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Infinite horizon indefinite stochastic linear quadratic control for discrete-time systems

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Abstract

This paper discusses discrete-time stochastic linear quadratic (LQ) problem in the infinite horizon with state and control dependent noise, where the weighting matrices in the cost function are assumed to be indefinite. The problem gives rise to a generalized algebraic Riccati equation (GARE) that involves equality and inequality constraints. The well-posedness of the indefinite LQ problem is shown to be equivalent to the feasibility of a linear matrix inequality (LMI). Moreover, the existence of a stabilizing solution to the GARE is equivalent to the attainability of the LQ problem. All the optimal controls are obtained in terms of the solution to the GARE. Finally, we give an LMI -based approach to solve the GARE via a semidefinite programming.

Keywords: Indefinite stochastic LQ control, discrete-time stochastic systems, generalized algebraic Riccati equation, linear matrix inequality, semidefinite programming

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1 Introduction

It is well known that the linear quadratic (LQ) optimal control problem of deterministic systems was first founded by Kalman [1], which has been playing important role in both theory and applications. The deterministic LQ problem has been discussed extensively by many researchers; see, e.g., [2–5]. The stochastic LQ problem was initiated by Wonham [6] and has been investigated; see [7–14] and the references therein. In the literature, it is a common assumption that the control weighting matrix should be positive definite and the state weighting matrix should be nonnegative. In this case, the solvability of the LQ problem is equivalent to that of the Riccati equations. However, a class of stochastic LQ problems with indefinite control weights

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may still be well-posed [15]. The solvability of indefinite stochastic LQ problem is closely linked to the solvability of the indefinite stochastic Riccati equations. There are many works focusing on this issue, we refer the reader to [16–18].

For the discrete-time LQ control, Y. Huang et al. [19, 20] studied a class of special cases, where the system is described by a difference equation with control and state dependent noise. In [21], the optimal control is obtained for the systems with only control dependent noise. These papers dealt with the LQ problem with the positive definite control weighting matrices in the cost functional. A discrete-time indefinite LQ control in a finite horizon with state and control dependent noise is studied in [22]. Analytical properties of the constrained discrete-time indefinite stochastic LQ control in finite time horizon were extensively studied in [23–25].

This paper considers the discrete-time indefinite stochastic LQ problem in the infinite horizon. Different from the finite horizon case, in order to guarantee the well-posedness of the LQ problem and the existence of the feedback stabilizing control, we have to define some concepts such as stabilizability. A generalized algebraic Riccati equation (GARE) is introduced. It turns out that the attainability of the LQ problem is necessary and sufficient for the existence of the stabilizing solution to the GARE. Meanwhile, we introduce a linear matrix inequality (LMI) condition and show that the well-posedness of the LQ problem is equivalent to the feasibility of the LMI. Furthermore, we present all optimal controls via the solution to the GARE. Finally, we give an LMI-based approach to solve the GARE via a semidefinite programming.

The remainder of this paper is organized as follows. In Section 2, we present the notions of stabilizability and some preliminaries. Section 3 shows that the solvability of the GARE is sufficient for the well-posedness of the LQ problem and the existence of an optimal control. Section 4 contains main results of the paper. In Section 5, we give an LMI-based approach to solve the GARE via a semidefinite programming. Section 6 ends this paper with some concluding remarks.

For convenience, we adopt the following notations. M' represents the transpose of a matrix M; $\mathrm{Tr}(M)$ is the trace of a square matrix M; $M > 0$ ($M \geqslant 0$) means that M is a positive definite (positive semi-definite) symmetric matrix; $\mathrm{E}[x]$ represents the mathematical expectation of a random variable x; $\mathbb{R}^k$ is the k-dimensional Euclidean space with the usual 2-norm $\|\cdot\|$; $\mathbb{R}^{m\times n}$ is the vector space of all $m\times n$ matrices with entries in $\mathbb{R}$; $M^\dagger$ means the Moore-Penrose pseudo inverse of a matrix M; I is the identity matrix with an appropriate dimension; $\mathcal{S}^n$ denotes the set of all real $n\times n$ symmetric matrices; and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2 Problem formulation and preliminaries

