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Robust state estimation for uncertain linear systems with deterministic input signals

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Abstract:

In this paper, we investigate state estimations of a dynamical system in which not only process and measurement noise, but also parameter uncertainties and deterministic input signals are involved. The sensitivity penalization based robust state estimation is extended to uncertain linear systems with deterministic input signals and parametric uncertainties which may nonlinearly affect a state-space plant model. The form of the derived robust estimator is similar to that of the well-known Kalman filter with a comparable computational complexity. Under a few weak assumptions, it is proved that though the derived state estimator is biased, the bound of estimation errors is finite and the covariance matrix of estimation errors is bounded. Numerical simulations show that the obtained robust filter has relatively nice estimation performances.

Keywords: Robust estimation; Deterministic input; Regularized least-squares

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1 Introduction

State estimation plays an important role in signal processing and control system design. It is known that the Kalman filter is the optimal estimator under the criterion of mean-squares and widely applied in numerous fields such as target tracking, global positioning systems, hydrological modelling, atmospheric observations, time-series analyses in systems biology and econometrics, automated drug delivery, and so on [1–3]. As modelling errors are generally unavoidable, robust state estimators such as H_2/H_∞ filtering, set-valued estimation, and guaranteed-cost designs, which do not vary appreciably when actual plant parameters deviate from their nominal ones in a reasonable way, have been developed, see [1, 4–8] and the references therein. Particularly

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worth mentioning is that a regularized least-squares (RLS) based framework is suggested in [1] for robust filter designs, whose attractive characteristic is that the filter shares the same form of the well known Kalman filter with corrected parameters. However, in this framework the plant parameters are required to depend linearly on an uncertainty block, which may be a restrictive condition. The other possible limitation is that the robust estimator needs to optimize a cost function at every estimation step, whose unique minimum has no analytic expression.

There is another paradigm in robust filter designs which is based on sensitivity penalization of estimation errors to parameter variations. In [9], it is employed for single-input single-output systems in the frequency domain with transfer function representation and spectral factorization. In [2, 10], it is adopted for multi-input multi-output time varying dynamical systems under state-space framework and the plant parameters are affected by modelling errors in a relatively arbitrary way. Based on the relationship between Kalman filter and regularized least-squares, as well as sensitivity penalization on estimation errors to parameter variations, an analytic expression of the robust state estimator has been derived [2, 10]. The estimator can be recursively implemented and has a comparable computational complexity with the widely applied Kalman filter.

The works aforementioned normally assume that the system is driven only by noise processes without deterministic input signals. It is true that the existence of a known deterministic input does not affect the estimation errors if the signal process does not involve parameter uncertainties, on the contrary it is not valid because the superposition principle is no longer established for the existence of parameter uncertainties [11]. On this occasion, it is of significance to take robust state estimation with deterministic input signals into account and analyze the asymptotic properties of the derived estimator. The H_∞ filtering approach [7] was extended to provide a guaranteed H_∞ bound for estimation errors in the presence of both parameter uncertainties and known input signals for continuous time varying uncertain systems [12] and discrete time varying uncertain systems [11], respectively. In [13] a robust H_∞ state estimator is investigated for a class of uncertain discrete time piecewise affine systems with partitioned state space based on which the filter implementation may not be synchronized with state trajectory transitions.

In this paper we generalize the robust state estimator sensitivity penalization based [2, 10] to cope with the cases where deterministic input signals are considered. An analytic expression has been derived for the robust estimator, which can be recursively implemented and has a similar form and a comparable computational complexity with the Kalman filter. In [2, 14] it is proved that under some assumptions, as well as conditions like detectability and stabilizability, the robust state estimator sensitivity penalization based is asymptotically unbiased when there are no deterministic input signals in process. Our main contribution lies in the fact that the estimation errors are proved to be bounded and have a bounded covariance matrix though the robust state filter is biased owing to the existence of modelling errors when there exist deterministic input signals. Some numerical simulations show that this robust estimator has relatively nice estimation performances and can be widely applied.

The rest of this paper is organized as follows. In Section 2, a state-space plant model is given and the robust state estimator sensitivity penalization based is derived. Some important properties such as convergence and boundedness are discussed in Section 3. Numerical simulation results are reported in Section 4. Finally, Section 5 concludes this paper. Two appendices are included to give a derivation of the recursive estimation procedure and a proof of the theoretical result.

