is sketched in [8] and [14] how to construct the distributed controller K such that (62) is feasible, which implies that the distributed controller K achieves the desired goal of stabilizing the NS $\boldsymbol{\Sigma}$.

Note also that the dimensions of the interconnection signals $v_j^{\mathrm{K}}(t,i)$ and $z_i^{\mathrm{K}}(t,j)$ between K_i and K_j are without any constraints in this paper. This is because each of the closed-loop scales Y_{ij}^{C} (or X_{ji}^{C}) is directly calculated according to the optimization problem (71), which is different from the reconstruction methods adopted in [8, 9].

5 Numerical examples

In this section, three numerical examples are given to demonstrated the efficacy of the proposed robust stability analysis and distributed control design methods of this paper.

For all the examples in the sequel, two IQC multipliers from [23] are employed to characterize the associated delay-difference operator $\mathcal{S}_\tau$, which are

$$\Pi_1 = [\star]^\sim \begin{bmatrix}(\mathcal{T}_u+1)X_1 & 0\\ 0 & -X_1\end{bmatrix}\begin{bmatrix}I_n & 0\\ I_n & I_n\end{bmatrix}, \tag{72}$$

$$\Pi_2 = [\star]^\sim \begin{bmatrix}\mathcal{T}_u^2X_2 & 0\\ 0 & -X_2\end{bmatrix}\begin{bmatrix}(1-\zeta^{-1})I_n & 0\\ 0 & I_n\end{bmatrix}, \tag{73}$$

where $X_1 = X_1^{\mathrm{T}} \geqslant 0$ and $X_2 = X_2^{\mathrm{T}} \geqslant 0$. $\mathcal{T}_u$ is the upper bound of delay duration. It can be proved that both of them are hard factorizations.

Example 1 Consider a network system $\boldsymbol{\Sigma}$ with time-varying communication delays. The system consists of N subsystems, each of them modeled as (1) and the interconnections among them are represented by (3). The following properties are satisfied:

- Let $n_{x_i} = n_{v_i} = n_{z_i} \equiv 2$ and $n_{d_i} = n_{e_i} \equiv 1$.
- Every parameter of the subsystems is independently and randomly generated from a continuous uniform distribution over the interval $[-0.5, 0.5]$.
- Each row of the SCM Φ is generated randomly and independently, in which the non-zero element is selected according to a discrete uniform distribution over all the possible locations.
- The upper bound $\mathcal{T}_{u_{ij}}$ of each time-varying delay τ_{ij} is independently and generated randomly from a discrete uniform distribution on the set $[0, 10]$.
- The induced ℓ_2 gain from d to e is set to $\gamma \equiv 1$.

Two approaches are utilized in verifying the robust stability of the generated system. One is based on LF-IQC by [19, Theorem 3], the other is on Theorem 1.

First, we give a general comment using the number of variables and the number of LMIs as measures of computational burden. Table 1 summarizes the number of independent variables for each approach. It is obvious that $N_\tau = 2$ because we have used two IQC multipliers (72) and (73) to describe the associated delay-difference operators. For simplicity, let $n_{v_{ij}} = 1$ for all $j \in \mathcal{N}_{\mathrm{T}}(i)$ and $i = 1, \dots, N$. The numbers of variables are calculated and plotted in Fig. 4 for $N = 1, 2, \dots, 30$. The numbers of variables for Theorem 1 do not grow much with N when compared with LF-IQC. However, note that Theorem 1 involves an LMI condition for each subsystem, and all the LMIs are partly coupled. Thus, it is hard to analyze theoretically which approach is computationally more demanding for a given N. Next, we will carry out specific robust stability verifications for the two approaches.

Table 1 Number of variables for each approach.

Method	Number of variables
LF-IQC	$N(n_{x_i}+n_{\psi_i})(N(n_{x_i}+n_{\psi_i})+1)/2$ $+N_\tau\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_{\rm T}(i)}[n_{v_{ij}}(n_{v_{ij}}+1)/2]$
Theorem 1	$N(n_{x_i}+n_{\psi_i})[(n_{x_i}+n_{\psi_i})+1]/2$ $+(N_\tau+1)\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_{\rm T}(i)}[n_{v_{ij}}(n_{v_{ij}}+1)/2]$

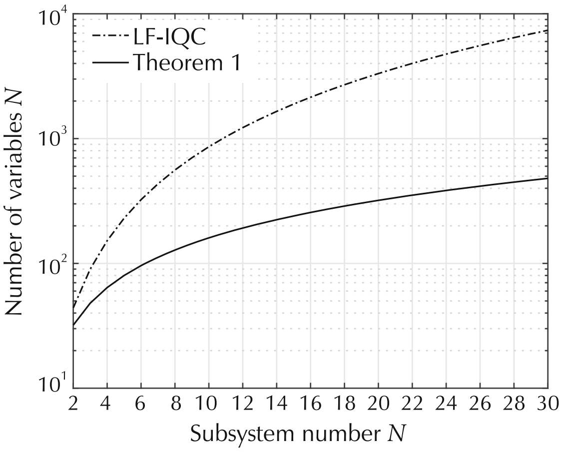

Fig. 4 Number of variables as a function of N.

