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Linear quadratic regulation for discrete-time systems with state delays and multiplicative noise

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Abstract

In this paper, the linear quadratic regulation problem for discrete-time systems with state delays and multiplicative noise is considered. The necessary and sufficient condition for the problem admitting a unique solution is given. Under this condition, the optimal feedback control and the optimal cost are presented via a set of coupled difference equations. Our approach is based on the maximum principle. The key technique is to establish relations between the costate and the state.

Keywords: Optimal control, time-delay system, multiplicative noise

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1 Introduction

Delay exists widely in areas of economics, physics, biology, chemistry and mechanics [1]. This makes the study of time-delay systems, which can model processes with delay, greatly significant in both theory and application. The research in this field includes stability and stabilization [2–4], optimal control [5,6], H_∞ control [7,8], etc. As an important part of the optimal control theory, the linear quadratic regulation (LQR) problem for time-delay systems has been extensively studied. For example, [9] focuses on continuous-time linear systems with state delay. A sufficient condition for a feedback control to be optimal is established via a set of differential equations. For other literature on this subject, see [10–12] and references therein.

Stochastic uncertainty is another important subject in the control theory. In practical situation, there exist various kinds of noises and disturbances. As a result, stochastic systems can characterize the real process more accurately. Stochastic systems can be naturally classified as continuous-time ones and discrete-time ones. Continuous-time stochastic systems are usu-

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ally described by Itô-type differential equations governed by Brownian motion. Systems which correspond to the discretization of Itô-type differential equations are those with multiplicative noises. For works concerning stochastic control, readers are referred to [13, 14] in continuous time and [15, 16] in discrete time.

Many control problems have been studied for systems with both time delay and stochastic uncertainty in the literature. These problems include stability and stabilization [17, 18], estimation [19], and optimal control [20–22]. Reference [20] considers discrete-time stochastic systems with a single input delay. By using the stochastic maximum principle, it presents a complete solution to the finite-horizon LQR problem. It establishes a necessary and sufficient condition for the existence of a unique optimal controller and gives an explicit optimal controller via a Riccati-ZXL difference equation. Reference [21] is concerned with the finite-horizon LQR problem for continuous-time stochastic systems with state and input delays. The optimal feedback controller is given by a new type of Riccati equations whose solvability is not easy to obtain. To the best of our knowledge, previous works on the LQR problem for state-delay systems in both deterministic setting and stochastic setting usually assume that the weighting matrix of the control in the quadratic cost function is strictly positive definite and only give sufficient conditions for the existence of an optimal control; see [9], [12] and [21]. Motivated by this, we are devoted to using the method in [20] to solve the LQR problem for stochastic systems with multiple state delays and aim to derive a necessary and sufficient condition for the existence of an explicit optimal control under the condition that the weighting matrix of the control is positive semi-definite. Readers may think that a possible way to settle this problem is to change it into a delay-free one by incorporating the history state into an augmented state. However, the resultant solution is a high-dimensional Riccati equation, which causes computational burden as pointed out by [7].

The contributions of the paper lie in that a necessary and sufficient condition for the LQR problem admitting a unique solution is given and under this condition, the optimal feedback control and the optimal cost are presented in terms of coupled difference equations. The main technique is to solve the maximum principle, which can be viewed as delayed forward (the state equation) and delayed backward (the costate equation) stochastic difference equations. The optimal costate is expressed as a linear function of a finite length of state and the corresponding coefficient matrices satisfy the above-mentioned coupled difference equations.

The rest of the paper is organized as follows. In Section 2, the stochastic LQR problem for state-delay systems is formulated. Section 3 presents the solution to the problem. Section 4 provides the proof of the main results. Section 5 uses numerical examples to illustrate the results. Section 6 makes a conclusion. Some details of proof are given in Appendix.

Notation $\mathbb{R}^n$ stands for the usual n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the space of real matrices with order $n\times m$; The superscript $'$ means the matrix transpose; I denotes the unit matrix; A symmetric matrix $M>0$ (reps. $\geqslant 0$) means that it is strictly positive definite (reps. positive semi-definite); For a random variable ξ and a σ-algebra $\mathcal{F}$, $\mathrm{E}(\xi)$ and $\mathrm{E}(\xi|\mathcal{F})$ represents the mathematical expectation of ξ and the conditional expectation of ξ with regards to $\mathcal{F}$, respectively; $\delta_{i,j}$ is the usual Kronecker function, i.e., $\delta_{i,i}=1$ and $\delta_{i,j}=0$ if $i\neq j$.

2 Problem statement

Consider the following discrete-time system with state delays and multiplicative noise:

$$\begin{aligned} x_{k+1} = &\sum_{i=0}^{d}[A_i+\sum_{l=1}^{r}\omega_k(l)\bar{A}_{i,l}]x_{k-i}\\ &+[B+\sum_{l=1}^{r}\omega_k(l)\bar{B}_l]u_k,\quad k=0,\dots,N, \end{aligned}\qquad (1)$$

where $x_k\in\mathbb{R}^p$ is the state; $u_k\in\mathbb{R}^q$ is the input control; the constant delay d is a positive integer; $\{x_0, x_{-1},\dots, x_{-d}\}$ is the deterministic initial value; $A_i,\bar{A}_{i,l},B$ and $\bar{B}_l$ with $i=0,\dots,d, l=1,\dots,r$ are constant matrices with compatible dimensions; and $v_k=(\omega_k(1)\ \cdots\ \omega_k(r))'$ is a r-dimensional white noise defined on a complete probability space $\{\Omega,\mathcal{P},\mathcal{F}\}$. The variance of v_k is σ, i.e.,

$$\mathrm{E}[v_kv_k'] = \sigma = \begin{pmatrix}\sigma_{11} & \cdots & \sigma_{1r}\\ \vdots & & \vdots\\ \sigma_{r1} & \cdots & \sigma_{rr}\end{pmatrix}\in\mathbb{R}^{r\times r},\quad \sigma\geqslant 0.$$

Let $\mathcal{F}_k$ be the natural filtration generated by v_k, i.e., $\mathcal{F}_k$ is the σ-algebra generated by $\{v_0,\dots,v_k\}$.

Consider the cost function

$$J=\mathrm{E}[\sum_{k=0}^{N}x_k'Qx_k+\sum_{k=0}^{N}u_k'Ru_k+x_{N+1}'Wx_{N+1}],\qquad (2)$$

where $Q \geqslant 0$, $R \geqslant 0$ and $W \geqslant 0$. Define the admissible control set as

$$\mathcal{U}_{\rm ad} = \{u_k, k = 0, \dots, N : u_k \text{ is } \mathcal{F}_{k-1}\text{-measurable}\}. \quad (3)$$

The problem to be addressed is stated as follows.

Problem 1 Find $u_k \in \mathcal{U}_{\rm ad}$ to minimize the cost function (2) subject to system (1).

Remark 1 (1) is a discrete-time system which admits two features: one is the time delay and another is the white noise v_k. Its possible application can be found in networked control systems (NCSs) with transmission delay as pointed out by [17]. Moreover, a special case of system (1) when v_k is an independent Bernoulli process is often used to describe packet dropout in NCSs; see [23] and [24].

3 Main results

3.1 A special case: scalar noise

In this section, we will focus on a special case of system (1) with $r = 1$:

$$x_{k+1} = \sum_{i=0}^{d}[A_i + \omega_k(1)\bar{A}_{i,1}]x_{k-i} + [B + \omega_k(1)\bar{B}_1]u_k. \quad (4)$$

In this context, notations $\omega_k(1), \bar{A}_{i,1}$, $i = 0, \dots, d$, and $\bar{B}_1$ will be re-denoted by $\omega_k, \bar{A}_i$ and $\bar{B}$, respectively. Accordingly, $\mathcal{F}_k$ is the σ-algebra generated by $\{\omega_0, \dots, \omega_k\}$ and the variance matrix σ is reduced to a scalar. The optimal control problem under consideration becomes the following one.