Notation Given a column vector x and a positive-definite matrix W, $\|x\|^2$ and $\|x\|_W^2$ are defined to denote the Euclidean norm and its weighted version, namely, $x^{\mathrm{T}}x$ and $x^{\mathrm{T}}Wx$, respectively.

2 Plant dynamics description and robust state estimator design

Consider the following uncertain linear system,

$$\begin{cases} x_{i+1} = A_i(\varepsilon_i)x_i + B_{1i}(\varepsilon_i)u_i + B_{2i}(\varepsilon_i)w_i, \\ y_i = C_i(\varepsilon_i)x_i + v_i, \quad i \geqslant 0, \end{cases} \tag{1}$$

where x is the state, w is the process noise, u is a deterministic input signal, y is the measurement, and v is the measurement noise. x_0, w_i and v_i are uncorrelated random vectors with $\mathrm{E}(x_0) = 0, \mathrm{E}(w_i) = 0, \mathrm{E}(v_i) = 0$ and $\mathrm{E}((\mathrm{col}(x_0, w_i, v_i))(*)^{\mathrm{T}}) = \mathrm{diag}\{\Pi_0, Q_i\delta_{ij}, R_i\delta_{ij}\}$, in which Π_0, Q_i and R_i are known positive definite matrices and δ_{ij} represents the Kronecker delta function. Moreover, ε_i denotes parametric modelling errors at the ith sampled instant which is composed of L real valued scalar uncertainties $\varepsilon_{i,k}, k = 1, \ldots, L$. It is assumed that the L

uncertainties are independent of each other and all the entries of matrices $A_i(\varepsilon_i), B_{1i}(\varepsilon_i), B_{2i}(\varepsilon_i)$ and $C_i(\varepsilon_i)$ are differentiable functions of ε_i.

Compared system (1) with the one in [2, 10], the known deterministic input is considered in this paper. It is obvious that system (1) collapses to the one in [2,10] when there is no deterministic input, therefore, system (1) can be regarded as a generalization of the one in [2,10].

From [1,3], we know that the Kalman filter admits a deterministic interpretation as the solution to a regularized least-squares problem, as follows:

$$\hat{x}_{i+1|i+1} = A_i(0)\hat{x}_{i|i+1} + B_{2i}(0)\hat{w}_{i|i+1} + B_{1i}(0)u_i, \quad (2)$$

$$\begin{bmatrix} \hat{x}_{i|i+1} \\ \hat{w}_{i|i+1} \end{bmatrix} = \arg\min_{x_i, w_i} [\|x_i - \hat{x}_{i|i}\|^2_{P^{-1}_{i|i}} + \|w_i\|^2_{Q^{-1}_i} + \|y_{i+1} - C_{i+1}(0)x_{i+1}\|^2_{R^{-1}_{i+1}}],$$

where $\hat{x}_{i|l}$ stands for the optimal estimator of x_i based on measurements $y_j|^l_{j=0}$, and $P_{i|l}$ the corresponding estimation errors covariance matrix. The cost function of the regularized least-squares problem is the regularized squares residual norm. The interpretation means that given an initial estimate $\hat{x}_{i|i}$ for x_i, one seeks to meliorate it by incorporating the additional information provided by the new measurement y_{i+1} and deterministic input u_i.

We improve the cost function of the regularized least-squares problem considering the estimation performances appreciable deterioration because of model uncertainties which are generally unavoidable. For notational simplicity, define matrices respectively as follows:

$$\begin{aligned}
&\Psi_i = R^{-1}_{i+1},\\
&H_i(\varepsilon_i, \varepsilon_{i+1}) = C_{i+1}(\varepsilon_{i+1})[A_i(\varepsilon_i)\ \ B_{2i}(\varepsilon_i)],\\
&\beta_i(\varepsilon_i, \varepsilon_{i+1}) = y_{i+1} - C_{i+1}(\varepsilon_{i+1})(B_{1i}(\varepsilon_i)u_i + A_i(\varepsilon_i)\hat{x}_{i|i}),\\
&\Phi_i = \mathrm{diag}\{P^{-1}_{i|i}, Q^{-1}_i\},\ \ \alpha_i = \mathrm{col}(x_i - \hat{x}_{i|i}, w_i).
\end{aligned}$$