Consider the following stochastic discrete-time system of the form:

$$x_{k+1} = Ax_k + Bu_k + [Cx_k + Du_k]w_k, \quad k \in \mathbb{N}, \ x_0 \in \mathbb{R}^n, \quad (1)$$

where A, B, C and D are constant matrices with appropriate dimensions. $x \in \mathbb{R}^n$ is called the system state, $u \in \mathbb{R}^m$ is the control input. $x_0 \in \mathbb{R}^n$ is the initial state which is deterministic. $\{w_k\}_{k\geqslant 0}$ are the one-dimensional independent random variables defined on the complete probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$, such that $\mathrm{E}[w_k] = 0$ and $\mathrm{E}[w_s w_t] = \delta_{st}$, where δ_{st} is the Kronecker delta.

We denote $\mathcal{F}_k$ the σ-algebra generated by w_k, $k \in \mathbb{N}$, i.e., $\mathcal{F}_k = \sigma(w_s : \ 1 \leqslant s \leqslant k)$. Let $\mathcal{L}^2(\Omega, \mathbb{R}^m)$ represent the space of $\mathbb{R}^m$-valued random vectors ξ with $\mathrm{E}\|\xi\|^2 < \infty$. $l^2_w(\mathbb{N}, \mathbb{R}^m)$ consists of all sequences $y = \{y_k : y_k \in \mathbb{R}^m\}_{k\in\mathbb{N}}$, such that $y_k \in \mathcal{L}^2(\Omega, \mathbb{R}^m)$ is $\mathcal{F}_{k-1}$ measurable for $k \in \mathbb{N}$, where we define $\mathcal{F}_{-1} = \{\phi, \Omega\}$, i.e., y_0 is a constant. The l^2-norm of $y \in l^2_w(\mathbb{N}, \mathbb{R}^m)$ is defined by $\|y\|^2_{l^2_w(\mathbb{N},\mathbb{R}^m)} = \sum\limits_{k=0}^{\infty} \mathrm{E}[\|y_k\|^2]$.

For simplicity of our discussion, we give the following definitions.

Definition 1 Consider system (1) with $u_k = 0$. System (1) is said to be mean-square stable [19] if for any $x_0 \in \mathbb{R}^n$, the corresponding state satisfies $\lim\limits_{k\to\infty} \mathrm{E}\|x_k\|^2 = 0$. $u = \{u_k : \ k \in \mathbb{N}\}$ (may be an open-loop control) is said to be a mean-square stabilizing control (with respect to x_0) if the corresponding state x_k of (1) satisfies $\lim\limits_{k\to\infty} \mathrm{E}\|x_k\|^2 = 0$. $u = \{u_k : k \in \mathbb{N}\}$ with $u_k = Kx_k$ is called a mean-square feedback stabilizing control law if for every x_0, the closed-loop system

$$\begin{aligned} &x_{k+1} = (A + BK)x_k + (C + DK)x_k w_k, \\ &k \in \mathbb{N}, \ \ x_0 \in \mathbb{R}^n \end{aligned}$$

is mean-square stable, where K is a constant matrix.

Definition 2 System (1) is said to be stabilizable in the mean square sense if there exists a mean-square feedback stabilizing control law $u_k = Kx_k$, where K is a constant matrix.

For system (1), the admissible control set $U_{\rm ad}(x_0)$ is defined as follows:

$$U_{\rm ad}(x_0) = \{u \in l^2_w(\mathbb{N}, \mathbb{R}^m) | u_k \text{ is mean-square stabilizing with respect to a given } x_0\}.$$

For any $(x_0, u_k) \in \mathbb{R}^n \times U_{\rm ad}(x_0)$, the associated cost functional to system (1) is defined as

$$J(x_0, u) = \sum_{k=0}^{\infty} \mathrm{E}[x_k' Q x_k + 2x_k' L u_k + u_k' R u_k], \tag{2}$$

where $L \in \mathbb{R}^{n\times m}$, $Q \in \mathcal{S}^n$ and $\mathbb{R} \in \mathcal{S}^m$ are given matrices.

The LQ optimal control problem is to find a control sequence $u^* = (u_0^*, \dots, u_n^*, \dots) \in U_{\rm ad}(x_0)$ such that

$$J(x_0, u^*) = V(x_0) = \inf_{u \in U_{\rm ad}(x_0)} J(x_0, u). \tag{3}$$

We call $V(x_0)$ the optimal cost value.