Problem 2 Find $u_k \in \mathcal{U}_{\rm ad}$ to minimize the cost function (2) subject to system (4).

Motivated by the approach proposed in [25], the maximum principle for Problem 2 is derived as

$$x_{k+1} = \sum_{i=0}^{d} A_i(k)x_{k-i} + B(k)u_k, \quad (5)$$

$$\lambda_N = Wx_{N+1}, \quad (6)$$

$$\lambda_{k-1} = \mathrm{E}[\sum_{m=0}^{d} A'_m(k+m)\lambda_{k+m}|\mathcal{F}_{k-1}] + Qx_k, \quad (7)$$

$$0 = \mathrm{E}[B'(k)\lambda_k|\mathcal{F}_{k-1}] + Ru_k, \quad k = 0, \dots, N, \quad (8)$$

where

$$\begin{cases} A_i(k) \doteq A_i + \omega_k\bar{A}_i, \quad B(k) \doteq B + \omega_k\bar{B}, \quad k = 0, \dots, N, \\ A_i(k) \doteq 0, \quad \lambda_k \doteq 0, \quad k > N. \end{cases}$$

Before presenting the solution to Problem 2, we define a set of matrix sequences R_k, L_k^j and P_k^j, $j = 0, \dots, d$, by the following backwards recursion for $k = N, \dots, 0$:

$$R_k = B'P_{k+1}^0B + \sigma\bar{B}'P_{k+1}^0\bar{B} + R, \quad (9)$$

$$L_k^j = B'P_{k+1}^0A_j + \sigma\bar{B}'P_{k+1}^0\bar{A}_j + B'P_{k+1}^{j+1}, \quad (10)$$

$$\begin{aligned} P_k^j = \sum_{i=0}^{d-j}[&A_i'P_{i+k+1}^0A_{i+j} + \sigma\bar{A}_i'P_{i+k+1}^0\bar{A}_{i+j} + A_i'P_{i+k+1}^{j+i+1} \\ &+(P_{i+k+1}^{i+1})'A_{i+j} - (L_{i+k}^i)'R_{i+k}^{-1}L_{i+k}^{j+i}] + \delta_{j,0}Q, \end{aligned} \quad (11)$$

where the terminal value is given by

$$P_{N+1}^0 = W, \ P_{N+i}^0 = 0, \quad i = 2, \dots, d+1, \quad (12)$$

$$P_{N+i}^j = 0, \quad j = 1, \dots, d+1, \ i = 1, \dots, d+1, \quad (13)$$

$$R_{N+i} = I, \ L_{N+i}^j = 0, \quad i = 1, \dots, d, \ j = 0, \dots, d. \quad (14)$$

Theorem 1 Problem 2 has a unique solution if and only if

$$R_k > 0 \quad (15)$$

for $k = N, \dots, 0$. In this case, the unique optimal control u_k and the optimal value of (2) are respectively

$$u_k = -R_k^{-1}\sum_{j=0}^{d} L_k^j x_{k-j}, \quad (16)$$

and

$$\begin{aligned} J^\star = x_0'P_0^0x_0 &+ 2x_0'\sum_{j=1}^{d}P_0^jx_{-j} + \sum_{j=1}^{d}\sum_{i=1}^{d}\sum_{f=0}^{d-1}x'_{-j} \\ &\times[A'_{f+j}P_{f+1}^0A_{f+i} + \sigma\bar{A}'_{f+j}P_{f+1}^0\bar{A}_{f+i} \\ &+A'_{f+j}P_{f+1}^{i+f+1} + (P_{f+1}^{j+f+1})'A_{f+i} \\ &-(L_f^{j+f})'R_f^{-1}L_f^{i+f}]x_{-i}. \end{aligned} \quad (17)$$

In addition, the following relation between the optimal costate and the state holds:

$$\lambda_{k-1} = \sum_{j=0}^{d} P_k^j x_{k-j}. \quad (18)$$

The proof of Theorem 1 will be provided in the next section.

Remark 2 In Theorem 1, we have extended the definition of the variables P_{k+1}^j, L_k^j, A_j and $\bar{A}_j$ as

$$P_k^j = 0, \ L_k^j = 0, \ A_j = \bar{A}_j = 0, \quad j > d,$$

for the convenience of simplicity.

Remark 3 From (9)–(11), it can be easily observed that P_k^j is completely determined by the following coupled difference equations:

$$\begin{aligned}P_k^j = &\sum_{i=0}^{d-j}[A_i'P_{i+k+1}^0A_{i+j} + \sigma\bar{A}_i'P_{i+k+1}^0\bar{A}_{i+j} + A_i'P_{i+k+1}^{j+i+1}\\&+(P_{i+k+1}^{i+1})'A_{i+j} - (B'P_{i+k+1}^0A_i + \sigma\bar{B}'P_{i+k+1}^0\bar{A}_i\\&+B'P_{i+k+1}^{i+1})'(B'P_{i+k+1}^0B + \sigma\bar{B}'P_{i+k+1}^0\bar{B} + R)^{-1}\\&\times(B'P_{i+k+1}^0A_{i+j} + \sigma\bar{B}'P_{i+k+1}^0\bar{A}_{i+j} + B'P_{i+k+1}^{j+i+1})]\\&+\delta_{j,0}Q,\ \ j = 0,\ldots,d, \qquad (19)\end{aligned}$$

with terminal value (12) and (13). The role of equation (19) in our problem is the same as that of the generalized difference Riccati equation in the standard stochastic LQR problem [15].

Remark 4 This remark is to make clear the differences of this paper from our previous one [20].

. First, problems considered in these two papers are completely different. Reference [20] studies the LQR problem for the following system with a single input delay:

$$x_{k+1} = (A + \omega_k\bar{A})x_k + (B + \omega_k\bar{B})u_{k-d}. \qquad (20)$$

While this paper focuses on system (1) which is with multiple state delays. Obviously, (20) and (1) are essentially different.

. Second, the maximum principle in [20] is given by

$$\lambda_{k-1} = \mathrm{E}[(A + \omega_k\bar{A})'\lambda_k|\mathcal{F}_{k-1}] + Qx_k, \qquad (21)$$
$$0 = \mathrm{E}[(B + \omega_k\bar{B})'\lambda_k|\mathcal{F}_{k-d-1}] + Ru_{k-d}, \qquad (22)$$

where the equilibrium equation is with a single input-delay. In this paper, the adjoint equation (7) is a backward difference equation with multiple delays. Equations (4)–(8) are more difficult to solve than (20)–(22) (see Section 4).

. Third, the results are different. In [20], the optimal controller is shown to be a predictor form as

$$u_k = -\Upsilon_{k+d}^{-1}M_{k+d}\mathrm{E}[x_{k+d}|\mathcal{F}_{k-1}],$$

where $\mathrm{E}[x_{k+d}|\mathcal{F}_{k-1}]$ is the conditional expectation of x_{k+d} with respect to $\mathcal{F}_{k-1}$ and can be expressed as

$$\mathrm{E}[x_{k+d}|\mathcal{F}_{k-1}] = A^dx_k + \sum_{i=1}^{d}A^{i-1}Bu_{k-i}.$$

The gain $-\Upsilon_{k+d}^{-1}M_{k+d}$ is given by the following Riccati-ZXL difference equation

$$Z_k = A'Z_{k+1}A + \sigma\bar{A}'X_{k+1}\bar{A} + Q - L_k,$$

$$X_k = Z_k + \sum_{i=0}^{d-1}(A')^iL_{k+i}A^i,$$

where

$$L_k = M_k'\Upsilon_k^{-1}M_k,$$
$$\Upsilon_k = B'Z_{k+1}B + \sigma\bar{B}'X_{k+1}\bar{B} + R,$$
$$M_k = B'Z_{k+1}A + \sigma\bar{B}'X_{k+1}\bar{A}.$$

While in this paper, the optimal controller has the form as (16) which involves a finite length of history states. In addition, the gains are determined by the coupled difference equations (19).