For each subsystem number N, one hundred systems are generated. Feasibility of the corresponding LMIs is verified using Matlab's LMILab toolbox, and computations are performed with a personal computer with an Intel(R) Core(TM) i5-8250U CPU. According to these computations, the average CPU time for each approach is plotted in Fig. 5 as a function of N.

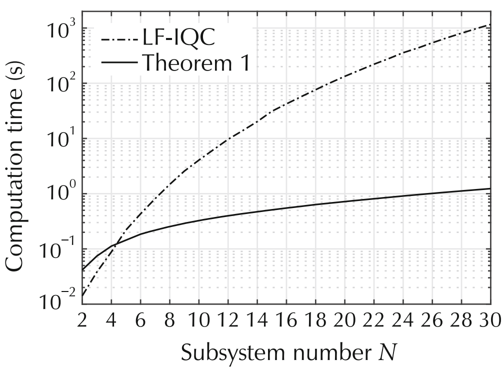

Fig. 5 Average CPU computation time versus N.

This figure shows that the growth rate in the average CPU time with respect to N is smaller for Theorem 1 than LF-IQC. It is also clear that for the NS with small size (less than approximately 5) LF-IQC is more efficient than Theorem 1. This is because in Theorem 1, even if the variables are less than those in LF-IQC, the LMI (18) is required to be verified for all $i=1,2,\ldots,N$ in a partly coupled manner, which makes the algorithmic complexity of Theorem 1 more costly from computation time point of view. However, According to Table 1, the numbers of variables in LF-IQC grow with the order of N^2, while Theorem 1 has the growth order of N. As the numbers of subsystem grow, LF-IQC is computationally more demanding because of the dramatic increase in the numbers of variables.

Example 2 In this example, we compare our approach with the existing approach based on the lumped formulations in terms of the degree of conservatism. The NS consists of two subsystems whose state-space model are given below.

$$\begin{bmatrix} x(t+1,1)\\ z(t,1)\\ e(t,1)\\ y(t,1)\end{bmatrix}=\begin{bmatrix}0.72 & -0.16 & 0.40 & 0.20\\ 0.45 & 0 & 0.30 & 0.10\\ 0.02 & 0 & 0 & 0.10\\ 1 & 0.10 & 1.40 & 0\end{bmatrix}\begin{bmatrix}x(t,1)\\ v(t,1)\\ d(t,1)\\ u(t,1)\end{bmatrix}, \quad (74)$$

$$\begin{bmatrix} x(t+1,2)\\ z(t,2)\\ e(t,2)\\ y(t,2)\end{bmatrix}=\begin{bmatrix}0.81 & -0.28 & 0.60 & 0.20\\ 0.37 & 0 & 0.40 & 0.10\\ 0.01 & 0 & 0 & 0.10\\ 1 & 0.10 & 1.60 & 0\end{bmatrix}\begin{bmatrix}x(t,2)\\ v(t,2)\\ d(t,2)\\ u(t,2)\end{bmatrix}. \quad (75)$$

These subsystems are connected through

$$v(t)=\left(\begin{bmatrix}\mathcal{D}_{\tau_1} & 0\\ 0 & \mathcal{D}_{\tau_2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}z\right)(t). \quad (76)$$

We are interested in computing the upper bounds of τ_1 and τ_2 such that the above system is robustly stable for all $\tau_i\in[0,\mathcal{T}_{u_i}], i=1,2$ and the ℓ_2 gain from d to e is no more than 1. Fig. 6 shows the estimated stability boundary as a function of delays τ_1 and τ_2 for each approach. The upper dot dash curve is for the approach LF-IQC and the lower solid curve is for Theorem 1. The areas to the lower left of the curves represent the corresponding stability regions. Evidently, using the same IQC multipliers to describe the delay-difference operator, Theorem 1 presents the more conservative results compared to LF-IQC. This is not a surprise, as Theorem 1 is only a sufficient condition for the robustness of the NS and utilizes less structural information of the NS than the approach of LF-IQC. However, when an NS has a very large scale, Theorem 1 provides a tradeoff between computation time and conservatism.