Then the cost function can be rewritten as $\|\alpha_i\|^2_{\Phi_i} + \|H_i(0,0)\alpha_i - \beta_i(0,0)\|^2_{\Psi_i}$ and an analytic solution to this regularized least-squares problem can be obtained. In our improvement, denote $y_{i+1} - C_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)\hat{x}_{i|i} - C_{i+1}(\varepsilon_{i+1})[A_i(\varepsilon_i)\ \ B_{2i}(\varepsilon_i)]\alpha_i$ as $e_i(\varepsilon_i, \varepsilon_{i+1})$ which is generally called innovation process, the new cost function of the RLS at every instant is suggested to be (3) to reduce the sensitivity of estimation performances to modelling errors.

$$\begin{aligned}
J(\alpha_i) =& \gamma_i[\|\alpha_i\|^2_{\Phi_i} + \|H_i(0,0)\alpha_i - \beta_i(0,0)\|^2_{\Psi_i}]\\
&+ (1-\gamma_i)\sum_{k=1}^{L}(\|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}}\|^2\\
&+ \|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}}\|^2)|_{\varepsilon_i=0, \varepsilon_{i+1}=0}.
\end{aligned} \quad (3)$$

From the cost function we can conclude that the deviations of the innovation process from $y_{i+1} - C_{i+1}(0)[A_i(0)\hat{x}_{i|i+1} + B_{1i}(0)u_i + B_{2i}(0)\hat{w}_{i|i+1}]$ reflect contributions of modelling errors to prediction errors based on y_{i+1}, and the design parameter γ_i takes account in both the importance of nominal estimation performances and that of estimation performance degradation due to modelling errors. Generally, the design parameter γ_i has an empirical value [2, 10] and can be adjusted according to the relative magnitude of modelling errors in practical application. The bigger the amplitude of modelling errors is, the smaller the parameter is. This means that the estimation performance degradation owing to modelling errors plays a more important role. It is consistent with physical intuitions. When there are no modelling errors and $\gamma_i = 1$, the state estimator through minimizing the cost function (3) collapses to the standard Kalman filter.

Define matrices S_i, T_{1i} and T_{2i} respectively as follows:

$$\begin{aligned}
S_i &= [\,S^{\mathrm{T}}_{i,1}(0,0)\ \ S^{\mathrm{T}}_{i,2}(0,0)\ \ \cdots\ \ S^{\mathrm{T}}_{i,L}(0,0)\,]^{\mathrm{T}},\\
T_{1i} &= [\,T^{\mathrm{T}}_{1i,1}(0,0)\ \ T^{\mathrm{T}}_{1i,2}(0,0)\ \ \cdots\ \ T^{\mathrm{T}}_{1i,L}(0,0)\,]^{\mathrm{T}},\\
T_{2i} &= [\,T^{\mathrm{T}}_{2i,1}(0,0)\ \ T^{\mathrm{T}}_{2i,2}(0,0)\ \ \cdots\ \ T^{\mathrm{T}}_{2i,L}(0,0)\,]^{\mathrm{T}},
\end{aligned}$$

where

$$\begin{aligned}
S_{i,k}(\varepsilon_i, \varepsilon_{i+1}) &= \begin{bmatrix} \dfrac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} A_i(\varepsilon_i) \\ C_{i+1}(\varepsilon_{i+1}) \dfrac{\partial A_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix},\\
T_{1i,k}(\varepsilon_i, \varepsilon_{i+1}) &= \begin{bmatrix} \dfrac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} B_{1i}(\varepsilon_i) \\ C_{i+1}(\varepsilon_{i+1}) \dfrac{\partial B_{1i}(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix},\\
T_{2i,k}(\varepsilon_i, \varepsilon_{i+1}) &= \begin{bmatrix} \dfrac{\partial C_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} B_{2i}(\varepsilon_i) \\ C_{i+1}(\varepsilon_{i+1}) \dfrac{\partial B_{2i}(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix},\\
k &= 1, 2, \dots, L.
\end{aligned}$$

Then, we can obtain

$$\begin{aligned}
&\sum_{k=1}^{L}(\|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}}\|^2 + \|\frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}}\|^2)|_{\varepsilon_i=0, \varepsilon_{i+1}=0}\\
&= ([\,S_i\ \ T_{2i}\,]\alpha_i + S_i\hat{x}_{i|i} + T_{1i}u_i)^{\mathrm{T}}([\,S_i\ \ T_{2i}\,]\alpha_i + S_i\hat{x}_{i|i} + T_{1i}u_i).
\end{aligned}$$