Remark 5 This paper concentrates on the finite-horizon LQR problem. By showing the convergence of the solution to equations (19) when N tends to $+\infty$, we can derive the corresponding results in infinite-horizon case. On the other hand, combination of this paper with [22], which is concerned with the LQR problem for systems with multiple input delays, will yield results for systems with both multiple state delays and multiple input delays.

3.2 Solution to Problem 1

Next, we shall extend the results in the previous section to system (1). The increase of the dimension of the white noise does not cause any essential changes. A counterpart of Theorem 1 will be presented without proof.

Theorem 2 Problem 1 admits a unique optimal control iff

$$R_k > 0 \qquad (23)$$

for $k = N,\ldots,0$, where R_k is given by the following coupled difference equations:

$$R_k = B'P_{k+1}^0B + \sum_{l=1}^{r}\sum_{f=1}^{r}\sigma_{lf}\bar{B}_l'P_{k+1}^0\bar{B}_f + R, \qquad (24)$$

$$L_k^j = B'P_{k+1}^0A_j + \sum_{l=1}^{r}\sum_{f=1}^{r}\sigma_{lf}\bar{B}_l'P_{k+1}^0\bar{A}_{j,f} + B'P_{k+1}^{j+1}, \qquad (25)$$

$$\begin{aligned}P_k^j = &\sum_{i=0}^{d-j}[A_i'P_{i+k+1}^0A_{i+j} + \sum_{l=1}^{r}\sum_{f=1}^{r}\sigma_{lf}\bar{A}_{i,l}'P_{i+k+1}^0\bar{A}_{i+j,f}\\&+A_i'P_{i+k+1}^{j+i+1} + (P_{i+k+1}^{i+1})'A_{i+j} - (L_{i+k}^i)'R_{i+k}^{-1}L_{i+k}^{j+i}]\\&+\delta_{j,0}Q,\ \ j = 0,\ldots,d, \qquad (26)\end{aligned}$$

with terminal value as (12)–(14). Under this condition, the unique optimal control and the optimal cost are as

$$u_k = -R_k^{-1}\sum_{j=0}^{d} L_k^j x_{k-j}, \tag{27}$$

and

$$\begin{aligned} J^\star = {} & x_0' P_0^0 x_0 + 2x_0' \sum_{j=1}^{d} P_0^j x_{-j} + \sum_{j=1}^{d}\sum_{i=1}^{d}\sum_{f=0}^{d-1} x_{-j}' \\ & \times[A_{f+j}' P_{f+1}^0 A_{f+i} + \sum_{l=1}^{r}\sum_{m=1}^{r} \sigma_{lm} \bar{A}_{f+j,l}' P_{f+1}^0 \bar{A}_{f+i,m} \\ & + A_{f+j}' P_{f+1}^{i+f+1} + (P_{f+1}^{j+f+1})' A_{f+i} \\ & -(L_f^{j+f})' R_f^{-1} L_f^{i+f}] x_{-i}. \end{aligned} \tag{28}$$

4 Derivation of the main results

4.1 Necessity of Theorem 1

Suppose that Problem 2 admits a unique solution. We will show that the matrix R_k defined by (9)–(14) is positive definite, the unique optimal control is as (16), and the optimal costate λ_{k-1} can be expressed like (18).

Lemma 1 Define a set of matrices $R_k, L_k^j, \Phi_k^{m,j}, S_k^{m-1,j}$ and P_k^j with $k = N, \dots, 0$, $j = 0, \dots, d$, and $m = 1, \dots, d$, by the following equations:

$$R_k = B' P_{k+1}^0 B + \sigma \bar{B}' P_{k+1}^0 \bar{B} + R, \tag{29}$$

$$L_k^j = B' P_{k+1}^0 A_j + \sigma \bar{B}' P_{k+1}^0 \bar{A}_j + B' P_{k+1}^{j+1}, \tag{30}$$

$$\Phi_k^{m,j} = \begin{cases} A_j(k) - B(k) R_k^{-1} L_k^j, & m = 1, \\ \sum_{f=1}^{m-1} \Phi_{k+m-1}^{1,m-1-f} \Phi_k^{f,j} + \Phi_{k+m-1}^{1,j+m-1}, & m > 1, \end{cases} \tag{31}$$

$$S_k^{m-1,j} = \sum_{f=1}^{m} P_{k+m}^{m-f} \Phi_k^{f,j} + P_{k+m}^{j+m}, \tag{32}$$

$$\begin{aligned} P_k^j = {} & A_0' P_{k+1}^0 A_j + \sigma \bar{A}_0' P_{k+1}^0 \bar{A}_j + A_0' P_{k+1}^{j+1} \\ & -(A_0' P_{k+1}^0 B + \sigma \bar{A}_0' P_{k+1}^0 \bar{B}) R_k^{-1} L_k^j \\ & + \sum_{m=1}^{d} \{ \mathrm{E}[A_m'(k+m) S_{k+1}^{m-1,0}] A_j \\ & - \mathrm{E}[A_m'(k+m) S_{k+1}^{m-1,0}] B R_k^{-1} L_k^j \\ & + \mathrm{E}[A_m'(k+m) S_{k+1}^{m-1,j+1}] \} + \delta_{j,0} Q, \end{aligned} \tag{33}$$

with terminal value given by

$$P_{N+1}^0 = W,\ P_{N+i}^0 = 0,\ i = 2, \dots, d+1, \tag{34}$$

$$P_{N+i}^j = 0,\ \ j = 1, \dots, d+1,\ i = 1, \dots, d+1, \tag{35}$$

$$\begin{aligned} & S_{N+1}^{m-1,j} = 0,\ \Phi_{N+i}^{1,j} = 0,\ \ m = 1, \dots, d, \\ & i = 1, \dots, d-1,\ j = 0, \dots, d. \end{aligned} \tag{36}$$

Suppose that Problem 2 has a unique optimal control, then R_k, which has been defined above, satisfies

$$R_k > 0 \tag{37}$$

for $k = N, \dots, 0$. The optimal control possesses the form of

$$u_k = -R_k^{-1} \sum_{j=0}^{d} L_k^j x_{k-j}. \tag{38}$$

When the control is optimal, the following relations hold

$$x_{k+m} = \sum_{j=0}^{d} \Phi_k^{m,j} x_{k-j}, \tag{39}$$

$$\lambda_{k+m-1} = \sum_{j=0}^{d} S_k^{m-1,j} x_{k-j},\ \ m = 1, \dots, d, \tag{40}$$

$$\lambda_{k-1} = \sum_{j=0}^{d} P_k^j x_{k-j}. \tag{41}$$

Proof See Appendix.

Remark 6 For $j > d$, $S_k^{m,j}$ and $\Phi_k^{m,j}$ are defined to be zero.

Remark 7 From (29)–(36), it can be observed that P_k^j is deterministic while $\Phi_k^{m,j}$ and $S_k^{m-1,j}$ contain noises at time $k, k+1, \dots, k+m-1$.

Lemma 2 For $k = 0, \dots, N$, there holds

$$\begin{aligned} & \sum_{m=1}^{d} \mathrm{E}[A_m'(k+m) S_{k+1}^{m-1,j}] \\ & = \sum_{i=1}^{d-j+1} [A_i' P_{i+k+1}^0 A_{i+j-1} + \sigma \bar{A}_i' P_{i+k+1}^0 \bar{A}_{i+j-1} \\ & \quad + A_i' P_{i+k+1}^{j+i} + (P_{i+k+1}^{i+1})' A_{i+j-1} \\ & \quad -(L_{i+k}^i)' R_{i+k}^{-1} L_{i+k}^{j-1+i}],\ \ j \geqslant 1, \end{aligned} \tag{42}$$

and

$$\sum_{m=1}^{d} \mathrm{E}[A_m'(k+m) S_{k+1}^{m-1,0}] = (P_{k+1}^1)'. \tag{43}$$

Proof See Appendix.

Finally, it can be shown that (33) can be rewritten as (11) with the help of (42) and (43). This process is completely similar to the derivation of (a16), so details are not provided here.