Definition 3 The LQ problem is called well-posed if

$$-\infty < V(x_0) < +\infty, \quad \forall x_0 \in \mathbb{R}^n.$$

A well-posed LQ problem is called attainable if there exists a control sequence $(u_0^*, \dots, u_n^* \dots)$ that achieves $V(x_0)$.

We suppose that system (1) is stabilizable throughout this paper. Hence, $U_{\rm ad}(x_0)$ is nonempty for each x_0.

Now, we present a new GARE as follows.

Definition 4 The constrained algebraic equation on $P \in \mathcal{S}^n$

$$\begin{cases} F(P) - H'(P)G^\dagger(P)H(P) = 0, \\ G(P)G^\dagger(P)H(P) - H(P) = 0,\ G(P) \geqslant 0 \end{cases} \tag{4}$$

with

$$\begin{cases} F(P) = A'PA + C'PC - P + Q, \\ H(P) = B'PA + D'PC + L', \\ G(P) = R + B'PB + D'PD \end{cases} \tag{5}$$

is called a constrained GARE.

Let us give some lemmas needed in the proof of our main results.

Lemma 1 [26] For any matrix $M \in \mathbb{R}^{m\times n}$, there is a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$, which satisfies

$$MM^\dagger M = M,\ M^\dagger M M^\dagger = M^\dagger,$$
$$(MM^\dagger)' = MM^\dagger,\ (M^\dagger M)' = M^\dagger M.$$

$M^\dagger$ is called the Moore-Penrose pseudo inverse of M.

Lemma 2 [26] Let a symmetric matrix M be given. Then

$$M'^\dagger = M^{\dagger\prime},\ MM^\dagger = M^\dagger M,$$
$$M \geqslant 0 \text{ if and only if } M^\dagger \geqslant 0.$$

Lemma 3 [22] Let matrices L, M, N be given, then the matrix equation $LXM = N$ has a solution X if and only if $LL^\dagger NMM^\dagger = N$. X is given by $X = L^\dagger N M^\dagger + Y - L^\dagger LYMM^\dagger$, where Y is a matrix with an appropriate dimension.

Lemma 4 (Extended Schur's lemma) [27] Let matrices $M = M'$, N, $R = R'$ be given with appropriate sizes. Then the following conditions are equivalent:

i) $M - NR^\dagger N' \geqslant 0, R \geqslant 0$, and $N(I - RR^\dagger) = 0$.

ii) $\begin{bmatrix} M & N \\ N' & R \end{bmatrix} \geqslant 0.$

iii) $\begin{bmatrix} R & N' \\ N & M \end{bmatrix} \geqslant 0.$

3 Sufficiency of the GARE

In this section, it is shown that the solvability of the GARE (4) is sufficient for the well-posedness of the LQ problem and the existence of an optimal control. Moreover, we show that any optimal control can be determined by means of the solution to the GARE.

Theorem 1 If the GARE (4) admits a solution P and there exist $Y_k \in \mathbb{R}^{m\times n}$ and $Z_k \in \mathbb{R}^m$ such that the following control:

$$\begin{aligned} u_k^{(Y_k,\ Z_k)} = & -(G^\dagger(P)H(P) - Y_k + G^\dagger(P)G(P)Y_k)x_k \\ & -G^\dagger(P)G(P)Z_k + Z_k \end{aligned} \tag{6}$$

is admissible for any initial x_0. Then LQ problem (1)–(3) is attainable. Furthermore, $u_k^{(Y_k,\ Z_k)}$ is the optimal control and the optimal cost value is uniquely determined by

$$V(x_0) = x_0' P x_0.$$

Proof Let P solve the GARE (4). It is clear that $\forall T \in \mathbb{N}$, $P \in \mathcal{S}^n$,

$$\begin{aligned} \mathrm{E}[x_T' P x_T - x_0' P x_0] &= \sum_{k=0}^{T-1} \mathrm{E}(x_{k+1}' P x_{k+1} - x_k' P x_k) \\ &= \sum_{k=0}^{T-1} \mathrm{E} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' Q(P) \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \end{aligned}$$

where

$$Q(P) = \begin{bmatrix} A'PA + C'PC - P & A'PB + C'PD \\ B'PA + D'PC & B'PB + D'PD \end{bmatrix}.$$