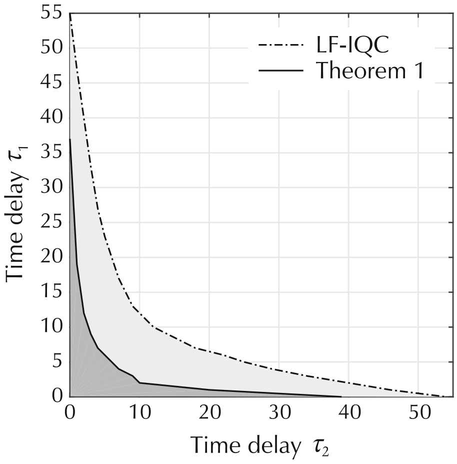

Fig. 6 Estimated boundary for the stability region in τ_1–τ_2 plain.

Example 3 Consider the same model as in Example 2. When the upper bound of τ_1 and τ_2 are set to $\mathcal{T}_{u_1} = 4$ and $\mathcal{T}_{u_2} = 8$, respectively, it can be verified that the robustness analysis conditions in Theorem 1 are not satisfied (note that this does not imply that the system is unstable, since the analysis LMIs in Theorem 1 is sufficient, but not necessary, conditions). The objective is to design a distributed, interconnected controller such that the closed-loop system is robustly stable for all $\tau_i, \tau_i^{\mathrm{K}} \in [0, \mathcal{T}_{u_i}], i = 1, 2$ and the ℓ_2 gain from d to e is no more than 1. To this end, we first confirm the existence of such a controller by verifying the conditions presented in Theorem 2. Then, a standard algorism is employed to reconstruct the distributed controller. The results are given in (77) and (78).

$$\mathcal{K}_1 = \left[\begin{array}{ccc|c|c} -0.1564 & -0.0017 & 0.0009 & 0.0014 & -0.1201 \\ 0.0001 & -0.0175 & -0.0012 & -0.0879 & 0.0001 \\ 0.0015 & 0.0000 & 0.0000 & 0.0000 & 0.0015 \\ \hline 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ \hline 0.0770 & -0.0147 & 0.0085 & 0.0019 & -2.7602 \end{array}\right], \tag{77}$$

$$\mathcal{K}_2 = \left[\begin{array}{ccc|c|c} -0.0583 & -0.0021 & 0.0038 & -0.0003 & -0.1072 \\ 0.0000 & -0.0067 & 0.0009 & -0.0641 & 0.0000 \\ 0.0042 & 0.0002 & -0.0003 & 0.0000 & 0.0084 \\ \hline 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ \hline 0.0026 & -0.0122 & 0.0143 & 0.0010 & -2.7217 \end{array}\right]. \tag{78}$$

Note that both of the controller subsystems have the order of 3, which is equal to the sum of the order of the associated subsystem and the order of the IQC-induced system. This achieves the design objective.

6 Conclusions

In this paper, we investigated the robustness and distributed control of a networked system with uncertain time-varying communication delays. By merging the dynamic IQC technique with the dissipation theory, we gave some sufficient conditions for the robust stability and performance of the NS against delay uncertainties. Furthermore, we derived some conditions for the existence of a distributed controller and employed an exist method to construct the controller parameters. Finally, we illustrated the usefulness of the analysis and synthesis results by several numerical examples. In a future research, we will consider the extension of this work using more general and tighten IQCs to cover the delay for the reduction of conservatism of the robustness analysis problem.

References

[1] B. Bamieh, F. Paganini, G. A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 2002, 47(7): 1091 – 1107.

[2] R. D'Andrea, G. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, 2003, 48(9): 1478 – 1495.

[3] G. Dullerud, R. D'Andrea. Distributed control of heterogeneous systems. *IEEE Transactions on Automatic Control*, 2004, 49(12): 2113 – 2128.

[4] T. Zhou. On the stability of spatially distributed systems. *IEEE Transactions on Automatic Control*, 2008, 53(10): 2385 – 2391.

[5] J. Lavaei. Decentralized implementation of centralized controllers for interconnected systems. *IEEE Transactions on Automatic Control*, 2012, 57(7): 1860 – 1865.

[6] P. Massioni, M. Verhaegen. Distributed control for identical dynamically coupled systems: A decomposition approach. *IEEE Transactions on Automatic Control*, 2009, 54(1): 124 – 135.