4.2 Sufficiency of Theorem 1

Proof Suppose $R_k > 0$ for $k = N, \dots, 0$ where R_k is determined by (9)–(14), then it will be proven that the unique solution to Problem 2 is (16) and the optimal value of (2) is (17). To this end, define

$$\begin{aligned}
&V(k,\bar{x}_k)\\
&=\mathrm{E}\{x_k'P_k^0x_k+2x_k'\sum_{j=1}^{d}P_k^jx_{k-j}+\sum_{j=1}^{d}\sum_{i=1}^{d}\sum_{f=0}^{d-1}x_{k-j}'\\
&\quad\times[A_{f+j}'P_{k+f+1}^0A_{f+i}+\sigma\bar{A}_{f+j}'P_{k+f+1}^0\bar{A}_{f+i}\\
&\quad+A_{f+j}'P_{k+f+1}^{i+f+1}+(P_{k+f+1}^{j+f+1})'A_{f+i}\\
&\quad-(L_{k+f}^{j+f})'R_{k+f}^{-1}L_{k+f}^{i+f}]x_{k-i}\},
\end{aligned}\tag{44}$$

where the notation $\bar{x}_k$ stands for the vector $(x_k' \ x_{k-1}' \ \cdots \ x_{k-d}')'$. By applying (5), (9) and (10), direct computation produces

$$\begin{aligned}
&V(k,\bar{x}_k)-V(k+1,\bar{x}_{k+1})\\
&=\mathrm{E}\{x_k'P_k^0x_k-x_k'[\sum_{f=0}^{d}(A_f'P_{k+f+1}^0A_f+\sigma\bar{A}_f'P_{k+f+1}^0\bar{A}_f\\
&\quad+A_f'P_{k+f+1}^{f+1}+(P_{k+f+1}^{f+1})'A_f)-\sum_{f=1}^{d}(L_{k+f}^f)'R_{k+f}^{-1}L_{k+f}^f]\\
&\quad\times x_k+2x_k'\sum_{j=1}^{d}P_k^jx_{k-j}-2x_k'\sum_{i=1}^{d}[\sum_{f=0}^{d-1}(A_f'P_{k+f+1}^0A_{f+i}\\
&\quad+\sigma\bar{A}_f'P_{k+f+1}^0\bar{A}_{f+i}+A_f'P_{k+f+1}^{i+f+1}+(P_{k+f+1}^{f+1})'A_{f+i})\\
&\quad-\sum_{f=1}^{d-1}(L_{k+f}^f)'R_{k+f}^{-1}L_{k+f}^{i+f}]x_{k-i}-\sum_{j=1}^{d}\sum_{i=1}^{d}x_{k-j}'(L_k^j)'\\
&\quad\times R_k^{-1}L_k^ix_{k-i}-u_k'(R_k-R)u_k-2u_k'\sum_{j=0}^{d}L_k^jx_{k-j}\}.
\end{aligned}$$

In view of the invertibility of R_k, we can complete the square in the above equation as

$$\begin{aligned}
&V(k,\bar{x}_k)-V(k+1,\bar{x}_{k+1})\\
&=\mathrm{E}\{x_k'P_k^0x_k-x_k'[\sum_{f=0}^{d}(A_f'P_{k+f+1}^0A_f+\sigma\bar{A}_f'P_{k+f+1}^0\bar{A}_f\\
&\quad+A_f'P_{k+f+1}^{f+1}+(P_{k+f+1}^{f+1})'A_f-(L_{k+f}^f)'R_{k+f}^{-1}L_{k+f}^f)]\\
&\quad\times x_k+2x_k'\sum_{j=1}^{d}P_k^jx_{k-j}-2x_k'\sum_{j=1}^{d}[\sum_{i=0}^{d-1}(A_i'P_{k+i+1}^0A_{i+j}\\
&\quad+\sigma\bar{A}_i'P_{k+i+1}^0\bar{A}_{i+j}+A_i'P_{k+i+1}^{j+i+1}+(P_{k+i+1}^{i+1})'A_{i+j}\\
&\quad-(L_{k+i}^i)'R_{k+i}^{-1}L_{k+i}^{j+i})]x_{k-j}+u_k'Ru_k-[u_k\\
&\quad+R_k^{-1}\sum_{j=0}^{d}L_k^jx_{k-j}]'R_k[u_k+R_k^{-1}\sum_{j=0}^{d}L_k^jx_{k-j}]\}.
\end{aligned}\tag{45}$$

By making use of (11), (45) is further rewritten as

$$\begin{aligned}
&V(k,\bar{x}_k)-V(k+1,\bar{x}_{k+1})\\
&=\mathrm{E}[x_k'Qx_k+u_k'Ru_k-(u_k+R_k^{-1}\sum_{j=0}^{d}L_k^jx_{k-j})'\\
&\quad\times R_k(u_k+R_k^{-1}\sum_{i=0}^{d}L_k^ix_{k-i})].
\end{aligned}\tag{46}$$

On both sides of (46), take sums from $k=0$ to $k=N$. It leads to

$$\begin{aligned}
&V(0,\bar{x}_0)-V(N+1,\bar{x}_{N+1})\\
&=\sum_{k=0}^{N}\mathrm{E}[x_k'Qx_k+u_k'Ru_k-(u_k+R_k^{-1}\sum_{j=0}^{d}L_k^jx_{k-j})'\\
&\quad\times R_k(u_k+R_k^{-1}\sum_{i=0}^{d}L_k^ix_{k-i})].
\end{aligned}$$

In view of (12)–(14), it can be readily derived that $V(N+1,\bar{x}_{N+1})=\mathrm{E}(x_{N+1}'Wx_{N+1})$. Then the above equation implies that the cost function (2) can be expressed as

$$\begin{aligned}
J=V(0,\bar{x}_0)+\sum_{k=0}^{N}\mathrm{E}[(u_k+R_k^{-1}\sum_{j=0}^{d}L_k^jx_{k-j})'\\
\times R_k(u_k+R_k^{-1}\sum_{i=0}^{d}L_k^ix_{k-i})].
\end{aligned}\tag{47}$$

Note that $V(0,\bar{x}_0)$ only depends on the initial value of system (5) and R_k is positive definite. Therefore, the unique controller minimizing J_N must be (16) and the optimal value of J is $V(0,\bar{x}_0)$, i.e., (17). □

5 Numerical examples

Example 1 Consider system (4) where both x_k and u_k are scalar and

$$\begin{aligned}
&d=2,\ \ \sigma=1,\ \ A_0=2,\ \ A_1=-1,\ \ A_2=-2,\\
&\bar{A}_0=0.5,\ \ \bar{A}_1=-1.5,\ \ \bar{A}_2=0.8,\ \ B=2,\ \ \bar{B}=1.
\end{aligned}$$

In the cost function (2), we set

$$N=0,\ \ W=1,\ \ R=1,\ \ Q=1.$$

By direct computation, we can obtain the solution to the backwards recursion (9)–(14) as

$$\begin{aligned}
&P_0^0=1.8750,\ \ P_0^1=-0.1250,\ \ P_0^2=-1.2000,\\
&P_1^0=1,\ \ P_1^1=0,\ \ P_1^2=0,\\
&R_0=6,\ \ L_0^0=4.5000,\ \ L_0^1=-3.5000,\ \ L_0^2=-3.2000.
\end{aligned}$$

Note that $R_0>0$. Thus from Theorem 1, it follows that the unique optimal control of Problem 2 is given by

$$u_0=-0.75x_0+0.5833x_{-1}+0.5333x_{-2}.\tag{48}$$

To show the effectiveness of our results, we will compare the values of (2) under the controller (48) and the following one

$$\hat{u}_0 = 1.3583 x_0. \tag{49}$$

Denote the value of (2) with (48) by $J^\star$ and that with (49) by J. Five cases with different initial values are considered below.