It is easy to know that the cost function $J(\alpha_i)$ is a strictly convex function when $0 < \gamma_i \leqslant 1$ from matrices Φ_i and Ψ_i, which has a global unique minimum expressed as α_{iopt} at $\frac{\partial J(\alpha_i)}{\partial \alpha_i} = 0$. It is determined by (4) as follows:

$$(\Phi_i + H_i^{\mathrm{T}}(0)\Psi_i H_i(0) + \frac{1-\gamma_i}{\gamma_i}[S_i \ \ T_{2i}]^{\mathrm{T}}[S_i \ \ T_{2i}])\alpha_{\text{iopt}} = H_i^{\mathrm{T}}(0)\Psi_i\beta_i(0) - \frac{1-\gamma_i}{\gamma_i}[S_i \ \ T_{2i}]^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i). \quad (4)$$

According to the above analysis we can provide the following recursive procedure to compute the estimate of the plant state when there exist a deterministic input signal and parameter uncertainties. The derivative details are provided in Appendix A. Denote $\lambda_i = (1-\gamma_i)/\gamma_i$.

1) Initialization. Designate $P_{0|0}$ and $\hat{x}_{0|0}$ as

$$P_{0|0} = (\hat{\Pi}_0^{-1} + C_0^{\mathrm{T}}(0)R_0^{-1}C_0(0))^{-1},$$
$$\hat{x}_{0|0} = P_{0|0}C_0^{\mathrm{T}}(0)R_0^{-1}y_0,$$

respectively, in which

$$\hat{\Pi}_0 = (\Pi_0^{-1} + \lambda_0 \sum_{k=1}^{L} (\frac{\partial C_0^{\mathrm{T}}(\varepsilon_0)}{\partial \varepsilon_{0,k}})(\frac{\partial C_0(\varepsilon_0)}{\partial \varepsilon_{0,k}})|_{\varepsilon_0=0})^{-1}.$$

2) Parameter modification. Define matrices $\hat{T}_{2i}, \hat{A}_i(0), \hat{B}_{1i}(0), \hat{B}_{2i}(0), \hat{P}_{i|i}$ and $\hat{Q}_i$ respectively as follows:

$$\hat{T}_{2i} = T_{2i} - \lambda_i S_i \hat{P}_{i|i} S_i^{\mathrm{T}} T_{2i},$$
$$\hat{A}_i(0) = (A_i(0) - \lambda_i \hat{B}_{2i}(0)\hat{Q}_i T_{2i}^{\mathrm{T}} S_i)(I - \lambda_i \hat{P}_{i|i} S_i^{\mathrm{T}} S_i),$$
$$\hat{B}_{2i}(0) = B_{2i}(0) - \lambda_i A_i(0)\hat{P}_{i|i} S_i^{\mathrm{T}} T_{2i},$$
$$\hat{B}_{1i}(0) = B_{1i}(0) - \lambda_i (A_i(0)\hat{P}_{i|i} S_i^{\mathrm{T}} + \hat{B}_{2i}(0)\hat{Q}_i \hat{T}_{2i}^{\mathrm{T}})T_{1i},$$
$$\hat{P}_{i|i} = (P_{i|i}^{-1} + \lambda_i S_i^{\mathrm{T}} S_i)^{-1},$$
$$\hat{Q}_i = (Q_i^{-1} + \lambda_i T_{2i}^{\mathrm{T}}(I + \lambda_i S_i P_{i|i} S_i^{\mathrm{T}})^{-1} T_{2i})^{-1}.$$

3) State estimate updating. Calculate $\hat{x}_{i+1|i+1}$ and $P_{i+1|i+1}$ respectively as

$$P_{i+1|i} = A_i(0)\hat{P}_{i|i}A_i^{\mathrm{T}}(0) + \hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^{\mathrm{T}}(0),$$
$$R_{\mathrm{e},i+1} = R_{i+1} + C_{i+1}(0)P_{i+1|i}C_{i+1}^{\mathrm{T}}(0),$$
$$P_{i+1|i+1} = P_{i+1|i} - P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{\mathrm{e},i+1}^{-1}C_{i+1}(0)P_{i+1|i},$$
$$\hat{x}_{i+1|i+1} = \hat{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{x}_{i|i} + P_{i+1|i+1}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}[y_{i+1} - C_{i+1}(0)(\hat{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{x}_{i|i})]. \quad (5)$$

Based on the above explanation, the form of the estimation procedure is consistent with the time and measurement update form of the robust estimator derived in [1] and has a similar form with the one in [2,10] increased by some terms relative to the deterministic input u_i. When $u_i = 0$, the derived state estimation procedure collapses to the robust filter in [2,10], which means that it is a generalization of the one in [2,10].