Let $T \to \infty$, then

$$\sum_{k=0}^{\infty} \mathrm{E} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' Q(P) \begin{bmatrix} x_k \\ u_k \end{bmatrix} = -x_0' P x_0.$$

By adding the above equality to the performance index, we have

$$\begin{aligned} J(x_0, u) &= \sum_{k=0}^{\infty} \mathrm{E}[x_k'(Q + A'PA + C'PC - P)x_k \\ &\quad + 2x_k'(A'PB + C'PD + L)u_k \\ &\quad + u_k'(R + B'PB + D'PD)u_k] + x_0'Px_0 \\ &= \sum_{k=0}^{\infty} \mathrm{E}[x_k'F(P)x_k + 2x_k'H'(P)u_k \\ &\quad + u_k'G(P)u_k] + x_0'Px_0. \end{aligned} \tag{7}$$

Define

$$M_k^1 = G^{\dagger}(P)G(P)Y_k - Y_k, \ \ M_k^2 = G^{\dagger}(P)G(P)Z_k - Z_k.$$

Hence, we can obtain that

$$G(P)M_k^1 = 0, \ \ G(P)M_k^2 = 0.$$

By completing squares, (7) can be rewritten as

$$\begin{aligned} J(x_0, u) &= \sum_{k=0}^{\infty} \mathrm{E}[(u_k + (G^{\dagger}(P)H(P) + M_k^1)x_k \\ &\quad + M_k^2)'G(P)(u_k + (G^{\dagger}(P)H(P) + M_k^1)x_k \\ &\quad + M_k^2)] + x_0'Px_0. \end{aligned} \tag{8}$$

This implies that the control sequence

$$u_k = u_k^{(Y_k,\, Z_k)} = -[(G^{\dagger}(P)H(P) + M_k^1)x_k + M_k^2], \ \ k \in \mathbb{N}$$

minimizes J with the optimal value given by $x_0'Px_0$. □

Definition 5 A solution P to the GARE (4) is called stabilizing if there exists an admissible control determined by (6).

Remark 1 A solution P to the GARE (4) is stabilizing if and only if for any x_0 there exists some $Z_k \in \mathbb{R}^m$ such that the following control:

$$u_k = -G^{\dagger}(P)H(P)x_k + [I - G^{\dagger}(P)G(P)]Z_k \tag{9}$$

is admissible, where x_k is the solution to (1) under the above control with the initial state x_0.

The following definition is concerned with the maximal solution.

Definition 6 A matrix P is called a maximal solution to the GARE (4) if $P \geqslant P^*$ for any P^* satisfying

$$\begin{cases} F(P^*) - H'(P^*)G^{\dagger}(P^*)H(P^*) \geqslant 0, \\ H(P^*) - G(P^*)G^{\dagger}(P^*)H(P^*) = 0, \ \ G(P^*) \geqslant 0. \end{cases} \tag{10}$$

By Definition 6, it is clear that the maximal solution must be unique if it exists. Now, let us turn to the GARE (4).

Theorem 2 There is at most one stabilizing solution to (4). Moreover, a stabilizing solution to (4) is also its maximal solution.

Proof Assume that P_1 and P_2 are different stabilizing solutions to (4). By Theorem 1, it follows that $x_0'P_1x_0 = x_0'P_2x_0$ for any x_0, so $P_1 = P_2$.

Let any P^* satisfy (10) and P be the stabilizing solution to (4). Putting P^* in (8) we assert that

$$J(x_0, u) \geqslant x_0'P^*x_0, \ \ \forall u \in U_{\mathrm{ad}}(x_0).$$

By Theorem 1, it is easy to show that $x_0'Px_0 = V(x_0) \geqslant x_0'P^*x_0$. Therefore, P is a maximal stabilizing solution to the GARE due to Definition 6. □

The following corollaries are special cases of the above result.