1) $x_0 = 2,\ x_{-1} = 0,\ x_{-2} = 0,\ J^\star = 7.5000,\ J = 16.3807$,

2) $x_0 = 2.2,\ x_{-1} = -3,\ x_{-2} = 1,\ J^\star = 25.6533,\ J = 25.7420$,

3) $x_0 = 1,\ x_{-1} = -2.8,\ x_{-2} = 1,\ J^\star = 18.5550,\ J = 20.0056$,

4) $x_0 = 0,\ x_{-1} = -5,\ x_{-2} = 2,\ J^\star = 63.2750,\ J = 83.8100$,

5) $x_0 = 3,\ x_{-1} = 1,\ x_{-2} = -4,\ J^\star = 156.4333,\ J = 311.6145$.

In all cases, controller (48) generates a smaller value for (2) than (49). This coincides with Theorem 1.

Example 2 Consider system (4) where $x_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}^2$, $d = 1, \sigma = 1$, and

$$A_0 = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix},\ \bar{A}_0 = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix},$$
$$\bar{A}_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ \bar{B} = \begin{pmatrix} 3 & 3 \\ 2 & 6 \end{pmatrix},$$

and the cost function (2) where

$$N = 2,\ Q = I,\ R = I,\ W = 0.$$

The solution to (9)–(14) is derived as

$$P_0^0 = \begin{pmatrix} 15.6887 & 1.3581 \\ 1.3581 & 3.7805 \end{pmatrix},\ P_0^1 = \begin{pmatrix} -4.7514 & 7.6065 \\ 2.0778 & 1.8842 \end{pmatrix},$$
$$P_1^0 = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix},\ P_1^1 = \begin{pmatrix} -4.5 & 3 \\ 2.5 & 0 \end{pmatrix},\ P_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$P_2^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ R_0 = \begin{pmatrix} 51 & 54 \\ 54 & 100 \end{pmatrix},\ L_0^0 = \begin{pmatrix} -4.5 & 1 \\ 20.5 & -21 \end{pmatrix},$$
$$L_0^1 = \begin{pmatrix} 10 & -8 \\ -18 & -9 \end{pmatrix},\ R_1 = \begin{pmatrix} 19 & 25 \\ 25 & 51 \end{pmatrix},\ L_1^0 = \begin{pmatrix} 3 & -3 \\ 13 & -13 \end{pmatrix},$$
$$L_1^1 = \begin{pmatrix} 4 & -2 \\ -1 & -6 \end{pmatrix},\ R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ L_2^0 = L_2^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It can be easily verified that R_0, R_1 and R_3 are all positive definite. Hence, according to Theorem 1, the unique optimal controller is

$$u_0 = \begin{pmatrix} 0.7129 & -0.5650 \\ -0.5900 & 0.5151 \end{pmatrix} x_0 + \begin{pmatrix} -0.9029 & 0.1438 \\ 0.6676 & 0.0124 \end{pmatrix} x_{-1},$$
$$u_1 = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} x_1 + \begin{pmatrix} -0.6657 & -0.1395 \\ 0.3459 & 0.1860 \end{pmatrix} x_0,$$
$$u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When the initial value is chosen to be

$$x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ x_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the optimal cost is

$$J^\star = 36.9361. \tag{50}$$

If the controller is changed into

$$\hat{u}_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x_0 + \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x_{-1},$$
$$\hat{u}_1 = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} x_1 + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} x_0,$$
$$\hat{u}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} x_1,$$

the associated value of the cost function is

$$J = 626.25,$$

which is larger than (50).

6 Conclusions

This paper solves the LQR problem for stochastic systems with state delays in discrete-time case. The coupled difference equations developed here play the same role in our problem as the generalized difference Riccati equation does in the standard stochastic LQR problem. The stabilization problem for this class of systems is worth considering in the future.

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Appendix

Proof of Lemma 1 Suppose that Problem 2 has a unique solution. (37)–(41) will be shown inductively on $k = N, \dots, 0$. For simplicity, denote the cost function starting from time k, $k = 0, \dots, N$, by

$$J(k) \doteq \sum_{i=k}^{N} \mathrm{E}(x_i' Q x_i + u_i' R u_i) + \mathrm{E}(x_{N+1}' W x_{N+1}). \tag{a1}$$

The verification of the case of $k = N$ is simple and similar to the discussion given below. Thus it will be omitted. Inductively, suppose (37)–(41) hold for $k \geqslant n + 1$. We shall show that they are true for $k = n$. First, $R_n > 0$ is to be verified. To this end, set n to be the initial time and let the initial value be

$$x_{n-i} \doteq 0, \quad i = 0, \dots, d. \tag{a2}$$

Take u_n to be any $\mathcal{F}_{n-1}$-measurable random variable and u_{n+1}, $\dots, u_N$ to be optimal. Now the optimal value of $J(n + 1)$ will be calculated. For $k = n + 1, \dots, N + 1$, denote

$$\alpha_k \doteq \mathrm{E}[x_k' \lambda_{k-1} + \sum_{j=1}^{d} x_{k-j}' \sum_{m=0}^{d-j} A_{j+m}'(k + m)\lambda_{k+m}]. \tag{a3}$$

By (7), (5) and (8), it can be derived

$$\begin{aligned}
&\alpha_k - \alpha_{k+1}\\
&= \mathrm{E}[x_k' \lambda_{k-1} - x_{k+1}' \lambda_k + \sum_{j=1}^{d} \sum_{m=0}^{d-j} x_{k-j}' A_{j+m}'(k + m)\lambda_{k+m}\\
&\quad - \sum_{j=0}^{d-1} \sum_{m=1}^{d-j} x_{k-j}' A_{j+m}'(k + m)\lambda_{k+m}]\\
&= \mathrm{E}\{x_k' \mathrm{E}[\sum_{m=0}^{d} A_m'(k + m)\lambda_{k+m} | \mathcal{F}_{k-1}] + x_k' Q x_k
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=0}^{d} x'_{k-j}A'_j(k)\lambda_k - u'_kB'(k)\lambda_k\\
&+\sum_{j=1}^{d} x'_{k-j}A'_j(k)\lambda_k - \sum_{m=1}^{d} x'_kA'_m(k+m)\lambda_{k+m}\}\\
&= \mathrm{E}(x'_kQx_k + u'_kRu_k), \quad k = n+1,\dots,N.
\end{aligned}
$$

Summing from $k = n+1$ to $k = N$ on the two sides of the above equation yields

$$\alpha_{n+1} - \alpha_{N+1} = \sum_{k=n+1}^{N} \mathrm{E}(x'_kQx_k + u'_kRu_k).$$

From (a3), (6) and $\lambda_k = 0$ for $k > N$, it follows that $\alpha_{N+1} = \mathrm{E}(x'_{N+1}Wx_{N+1})$. Thus the optimal value of $J(n+1)$ is

$$
\begin{aligned}
&J^\star(n+1)\\
&= \alpha_{n+1}\\
&= \mathrm{E}[x'_{n+1}\lambda_n + \sum_{j=1}^{d} x'_{n+1-j}\sum_{m=0}^{d-j} A'_{j+m}(n+m+1)\lambda_{n+m+1}]. \qquad \text{(a4)}
\end{aligned}
$$

(41) is assumed to be true for $k = n+1$, i.e., λ_n is as

$$\lambda_n = \sum_{j=0}^{d} P^j_{n+1}x_{n+1-j}. \qquad \text{(a5)}$$

By applying (a4), (a5), (a2) and (5), it leads to

$$
\begin{aligned}
J^\star(n+1) &= \mathrm{E}(x'_{n+1}P^0_{n+1}x_{n+1})\\
&= \mathrm{E}[u'_nB'(n)P^0_{n+1}B(n)u_n]\\
&= \mathrm{E}[u'_n(B'P^0_{n+1}B + \sigma\bar{B}'P^0_{n+1}\bar{B})u_n], \qquad \text{(a6)}
\end{aligned}
$$

where the fact that the matrix P is deterministic has been employed (see Remark 7). That Problem 2 has a unique solution implies the solution to the optimization problem $\min\limits_{u_n} \hat{J}(n)$ is unique. Therein,

$$\hat{J}(n) = \mathrm{E}(x'_nQx_n + u'_nRu_n) + J^\star(n+1). \qquad \text{(a7)}$$

Furthermore, the weighting matrix of u_n in $\hat{J}(n)$ must be positive definite. By substituting (a6) into (a7), it can be easily obtained that the weighting matrix is just R_n. Hence, $R_n > 0$ has been shown.