3 Some properties of the estimator

In this section, some important asymptotic properties of the derived state estimator are investigated. Suppose $\varepsilon_{i,k}$ is normalized in magnitude to be contractive and the set $\mathcal{E}$ is composed of these modelling errors. That is, $\mathcal{E} = \{\varepsilon \| \varepsilon_{i,k}| \leqslant 1, k = 1,\dots,L\}$. Moreover, we adopt two assumptions for the asymptotic behaviours analysis of the robust state estimator.

A1) $A_i(0), B_{1i}(0), B_{2i}(0), C_i(0), R_i, Q_i, S_i, T_{1i}, T_{2i}$ and γ_i are time invariant.

A2) The uncertain linear system of (1) is exponentially stable in the sense of Lyapunov. Moreover, matrices $A_i(0), B_{1i}(0), B_{2i}(0), C_i(0), R_i, Q_i$ and Π_0 are bounded for $i > 0$ and $\varepsilon_i \in \mathcal{E}$.

Equation (5) can be rewritten as follows:

$$\hat{x}_{i+1|i+1} = A_{\mathrm{f}i}\hat{x}_{i|i} + [B_{\mathrm{f}i} \ \ P_{i+1|i+1}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}]\begin{bmatrix} u_i \\ y_{i+1} \end{bmatrix},$$

where

$$A_{\mathrm{f}i} = [I - P_{i+1|i+1}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)]\hat{A}_i(0),$$
$$B_{\mathrm{f}i} = [I - P_{i+1|i+1}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)]\hat{B}_{1i}(0).$$

This expression is similar to equation (6) in [14] and the difference is that "the input" of (6) is y_{i+1} instead of $[u_i; y_{i+1}]$, therefore Theorem 1 in [14] can be generalized directly to system (5) as follows when the convergence of system (1) is considered.

Theorem 1 Assume that condition A1) is satisfied, $(A_i(0), \bar{C}_i)$ is detectable and $(A_i(0) - \lambda_i B_i(0)Q_i T_{2i}^{\mathrm{T}}(I + \lambda_i T_{2i}Q_i T_{2i}^{\mathrm{T}})^{-1}S_i, B_i(0)Q_i^{\frac{1}{2}}(I + \lambda_i Q_i^{\frac{1}{2}}T_{2i}^{\mathrm{T}}T_{2i}Q_i^{\frac{1}{2}})^{-\frac{1}{2}})$ is stabilizable. Then, for arbitrary $\Pi_0 > 0$ and $0 < \gamma_i \leqslant 1$, $P_{i|i-1}$ converges exponentially to a unique positive semidefinite matrix P, while $A_{\mathrm{p}i}$ converges to a constant stable matrix A_{p}. Here,

$$A_{\mathrm{p}i} = A_i(0) - (A_i(0)P_{i|i-1}\bar{C}_i^{\mathrm{T}} + \bar{B}_i J_i)(W_i + \bar{C}_i P_{i|i-1}\bar{C}_i^{\mathrm{T}})^{-1}\bar{C}_i,$$
$$A_{\mathrm{p}} = A_i(0) - (A_i(0)P\bar{C}_i^{\mathrm{T}} + \bar{B}_i J_i)(W_i + \bar{C}_i P\bar{C}_i^{\mathrm{T}})^{-1}\bar{C}_i,$$
$$\bar{B}_i = B_i(0)Q_i^{\frac{1}{2}}, \quad J_i = [0 \ \ \sqrt{\lambda_i}Q_i^{\frac{1}{2}}T_{2i}^{\mathrm{T}}],$$
$$W_i = \begin{bmatrix} I & 0 \\ 0 & I + \lambda_i T_{2i}Q_i T_{2i}^{\mathrm{T}} \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} R_i^{-\frac{1}{2}}C_i(0) \\ \sqrt{\lambda_i}S_i \end{bmatrix}.$$

We know that the convergence of $P_{i|i-1}$ is equivalent to that of $P_{i|i}$ from the relation between $P_{i|i-1}$ and $P_{i|i}$. Therefore, the derived robust state estimator converges to a time-invariant stable system when the conditions of Theorem 1 are satisfied.