Corollary 1 Suppose that the GARE (4) admits a stabilizing solution P. If $G(P) = 0$, then any admissible control is optimal and the GARE (4) reduces to

$$\begin{cases} P = A'PA + Q + C'PC, \\ B'PA + D'PC + L' = 0, \\ R + B'PB + D'PD = 0. \end{cases} \tag{11}$$

Proof By (7) and $G(P) = 0$, we can show that

$$J(x_0, u) = x_0'Px_0,$$

which implies that $V(x_0) = x_0'Px_0$ for any $u_k \in U_{\mathrm{ad}}(x_0)$. □

Corollary 2 Let P be a stabilizing solution to the GARE (4). If $G(P) > 0$, then the LQ problem(1)–(3) is uniquely solvable. The unique optimal control is given by

$$u_k = -G^{-1}(P)H(P)x_k.$$

Proof Using Theorem 1, we immediately obtain Corollary 2. □

4 Well-posedness and attainability of LQ problem

In this section, we first present the connection between the well-posedness of the LQ problem and the solvability of the GARE. Then, we study the well-posedness via the LMI condition. Finally, we establish the link between the attainability of the LQ problem and the solvability of the GARE.

Lemma 5 The LQ problem (1)–(3) is well-posed if and only if there exists a symmetric constant matrix P such that

$$V(x_0) = x_0' P x_0, \quad \forall x_0 \in \mathbb{R}^n. \tag{12}$$

Proof (12) can be shown by a simple adaptation of the well-known result in the deterministic case [2].

We introduce the following convex set $\mathcal{P}$ of $\mathcal{S}^n$:

$$\mathcal{P} = \left\{P \in \mathcal{S}^n \middle| \begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \geqslant 0\right\}. \tag{13}$$

□

Theorem 3 The LQ problem (1)–(3) is well-posed if and only if the set $\mathcal{P}$ is nonempty. Moreover, there exists $P \in \mathcal{P}$ such that $P \geqslant \tilde{P}$, $\forall \tilde{P} \in \mathcal{P}$.

Proof (Sufficiency) Assume that the set $\mathcal{P}$ is nonempty, let $\forall \tilde{P} \in \mathcal{P}$. Then, adding the following equality:

$$\sum_{k=0}^{+\infty} \mathrm{E}[x_{k+1}' \tilde{P} x_{k+1} - x_k' \tilde{P} x_k] = -\mathrm{E}[x_0' \tilde{P} x_0]$$

to the cost function

$$J(x_0, u) = \sum_{k=0}^{+\infty} \mathrm{E}[x_k' Q x_k + 2x_k' L u_k + u_k' R u_k]$$

and applying (1), we can see that for any $(x_0, u_k) \in \mathbb{R}^n \times U_{\mathrm{ad}}(x_0)$,

$$\begin{aligned} J(x_0, u) = &\sum_{k=0}^{+\infty} \mathrm{E}\Big(\begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} F(\tilde{P}) & H'(\tilde{P}) \\ H(\tilde{P}) & G(\tilde{P}) \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\Big) + x_0' \tilde{P} x_0 \\ \geqslant & x_0' \tilde{P} x_0. \end{aligned}$$

Since x_0 and u_k are arbitrary, thus

$$V(x_0) \geqslant x_0' \tilde{P} x_0 \tag{14}$$

implies the well-posedness of the LQ problem.

(Necessity) Assume that the LQ problem (1)–(3) is well-posed, Lemma 5 yields that there exists a symmetric matrix P such that $V(x_0) = x_0' P x_0$, $\forall x_0 \in \mathbb{R}^n$.

By the dynamic programming principle, we obtain

$$\begin{aligned} x_0' P x_0 \leqslant &\sum_{k=0}^{h} \mathrm{E}[x_k' Q x_k + 2x_k' L u_k + u_k' R u_k \\ &+ x_{h+1}' P x_{h+1}], \quad \forall h \geqslant 0, \ \forall u \in U_{\mathrm{ad}}(x_0). \end{aligned}$$

Based on (1) and the above inequality, we conclude that

$$\begin{aligned} &\mathrm{E}[x_{h+1}' P x_{h+1}] - x_0' P x_0 \\ &+ \sum_{k=0}^{h} \mathrm{E}[x_k' Q x_k + 2x_k' L u_k + u_k' R u_k] \\ &= \sum_{k=0}^{h} \mathrm{E}[x_{k+1}' P x_{k+1} - x_k' P x_k \\ &\quad + x_k' Q x_k + 2x_k' L u_k + u_k' R u_k] \\ &= \sum_{k=0}^{h} \mathrm{E}\Big(\begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\Big) \geqslant 0. \end{aligned} \tag{15}$$