Second, the optimal u_n is to be solved. Substitution of (a5) and (5) into (8) produces

$$
\begin{aligned}
0 &= \mathrm{E}[\sum_{j=0}^{d} B(n)'P^j_{n+1}x_{n+1-j}|\mathcal{F}_{n-1}] + Ru_n\\
&= \sum_{j=0}^{d}(B'P^0_{n+1}A_j + \sigma\bar{B}'P^0_{n+1}\bar{A}_j)x_{n-j} + \sum_{j=0}^{d-1} B'P^{j+1}_{n+1}x_{n-j}\\
&\quad +R_nu_n\\
&= \sum_{j=0}^{d} L^j_nx_{n-j} + R_nu_n.
\end{aligned}
$$

Combined with $R_n > 0$, it is readily seen that the optimal u_n is given by

$$u_n = -R_n^{-1}\sum_{j=0}^{d} L^j_nx_{n-j}. \qquad \text{(a8)}$$

Third, let us verify (39) for $k = n$, i.e., the following relation

$$x_{n+t} = \sum_{j=0}^{d} \Phi^{t,j}_n x_{n-j}, \qquad \text{(a9)}$$

holds for $t = 1,\dots,d$. The analysis will be made inductively on t. By substituting (a8) into (5), x_{n+1} becomes

$$
\begin{aligned}
x_{n+1} &= \sum_{j=0}^{d}(A_j(n) - B(n)R_n^{-1}L^j_n)x_{n-j}\\
&= \sum_{j=0}^{d} \Phi^{1,j}_n x_{n-j}, \qquad \text{(a10)}
\end{aligned}
$$

which is the case of $t = 1$. Inductively, suppose (a9) is true for $t = 1,\dots,s$. By applying (39) with $k = n+s$ and $m = 1$, one gets

$$
\begin{aligned}
x_{n+s+1} &= \sum_{j=0}^{d} \Phi^{1,j}_{n+s}x_{n+s-j}\\
&= \sum_{j=0}^{d} \Phi^{1,j+s}_{n+s}x_{n-j} + \sum_{f=1}^{s} \Phi^{1,s-f}_{n+s}x_{n+f}, \qquad \text{(a11)}
\end{aligned}
$$

where $\Phi^{1,j+s}_{n+s} = 0$ if $j+s > d$ has been employed (See Remark 6). By making use of (a9) for $t = 1,\dots,s$ in (a11), it yields

$$
\begin{aligned}
x_{n+s+1} &= \sum_{j=0}^{d}[\Phi^{1,j+s}_{n+s} + \sum_{f=1}^{s} \Phi^{1,s-f}_{n+s}\Phi^{f,j}_n]x_{n-j}\\
&= \sum_{j=0}^{d} \Phi^{s+1,j}_n x_{n-j},
\end{aligned}
$$

which is (a9) for $t = s+1$. Hence, (a9) has been shown inductively.

Next, (40) is to be proven for $k = n$ and $m = 1,\dots,d$. According to the inductive hypothesis, (41) is true for $k = n+m$, i.e.,

$$
\begin{aligned}
\lambda_{n+m-1} &= \sum_{j=0}^{d} P^j_{n+m}x_{n+m-j}\\
&= \sum_{j=0}^{d} P^{j+m}_{n+m}x_{n-j} + \sum_{f=1}^{m} P^{m-f}_{n+m}x_{n+f}, \qquad \text{(a12)}
\end{aligned}
$$

where $P^{j+m}_{n+m} = 0$ if $j+m > d$ has been used (See Remark 2). By employing (39) with $k = n$ in (a12), we get

$$
\begin{aligned}
\lambda_{n+m-1} &= \sum_{j=0}^{d}[P^{j+m}_{n+m} + \sum_{f=1}^{m} P^{m-f}_{n+m}\Phi^{f,j}_n]x_{n-j}\\
&= \sum_{j=0}^{d} S^{m-1,j}_n x_{n-j},
\end{aligned}
$$

which is indeed (40) with $k = n$.

Finally, let us show (41) for $k = n$. In (40) and (41), setting $k = n + 1$ produces

$$\lambda_{n+m} = \sum_{j=0}^{d} S_{n+1}^{m-1,j} x_{n+1-j} = S_{n+1}^{m-1,0} x_{n+1} + \sum_{j=0}^{d} S_{n+1}^{m-1,j+1} x_{n-j}, \quad m = 1, \dots, d,$$

$$\lambda_n = \sum_{j=0}^{d} P_{n+1}^{j} x_{n+1-j} = P_{n+1}^{0} x_{n+1} + \sum_{j=0}^{d} P_{n+1}^{j+1} x_{n-j},$$

where zeros terms $S_{n+1}^{m-1,d+1}$ and P_{n+1}^{d+1} have been added on purpose. Combined with (a10), the above equations become

$$\lambda_{n+m} = \sum_{j=0}^{d} [S_{n+1}^{m-1,0}(A_j(n) - B(n)R_n^{-1}L_n^j) + S_{n+1}^{m-1,j+1}] x_{n-j}, \quad \text{(a13)}$$

$$\lambda_n = \sum_{j=0}^{d} [P_{n+1}^{0}(A_j(n) - B(n)R_n^{-1}L_n^j) + P_{n+1}^{j+1}] x_{n-j}. \quad \text{(a14)}$$

Substitution of (a13) and (a14) into (7) leads to

$$\begin{aligned}\lambda_{n-1} &= \mathrm{E}\{\sum_{j=0}^{d}[A_0'(n)P_{n+1}^0 A_j(n) - A_0'(n)P_{n+1}^0 B(n)R_n^{-1}L_n^j \\ &\quad + A_0'(n)P_{n+1}^{j+1}] x_{n-j} + \sum_{m=1}^{d}\sum_{j=0}^{d}[A_m'(n+m)S_{n+1}^{m-1,0}A_j(n) \\ &\quad - A_m'(n+m)S_{n+1}^{m-1,0}B(n)R_n^{-1}L_n^j \\ &\quad + A_m'(n+m)S_{n+1}^{m-1,j+1}] x_{n-j} | \mathcal{F}_{n-1}\} + Qx_n.\end{aligned}$$

In view of Remark 7, P_{n+1}^j is deterministic and $S_{n+1}^{m-1,j}$ contain noises at time $n+1, \dots, n+m$. Therefore, the above equation can be further computed as

$$\begin{aligned}\lambda_{n-1} &= \sum_{j=0}^{d} \mathrm{E}[A_0'(n)P_{n+1}^0 A_j(n) - A_0'(n)P_{n+1}^0 B(n)R_n^{-1}L_n^j \\ &\quad + A_0'(n)P_{n+1}^{j+1}] x_{n-j} + \sum_{m=1}^{d}\sum_{j=0}^{d}\mathrm{E}[A_m'(n+m)S_{n+1}^{m-1,0}A_j(n) \\ &\quad - A_m'(n+m)S_{n+1}^{m-1,0}B(n)R_n^{-1}L_n^j + A_m'(n+m)S_{n+1}^{m-1,j+1}] \\ &\quad \times x_{n-j} + Qx_n \\ &= \sum_{j=0}^{d}[A_0'P_{n+1}^0 A_j + \sigma\bar{A}_0'P_{n+1}^0\bar{A}_j + A_0'P_{n+1}^{j+1} \\ &\quad - (A_0'P_{n+1}^0 B + \sigma\bar{A}_0'P_{n+1}^0\bar{B})R_n^{-1}L_n^j] x_{n-j} \\ &\quad + \sum_{j=0}^{d}\sum_{m=1}^{d}\{\mathrm{E}[A_m'(n+m)S_{n+1}^{m-1,0}]A_j \\ &\quad - \mathrm{E}[A_m'(n+m)S_{n+1}^{m-1,0}]BR_n^{-1}L_n^j \\ &\quad + \mathrm{E}[A_m'(n+m)S_{n+1}^{m-1,j+1}]\} x_{n-j} + Qx_n \\ &= \sum_{j=0}^{d} P_n^j x_{n-j},\end{aligned}$$