We then consider the boundedness and biasness of estimation errors for this robust state filter. For simple denotations, define $\bar{x}_i, \hat{\bar{x}}_{i|i}$ and $\tilde{\bar{x}}_{i|i}$ respectively as $\bar{x}_i = [I + \Omega_i(0)]x_i, \hat{\bar{x}}_{i|i} = [I + \Omega_i(0)]\hat{x}_{i|i}$ and $\tilde{\bar{x}}_{i|i} = \bar{x}_i - \hat{\bar{x}}_{i|i}$, where $\Omega_i(\varepsilon_i) = P_{i|i-1}C_i^{\mathrm{T}}(0)R_i^{-1}C_i(\varepsilon_i)$. It is obvious that $\tilde{\bar{x}}_{i|i} = [I+\Omega_i(0)](x_i - \hat{x}_{i|i})$. Then from equation (5) we can directly prove that

$$\begin{bmatrix}\tilde{\bar{x}}_{i+1|i+1}\\ \hat{\bar{x}}_{i+1|i+1}\end{bmatrix} = \tilde{A}_i(\varepsilon_i)\begin{bmatrix}\tilde{\bar{x}}_{i|i}\\ \hat{\bar{x}}_{i|i}\end{bmatrix} + \tilde{B}_{2i}(\varepsilon_i)\begin{bmatrix}w_i\\ v_{i+1}\end{bmatrix} + \tilde{B}_{1i}(\varepsilon_i)u_i, \quad (6)$$

where matrices $\tilde{A}_i(\varepsilon_i, \varepsilon_{i+1}), \tilde{B}_{2i}(\varepsilon_i, \varepsilon_{i+1})$ and $\tilde{B}_{1i}(\varepsilon_i, \varepsilon_{i+1})$ are shown as

$$\tilde{A}_i(\varepsilon_i,\varepsilon_{i+1}) = \left[\begin{array}{c|c} \begin{matrix}(I+\Omega_{i+1}(0))A_i(\varepsilon_i)(I+\Omega_i(0))^{-1}\\ -\Omega_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)(I+\Omega_i(0))^{-1}\end{matrix} & \begin{matrix}(I+\Omega_{i+1}(0))A_i(\varepsilon_i)(I+\Omega_i(0))^{-1}\\ -\Omega_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)(I+\Omega_i(0))^{-1}\\ -\hat{A}_i(0)(I+\Omega_i(0))^{-1}\end{matrix} \\ \hline \Omega_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)(I+\Omega_i(0))^{-1} & \begin{matrix}\Omega_{i+1}(\varepsilon_{i+1})A_i(\varepsilon_i)(I+\Omega_i(0))^{-1}\\ +\hat{A}_i(0)(I+\Omega_i(0))^{-1}\end{matrix}\end{array}\right],$$

$$\tilde{B}_{2i}(\varepsilon_i,\varepsilon_{i+1}) = \begin{bmatrix}((I+\Omega_{i+1}(0))-\Omega_{i+1}(\varepsilon_{i+1}))B_{2i}(\varepsilon_i) & -P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}\\ \Omega_{i+1}(\varepsilon_{i+1})B_{2i}(\varepsilon_i) & P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}\end{bmatrix},$$

$$\tilde{B}_{1i}(\varepsilon_i,\varepsilon_{i+1}) = \begin{bmatrix}((I+\Omega_{i+1}(0))-\Omega_{i+1}(\varepsilon_{i+1}))B_{1i}(\varepsilon_i) - B_{1i}(0)\\ \Omega_{i+1}(\varepsilon_{i+1})B_{1i}(\varepsilon_i) + \hat{B}_{1i}(0)\end{bmatrix}.$$