Setting $u_k = \bar{u}$ and letting $h = 0$, we have

$$\begin{bmatrix} x_0 \\ \bar{u} \end{bmatrix}' \begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{u} \end{bmatrix} \geqslant 0.$$

Because x_0 and $\bar{u}$ are arbitrary, it is easy to see that

$$\begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \geqslant 0.$$

This means that $P \in \mathcal{P}$. Employing (14), it follows that $P \geqslant \tilde{P}$, $\forall \tilde{P} \in \mathcal{P}$. □

The following theorem can be viewed as the converse of Theorem 1, which plays an essential role in this paper.

Theorem 4 The LQ problem (1)–(3) is attainable for any x_0, then the GARE (4) has a stabilizing solution. Moreover, any optimal control is given by (6).

Proof If the LQ problem (1)–(3) is attainable, it is also well-posed. By Theorem 5, there exists a maximal element $P \in \mathcal{P}$ satisfying $V(x_0) = x_0' P x_0$ and

$$\begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \geqslant 0.$$

Using Lemma 4, it is obvious that

$$\begin{cases} F(P) - H'(P)G^{\dagger}(P)H(P) \geqslant 0, \\ H'(P)(I - G(P)G^{\dagger}(P)) = 0, \quad G(P) \geqslant 0. \end{cases} \tag{16}$$

Let u^* be an optimal control for any initial x_0. As same as Theorem 5, the following (17) is derived:

$$\begin{aligned} V(x_0) =& J(x_0, u^*) \\ =& \sum_{k=0}^{+\infty} \mathrm{E}[x_k'(F(P) - H'(P)G^{\dagger}(P)H(P))x_k] \\ &+ \sum_{k=0}^{+\infty} \mathrm{E}[(u_k^* + G^{\dagger}(P)H(P)x_k)'G(P) \\ &\times (u_k^* + G^{\dagger}(P)H(P)x_k)] + x_0'Px_0. \end{aligned} \tag{17}$$

Combining (16) with $V(x_0) = x_0'Px_0$, it can be shown that

$$\sum_{k=0}^{+\infty} \mathrm{E}[x_k'(F(P) - H'(P)G^{\dagger}(P)H(P))x_k] = 0,$$
$$\sum_{k=0}^{+\infty} \mathrm{E}[(u_k^*+G^{\dagger}(P)H(P)x_k)'G(P)(u_k^*+G^{\dagger}(P)H(P)x_k)] = 0.$$

Hence, we have

$$F(P) - H'(P)G^{\dagger}(P)H(P) = 0, \quad k \in \mathbb{N}.$$

Together with (15), we obtain that P is a solution to the GARE (4).

In what follows, we show that any optimal control u_k^* can be given by (6). From (16), it yields that

$$G(P)^{\frac{1}{2}}[u_k^* + G^{\dagger}(P)H(P)x_k] = 0,$$

which implies

$$G(P)u_k^* + G(P)G^{+}(P)H(P)x_k = 0.$$

By Lemma 3 with

$$L = G(P), \quad M = I, \quad N = -G(P)G^{+}(P)H(P)x_k,$$

we solve the above equation and have the following solution $u_k^* = -G^{+}(P)H(P)x_k + Y - G^{+}(P)G(P)Y$. Thus u_k^* can be represented by (6) with $Z_k = Y$ and $Y_k = 0$. On the other hand, from Definition 5, it follows that P is a stabilizing solution to the GARE (4). □

5 Characterizing LQ problem via SDP

In this section, we develop an approach based on semidefinite programming (SDP). We show that the stabilizing of the feedback control can be examined via solving a SDP problem. We establish several relations among the GARE, the SDP and the optimality of the LQ problem.

First, we introduce the following definition.

Definition 7 [28] Let a vector $c = (c_1, \dots, c_m)' \in \mathbb{R}^m$ and matrices $F_0, F_1, \dots, F_m \in \mathcal{S}^n$ be given. The following optimization problem:

$$\begin{aligned} &\min\ c'x \\ &\text{s.t. } F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geqslant 0 \end{aligned} \tag{18}$$

is called a semidefinite programming (SDP). The SDP is feasible if there exists an x such that $F(x) \geqslant 0$.