which is (41) for $k = n$. Until now, the proof of this Lemma is completed. □

Proof of Lemma 2 The lemma will be shown inductively on $k = N, \dots, 0$. Since all the variables S_{k+1}^j, P_{i+k+1}^j and L_{i+k}^j are zero with $k \geqslant N$ and $i \geqslant 1$, the case of $k = N$ is trivial. Suppose that the claim is true for $k = N, \dots, n$. By applying (42) and (43) with $k = n, \dots, N$ in (33), it yields

$$\begin{aligned}P_k^j &= A_0'P_{k+1}^0 A_j + \sigma\bar{A}_0'P_{k+1}^0\bar{A}_j + A_0'P_{k+1}^{j+1} + (P_{k+1}^1)'A_j \\ &\quad - [A_0'P_{k+1}^0 B + \sigma\bar{A}_0'P_{k+1}^0\bar{B} + (P_{k+1}^1)'B]R_k^{-1}L_k^j \\ &\quad + \sum_{i=1}^{d-j}[A_i'P_{i+k+1}^0 A_{i+j} + \sigma\bar{A}_i'P_{i+k+1}^0\bar{A}_{i+j} + A_i'P_{i+k+1}^{j+1+i} \\ &\quad + (P_{i+k+1}^{i+1})'A_{i+j} - (L_{i+k}^i)'R_{i+k}^{-1}L_{i+k}^{j+i}] + \delta_{j,0}Q. \quad \text{(a15)}\end{aligned}$$

From (30), it follows

$$(L_k^0)' = A_0'P_{k+1}^0 B + \sigma\bar{A}_0'P_{k+1}^0\bar{B} + (P_{k+1}^1)'B.$$

Employ the above equation in (a15). It leads to

$$\begin{aligned}P_k^j &= \sum_{i=0}^{d-j}[A_i'P_{i+k+1}^0 A_{i+j} + \sigma\bar{A}_i'P_{i+k+1}^0\bar{A}_{i+j} + A_i'P_{i+k+1}^{j+1+i} \\ &\quad + (P_{i+k+1}^{i+1})'A_{i+j} - (L_{i+k}^i)'R_{i+k}^{-1}L_{i+k}^{j+i}] + \delta_{j,0}Q, \\ &\quad j = 0, \dots, d, \quad k = n, \dots, N. \quad \text{(a16)}\end{aligned}$$

Now we show (42) and (43) for $k = n - 1$. The following relation will be verified inductively on $t = d, \dots, 1$:

$$\begin{aligned}&\sum_{m=t}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}] \\ &= \sum_{f=1}^{t-1}\sum_{i=t}^{d}[A_i'P_{i+n}^0 A_{i-1-f} + \sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1-f} + A_i'P_{i+n}^{i-f} \\ &\quad + (P_{i+n}^{i+1})'A_{i-1-f} - (L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-f-1}]\mathrm{E}(\Phi_n^{f,j}) \\ &\quad + \sum_{i=t}^{d}[A_i'P_{i+n}^0 A_{i-1+j} + \sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1+j} + A_i'P_{i+n}^{j+i} \\ &\quad + (P_{i+n}^{i+1})'A_{i-1+j} - (L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{j+i-1}]. \quad \text{(a17)}\end{aligned}$$

First, consider the case of $t = d$. In (32) and (31), setting $k = n$ and $m = d$ produces

$$\Phi_n^{d,j} = \sum_{f=1}^{d-1}\Phi_{n+d-1}^{1,d-1-f}\Phi_n^{f,j} + \Phi_{n+d-1}^{1,j+d-1}, \quad \text{(a18)}$$

$$S_n^{d-1,j} = \sum_{f=1}^{d} P_{n+d}^{d-f}\Phi_n^{f,j} + P_{n+d}^{j+d}, \quad j \geqslant 0. \quad \text{(a19)}$$

Substitution of (a18) into (a19) generates

$$S_n^{d-1,j} = \sum_{f=1}^{d-1}(P_{n+d}^0\Phi_{n+d-1}^{1,d-1-f} + P_{n+d}^{d-f})\Phi_n^{f,j} + P_{n+d}^0\Phi_{n+d-1}^{1,j+d-1} + P_{n+d}^{j+d},$$

which means

$$\begin{aligned}&\mathrm{E}[A_d'(n+d-1)S_n^{d-1,j}] \\ &= \sum_{f=1}^{d-1}\mathrm{E}[A_d'(n+d-1)(P_{n+d}^0\Phi_{n+d-1}^{1,d-1-f} + P_{n+d}^{d-f})\Phi_n^{f,j}]\end{aligned}$$

$$+\mathrm{E}[A_d'(n+d-1)P_{n+d}^0\Phi_{n+d-1}^{1,j+d-1}]+\mathrm{E}[A_d'(n+d-1)P_{n+d}^{j+d}].$$

Based upon Remark 7, it is known that $A_d(n+d-1)$ and $\Phi_{n+d-1}^{1,d-1-f}$ are independent of $\Phi_n^{f,j}$. Thus the above equation can be further computed as

$$\begin{aligned}&\mathrm{E}[A_d'(n+d-1)S_n^{d-1,j}]\\&=\sum_{f=1}^{d-1}\{\Upsilon_f+A_d'P_{n+d}^{d-f}\}\mathrm{E}(\Phi_n^{f,j})+\Upsilon_{-j}+A_d'P_{n+d}^{j+d}\end{aligned}\quad\text{(a20)}$$

with

$$\Upsilon_f=\mathrm{E}[A_d'(n+d-1)P_{n+d}^0\Phi_{n+d-1}^{1,d-1-f}].\quad\text{(a21)}$$

By applying (31) with $k=n+d-1$ and $j=d-1-f$ in (a21), it results in

$$\begin{aligned}\Upsilon_f=&A_d'P_{n+d}^0A_{d-1-f}+\sigma\bar{A}_d'P_{n+d}^0\bar{A}_{d-1-f}\\&-(A_d'P_{n+d}^0B+\sigma\bar{A}_d'P_{n+d}^0\bar{B})R_{n+d-1}^{-1}L_{n+d-1}^{d-1-f}.\end{aligned}$$

In view of (30), there holds

$$(L_{n+d-1}^d)'=A_d'P_{n+d}^0B+\sigma\bar{A}_d'P_{n+d}^0\bar{B}.$$

Therefore, Υ_f becomes

$$\begin{aligned}\Upsilon_f=&A_d'P_{n+d}^0A_{d-1-f}+\sigma\bar{A}_d'P_{n+d}^0\bar{A}_{d-1-f}\\&-(L_{n+d-1}^d)'R_{n+d-1}^{-1}L_{n+d-1}^{d-1-f}.\end{aligned}$$

Υ_{-j} can be obtained by replacing f with $-j$ in the above equation. Employ Υ_f and Υ_{-j} in (a20). It yields

$$\begin{aligned}&\mathrm{E}[A_d'(n+d-1)S_n^{d-1,j}]\\&=\sum_{f=1}^{d-1}\{A_d'P_{n+d}^0A_{d-1-f}+\sigma\bar{A}_d'P_{n+d}^0\bar{A}_{d-1-f}\\&\quad-(L_{n+d-1}^d)'R_{n+d-1}^{-1}L_{n+d-1}^{d-1-f}+A_d'P_{n+d}^{d-f}\}\mathrm{E}(\Phi_n^{f,j})\\&\quad+A_d'P_{n+d}^0A_{d-1+j}+\sigma\bar{A}_d'P_{n+d}^0\bar{A}_{d-1+j}\\&\quad-(L_{n+d-1}^d)'R_{n+d-1}^{-1}L_{n+d-1}^{d-1+j}+A_d'P_{n+d}^{j+d},\end{aligned}$$

which is indeed (a17) for $t=d$. So far, the case of $t=d$ has been clarified.