From equation (6), we have with the whiteness of w_i and v_i, as well as the assumption of unrelated w_i and v_i, that

$$\begin{aligned}
&\|\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{i+1|i+1}\\ \hat{\bar{x}}_{i+1|i+1}\end{bmatrix}\}\|\\
&= \|\mathrm{E}\{\tilde{A}_i(\varepsilon_i)\begin{bmatrix}\tilde{\bar{x}}_{i|i}\\ \hat{\bar{x}}_{i|i}\end{bmatrix} + \tilde{B}_{2i}(\varepsilon_i)\begin{bmatrix}w_i\\ v_{i+1}\end{bmatrix} + \tilde{B}_{1i}(\varepsilon_i)u_i\}\|\\
&= \|\mathrm{E}\{\tilde{A}_i(\varepsilon_i)\begin{bmatrix}\tilde{\bar{x}}_{i|i}\\ \hat{\bar{x}}_{i|i}\end{bmatrix}\} + \tilde{B}_{1i}(\varepsilon_i)u_i\|\\
&= \|[\prod_{l=0}^{i}\tilde{A}_l(\varepsilon_l)]\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{0|0}\\ \hat{\bar{x}}_{0|0}\end{bmatrix}\} + [\prod_{l=1}^{i}\tilde{A}_l(\varepsilon_l)]u_0\\
&\quad + [\prod_{l=2}^{i}\tilde{A}_l(\varepsilon_l)]\tilde{B}_{11}(\varepsilon_i)u_1 + \dots\\
&\quad + [\prod_{l=i}^{i}\tilde{A}_l(\varepsilon_l)]\tilde{B}_{1(i-1)}(\varepsilon_i)u_{i-1} + \tilde{B}_{1i}(\varepsilon_i)u_i\|\\
&\leqslant \|[\prod_{l=0}^{i}\tilde{A}_l(\varepsilon_l)]\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{0|0}\\ \hat{\bar{x}}_{0|0}\end{bmatrix}\}\| + \|[\prod_{l=1}^{i}\tilde{A}_l(\varepsilon_l)]u_0\|\\
&\quad + \|[\prod_{l=2}^{i}\tilde{A}_l(\varepsilon_l)]\tilde{B}_{11}(\varepsilon_i)u_1\| + \dots\\
&\quad + \|[\prod_{l=i}^{i}\tilde{A}_l(\varepsilon_l)]\tilde{B}_{1(i-1)}(\varepsilon_i)u_{i-1}\| + \|\tilde{B}_{1i}(\varepsilon_i)u_i\|\\
&\leqslant \|[\prod_{l=0}^{i}\tilde{A}_l(\varepsilon_l)]\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{0|0}\\ \hat{\bar{x}}_{0|0}\end{bmatrix}\}\| + \|[\prod_{l=1}^{i}\tilde{A}_l(\varepsilon_l)]u_0\|\\
&\quad + \|[\prod_{l=2}^{i}\tilde{A}_l(\varepsilon_l)]\|\,\|\tilde{B}_{11}(\varepsilon_i)u_1\| + \dots + \|[\prod_{l=i}^{i}\tilde{A}_l(\varepsilon_l)]\|\\
&\quad \times \|\tilde{B}_{1(i-1)}(\varepsilon_i)u_{i-1}\| + \|\tilde{B}_{1i}(\varepsilon_i)u_i\|\\
&\leqslant \|[\prod_{l=0}^{i}\tilde{A}_l(\varepsilon_l)]\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{0|0}\\ \hat{\bar{x}}_{0|0}\end{bmatrix}\}\| + \|[\prod_{l=1}^{i}\tilde{A}_l(\varepsilon_l)]u_0\|\\
&\quad + (\|[\prod_{l=2}^{i}\tilde{A}_l(\varepsilon_l)]\| + \dots + \|[\prod_{l=i}^{i}\tilde{A}_l(\varepsilon_l)]\| + 1)\\
&\quad \times \max\{\|\tilde{B}_{11}(\varepsilon_i)u_1\|, \dots, \|\tilde{B}_{1i}(\varepsilon_i)u_i\|\}. \quad (7)
\end{aligned}$$

It is proved in [14] that there are finite positive constants M_1, M_2, M_3 and $0 \leqslant \rho_3 < 1$ such that $\|[\prod_{l=k_1}^{k_2}\tilde{A}_l(\varepsilon_l)]\| \leqslant \dfrac{(3+\sqrt{5})\sqrt{M_1^2+M_2^2+(k_2-k_1+1)^2M_3^2}}{2}\rho_3^{k_2-k_1}$ if the uncertain linear system is exponentially stable, then

$$\lim_{i\to\infty}\|[\prod_{l=0}^{i}\tilde{A}_l(\varepsilon_l)]\mathrm{E}\{\begin{bmatrix}\tilde{\bar{x}}_{0|0}\\ \hat{\bar{x}}_{0|0}\end{bmatrix}\}\| = 0,$$