[7] F. Borrelli, T. Keviczky. Distributed LQR design for identical dynamically decoupled systems. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1901 – 1912.

[8] C. Lanbort, R. S. Chandra, R. D'Andrea. Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1502 – 1519.

[9] R. S. Chandra, C. Lanbort, R. D'Andrea. Distribued control design with robustness to small time delays. *Systems & Control Letters*, 2009, 58(4): 296 – 303.

[10] X. Qi, M. V. Salapaka, P. G. Voulgaris, et al. Structured optimal and robust control with multiple criteria: A convex solution. *IEEE Transactions on Automatic Control*, 2004, 49(10): 1623 – 1640.

[11] L. Lessard, S. Lall. Reduction of decentralized control problems to tractable representations. *Proceedings of the 48th IEEE Conference on Decision and Control*. Shanghai: IEEE, 2009: 1621 – 1626.

[12] E. Summers, M. Arcak, R. A. Packard. Delay robustness of interconnected passive system: An integral quadratic constraint approach. *IEEE Transactions on Automatic Control*, 2013, 58(3): 712 – 724.

[13] A. Helmersson. IQC synthesis based on inertial constraints. *Proceedings of the 14th IFAC World Congress*, Beijing: Elsevier, 1999: 163 – 168.

[14] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 2001, 37(3): 361 – 375.

[15] T. Zhou. Coordinated one-step optimal distributed state prediction for a networked dynamical system. *IEEE Transactions on Automatic Control*, 2013, 58(11): 2756 – 2771.

[16] T. Zhou, Y. Zhang. On the stability and robust stability of networked dynamic systems. *IEEE Transactions on Automatic Control*, 2016, 61(6): 1595 – 1600.

[17] Z.K. Wang, T. Zhou. IQC based robust stability verification for a networked system with communication delays. *Science China Information Science*, 2018, 61(12): 108 – 122.

[18] A. Megretski, A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 1997, 42(6): 819 – 830.

[19] P. Seiler. Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE Transactions on Automatic Control*, 2015, 60(6): 1704 – 1709.

[20] C. Yuan, F. Wu. Exact-memory and memoryless control of linear systems with time-varying input delay using dynamic IQCs. *Automatica*, 2017, 77: 246 – 253.

[21] N. Kottenstette, M. J. McCourt, M. Xia, et al. On relationships among passivity, positive realness, and dissipativity in linear systems. *Automatica*, 2014, 50(4): 1003 – 1016.

[22] C-Y. Kao, A. Rantzer. Stability analysis of systems with uncertain time-varying delays. *Automatica*, 2007, 43(6): 959 – 970.

[23] C-Y. Kao. On stability of discrete-time LTI systems with varying time delays. *IEEE Transactions on Automatic Control*, 2012, 57(5): 1243 – 1248.

[24] L. El Ghaoui, F. Oustry, M. Ait Rami. A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, 1997, 42(8): 1171 – 1176.

Zhike WANG received the B.Sc. degree in Electrical Engineering and Automation and the M.Sc. degree in Navigation, Guidance and Control from Air Force Engineering University, Xi'an, China, in 2005 and 2009, respectively. He is currently working toward the Ph.D. degree at the Department of Automation, Tsinghua University, Beijing, China. His main research interest includes the fields of robust stability analysis and distributed control of networked systems. E-mail: wangzk14@mails.tsinghua.edu.cn.

Tong ZHOU received the B.Sc. degree in Automatic Control and the M.Sc. degree in Control Theory and Applications from the University of Electronic Science and Technology of China, Chengdu, China, in 1984 and 1989, respectively, and the M.Sc. degree in Electrical and Computer Engineering from Kanazawa University, Kanazawa, Japan, in 1991, and the Ph.D. degree in Industrial Machinery from Osaka University, Osaka, Japan, in 1994. After visiting several universities in the Netherlands, China, and Japan, he, in 1999, joined Tsinghua University, Beijing, China, where he is currently a Professor of control theory and control engineering. His current research interests include robust estimation and control, system identification, signal processing, hybrid systems, and their applications to real-world problems in molecular cellbiology, spatiotemporal systems, magnetic levitation systems, and communication systems. Dr. Zhou was a recipient of the First-Class Natural Science Prize in 2003 from the Ministry of Education, China, and a recipient of the National Outstanding Youth Foundation of China in 2006. He was an Associate Editor for the IEEE Transactions on Automatic Control, and is currently on the Editorial Board of Automatica. E-mail: tzhou@mail.tsinghua.edu.cn.