Suppose that (a17) holds for $t=h+1$ with $1\leqslant h\leqslant d-1$, i.e.,

$$\begin{aligned}&\sum_{m=h+1}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}]\\&=\sum_{f=1}^{h}\sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-1-f}+\sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1-f}+A_i'P_{i+n}^{i-f}\\&\quad+(P_{i+n}^{i+1})'A_{i-1-f}-(L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-f-1}]\mathrm{E}(\Phi_n^{f,j})\\&\quad+\sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-1+j}+\sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1+j}+A_i'P_{i+n}^{j+i}\\&\quad+(P_{i+n}^{i+1})'A_{i-1+j}-(L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{j+i-1}].\end{aligned}\quad\text{(a22)}$$

Now we show it is true for $t=h$. Note that

$$\begin{aligned}&\sum_{m=h}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}]\\&=\mathrm{E}[A_h'(n-1+h)S_n^{h-1,j}]\\&\quad+\sum_{m=h+1}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}].\end{aligned}\quad\text{(a23)}$$

According to (32) and (31), $S_n^{h-1,j}$ and $\Phi_n^{h,j}$ are respectively as

$$S_n^{h-1,j}=\sum_{f=1}^{h}P_{n+h}^{h-f}\Phi_n^{f,j}+P_{n+h}^{j+h},\quad\text{(a24)}$$

$$\Phi_n^{h,j}=\sum_{f=1}^{h-1}\Phi_{n+h-1}^{1,h-1-f}\Phi_n^{f,j}+\Phi_{n+h-1}^{1,j+h-1},\quad\text{(a25)}$$

which yields

$$\begin{aligned}&\mathrm{E}[A_h'(n-1+h)S_n^{h-1,j}]\\&=\sum_{f=1}^{h-1}\{A_h'P_{n+h}^{h-f}+\mathrm{E}[A_h'(n-1+h)P_{n+h}^0\Phi_{n+h-1}^{1,h-1-f}]\}\mathrm{E}[\Phi_n^{f,j}]\\&\quad+\mathrm{E}[A_h'(n-1+h)P_{n+h}^0\Phi_{n+h-1}^{1,j+h-1}]+A_h'P_{n+h}^{j+h}.\end{aligned}\quad\text{(a26)}$$

Therein, the independence of $\{A_h(n-1+h),\ \Phi_{n+h-1}^{1,h-1-f}\}$ and $\Phi_n^{f,j}$ with $f\leqslant h-1$ has been applied. On the other hand, (a25) implies

$$\mathrm{E}(\Phi_n^{h,j})=\sum_{f=1}^{h-1}\mathrm{E}(\Phi_{n+h-1}^{1,h-1-f})\mathrm{E}(\Phi_n^{f,j})+\mathrm{E}(\Phi_{n+h-1}^{1,j+h-1}).\quad\text{(a27)}$$

Combine (a22), (a23), (a26) and (a27). It results in

$$\begin{aligned}&\sum_{m=h}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}]\\&=\sum_{f=1}^{h-1}\{\Upsilon_f+A_h'P_{n+h}^{h-f}+\sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-1-f}\\&\quad+\sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1-f}+A_i'P_{i+n}^{i-f}+(P_{i+n}^{i+1})'A_{i-1-f}\\&\quad-(L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-f-1}]\}\mathrm{E}(\Phi_n^{f,j})\\&\quad+\Upsilon_{-j}+A_h'P_{n+h}^{j+h}+\sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-1+j}\\&\quad+\sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1+j}+A_i'P_{i+n}^{j+i}+(P_{i+n}^{i+1})'A_{i-1+j}\\&\quad-(L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{j+i-1}],\end{aligned}\quad\text{(a28)}$$

where

$$\begin{aligned}\Upsilon_f=&\mathrm{E}[A_h'(n-1+h)P_{n+h}^0\Phi_{n+h-1}^{1,h-1-f})]\\&+\sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-1-h}+\sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1-h}+A_i'P_{i+n}^{i-h}\\&+(P_{i+n}^{i+1})'A_{i-1-h}-(L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-h-1}]\mathrm{E}(\Phi_{n+h-1}^{1,h-1-f}).\end{aligned}\quad\text{(a29)}$$

In (a16), take $k = n+h$ and $j = h+1$. Then $(P_{n+h}^{h+1})'$ is derived as

$$\begin{aligned}&(P_{n+h}^{h+1})'\\&= \sum_{i=h+1}^{d}[A_i'P_{i+n}^0A_{i-h-1} + \sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-h-1} + (P_{i+n}^{i+1})'A_{i-h-1}\\&\quad + A_i'P_{i+n}^{i-h} - (L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-h-1}].\end{aligned}$$

Apply the above equation in (a29). Thus Υ_f becomes

$$\begin{aligned}\Upsilon_f &= \mathrm{E}[A_h'(n-1+h)P_{n+h}^0\Phi_{n+h-1}^{1,h-1-f})]\\&\quad + (P_{n+h}^{h+1})'\mathrm{E}(\Phi_{n+h-1}^{1,h-1-f}).\end{aligned} \tag{a30}$$

Furthermore, employ (31) and (30) in (a30). We get

$$\begin{aligned}\Upsilon_f &= A_h'P_{n+h}^0A_{h-f-1} + \sigma\bar{A}_h'P_{n+h}^0\bar{A}_{h-f-1}\\&\quad + (P_{n+h}^{h+1})'A_{h-f-1} - [A_h'P_{n+h}^0B + \sigma\bar{A}_h'P_{n+h}^0\bar{B}\\&\quad + (P_{n+h}^{h+1})'B]R_{n+h-1}^{-1}L_{n+h-1}^{h-f-1}\\&= A_h'P_{n+h}^0A_{h-f-1} + \sigma\bar{A}_h'P_{n+h}^0\bar{A}_{h-f-1} + (P_{n+h}^{h+1})'A_{h-f-1}\\&\quad - (L_{n+h-1}^h)'R_{n+h-1}^{-1}L_{n+h-1}^{h-f-1}.\end{aligned} \tag{a31}$$

Replacing f with $-j$ in (a31) yields Υ_{-j}. Substitute Υ_f and Υ_{-j} into (a28). Then (a17) for $t = h$ can be directly obtained. So far, it has been shown that (a17) is true for $t = 1, \ldots, d$ in an inductive way. In particular, setting $t = 1$ in (a17) generates

$$\begin{aligned}&\sum_{m=1}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,j}]\\&= \sum_{i=1}^{d}[A_i'P_{i+n}^0A_{i-1+j} + \sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1+j} + A_i'P_{i+n}^{j+i} + (P_{i+n}^{i+1})'\\&\quad \times A_{i-1+j} - (L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{j+i-1}],\quad j = 1, \ldots, d,\end{aligned} \tag{a32}$$

$$\begin{aligned}&\sum_{m=1}^{d}\mathrm{E}[A_m'(n-1+m)S_n^{m-1,0}]\\&= \sum_{i=1}^{d}[A_i'P_{i+n}^0A_{i-1} + \sigma\bar{A}_i'P_{i+n}^0\bar{A}_{i-1} + A_i'P_{i+n}^i\\&\quad + (P_{i+n}^{i+1})'A_{i-1} - (L_{i+n-1}^i)'R_{i+n-1}^{-1}L_{i+n-1}^{i-1}].\end{aligned} \tag{a33}$$

Note that in (a32), if $i > d-j+1$, the variables A_{i-1+j}, $\bar{A}_{i-1+j}$, P_{i+n}^{j+i} and L_{i+n-1}^{j+i-1} are all zero. Therefore, (a32) is actually (42) with $k = n-1$. In view of (a16), it can be easily observed that the right side of (a33) is $(P_n^1)'$. This verifies (43) for $k = n-1$. Therefore, the proof is completed. □

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