Consider the following SDP problem:

$$\begin{aligned} &\min\ -\mathrm{Tr}(P) \\ &\text{s.t. } \begin{bmatrix} F(P) & H'(P) \\ H(P) & G(P) \end{bmatrix} \geqslant 0. \end{aligned} \tag{19}$$

The following assertions provide connections among the well-posedness of the LQ problem, the feasibility of the SDP and the solvability of the GARE.

Theorem 5 The SDP (19) is feasible if and only if the LQ problem (1)–(3) is well-posed.

Proof By Theorem 5, we easily get the desired result. □

Theorem 6 If the SDP (19) is feasible, then it has a unique optimal solution P^* satisfying $V(x_0) = x_0'P^*x_0$, $\forall x_0 \in \mathbb{R}^n$.

Proof By Theorem 5, it follows that the SDP (19) has a maximal solution P such that $V(x_0) = x_0'Px_0$, $\forall x_0 \in \mathbb{R}^n$, which is also an optimal solution to (19). Let P^* be arbitrary optimal solution to (19). It is evident that $\mathrm{Tr}(P - P^*) = 0$. Moreover, the maximality of P results in $P - P^* \geqslant 0$. Therefore, $P - P^* = 0$, i.e., $P = P^*$. □

Theorem 7 If the LQ problem (1)–(3) is attainable, then the unique optimal solution to (19) is the stabilizing solution to the GARE (4).

Proof From Theorems 2 and 6, the assertion is immediately obtained. □

Similar to the discussion in [29], we give a computational approach to determine the lower bound of the control weighting matrix R for the LQ problem to be well-posed.

Definition 8 Let Q and L be given. The smallest

$r^* \in \mathbb{R}$ is called the well-posed margin if the LQ problem (1)–(3) is well-posed for any $R \geqslant r^* I$.

Remark 2 By the above definition, if the smallest eigenvalue $\lambda_{\min}(R)$ of R satisfies $\lambda_{\min}(R) \geqslant r^*$, then the LQ problem is well-posed. Otherwise, the LQ problem is ill-posed. In particular, if $r^* = 0$, the LQ problem is ill-posed for any indefinite R.

The following result shows that the well-posedness margin r^* can be obtained numerically.

Theorem 8 The well-posedness margin r^* can be derived by solving the following SDP problem:

$$\begin{aligned} &\min \ r \\ &\text{s.t.} \begin{bmatrix} F(P) & H'(P) \\ H(P) & B'PB + D'PD + rI \end{bmatrix} \geqslant 0. \end{aligned} \tag{20}$$

Proof From Theorem 7, Theorem 8 is easily proved. □

Example 1 System (1)–(3) is specified by the following matrices:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ Q = \begin{bmatrix} -21 & 0 \\ 0 & 6/5 \end{bmatrix}, \ L = 0, \ R = -1.$$

Solving the corresponding GARE (4) yields $G = 5$, $H = [0\ 1]$, $P = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$. Finally, we can calculate the optimal control sequence $u_k = -[0\ \ 1/5]x_k$ and the optimal cost value $V(x_0) = x_0' P x_0 = 8$. By Theorem 8, the well-posedness margin $r^* = 6$.

Remark 3 In stochastic systems, there is no similar definition to transfer function as in linear system theory. Therefore, we cannot define an optimal feedback control to be proper or improper as in [30]. In this paper, the admissible control set $U_{\mathrm{ad}}(x_0)$ is not limited to a static state feedback form, which may include other forms such as open-loop controls. However, from Theorem 4, if the LQ problem is attainable, all optimal controls take the form of (6), which completely characterizes all the optimal control forms.

6 Conclusions

This paper has investigated the infinite horizon indefinite LQ control for discrete-time stochastic systems with state and control dependent noise. A GARE has been introduced. The well-posedness of the LQ problem is equivalent to the feasibility of an LMI. Moreover, the attainability of the LQ problem is equivalent to the existence of a stabilizing solution to the GARE. All the optimal controls are obtained in terms of the solution to the GARE. To some extent, the results of this paper may be viewed as a discrete-time version of [29].

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