$$\lim_{i\to\infty}\|[\prod_{l=1}^{i}\tilde{A}_l(\varepsilon_l)]u_0\| = 0.$$

From $\lim_{n\to\infty}\Big(\dfrac{(3+\sqrt{5})\sqrt{M_1^2+M_2^2+(n+1)^2M_3^2}}{2}\rho_3^n\Big)^{\frac{1}{n}} = \rho_3\lim_{n\to\infty}\Big((n+1)\dfrac{(3+\sqrt{5})}{2}\sqrt{\dfrac{M_1^2+M_2^2}{(n+1)^2}+M_3^2}\Big)^{\frac{1}{n}} = \rho_3 < 1,$

we can conclude that

$$\sum_{n=0}^{+\infty} \frac{(3+\sqrt{5})\sqrt{M_1^2+M_2^2+(n+1)^2M_3^2}}{2}\rho_3^n = N_1 < +\infty,$$

where N_1 is a finite positive constant. It means that the summation of series in the last term of equation (7) is finite. The fact that $A_i(\varepsilon_i)$ and $B_{1i}(\varepsilon_i)$ are bounded leads to that $\tilde{A}_i(\varepsilon_i, \varepsilon_{i+1})$ and $\tilde{B}_{1i}(\varepsilon_i, \varepsilon_{i+1})$ are bounded, which implies that there exists a finite positive constant $N_2 \geqslant 0$ such that $\lim_{i\to\infty} \|\mathrm{E}\{\begin{bmatrix} \tilde{\tilde{x}}_{i+1|i+1} \\ \hat{\hat{x}}_{i+1|i+1} \end{bmatrix}\}\| \leqslant N_2$. Therefore, the estimation errors of the robust filter are bounded. When there are no deterministic input signals, that is, $u_i = 0$, we can obtain that $\lim_{i\to\infty} \|\mathrm{E}\{\begin{bmatrix} \tilde{\tilde{x}}_{i+1|i+1} \\ \hat{\hat{x}}_{i+1|i+1} \end{bmatrix}\}\| = 0$, which means that the robust state estimator sensitivity penalization based is asymptotically unbiased. It is consistent with the result derived in [2, 14].

Based on the relations in (6) and the stability of matrix $A_i(\varepsilon_i)$, as well as the aforementioned derivation, we achieve a condition for the boundedness of estimation errors of the robust filter as follows. Its proof is deferred to Appendix B.

Theorem 2 Suppose that assumptions A1) and A2) are simultaneously satisfied, $(A_i(0), \bar{C}_i)$ is detectable and $(A_i(0) - \lambda_i B_i(0) Q_i T_{2i}^{\mathrm{T}} (I + \lambda_i T_{2i} Q_i T_{2i}^{\mathrm{T}})^{-1} S_i, B_i(0) Q_i^{\frac{1}{2}} (I + \lambda_i Q_i^{\frac{1}{2}} T_{2i}^{\mathrm{T}} T_{2i} Q_i^{\frac{1}{2}})^{-\frac{1}{2}})$ is stabilizable, then, the estimation errors of the robust filter have a finite covariance matrix at every sampled instant.

Remark When $B_{1i}(\varepsilon_i) = B_{2i}(\varepsilon_i)$ in system (1), the deterministic input u_i together with w_i can be regarded as process noises whose expectations are not equal to zeros. Therefore, the derived robust state estimator can be taken as a generalization of the one in [2, 10].

4 Numerical simulations

In this section, we compare the performances of the derived state estimator with those of the Kalman filter based on actual parameters and nominal parameters by some examples. In these simulations, it is assumed that modelling errors are time-invariant, and every uncertainty parameter is contractive, that is, it belongs to the interval $[-1, 1]$. Furthermore, 1000 time-domain input-output data pairs are generated for plant state estimation, in which all the initial states are set to zero, while disturbances w_i and v_i are produced according to normal distributions. The deterministic input signal u_i is fixed or produced according to normal distributions.

500 simulations are performed for each set of numerical experiment settings to calculate the ensemble-average estimation error variance at every sampled instant. The size of the ensemble-average is approximated by the averaged value of the square of the Euclidean distance from the actual plant state to its estimate, that is $\mathrm{E}\|x_i - \hat{x}_{i|i}\|^2 \approx \frac{1}{500}\sum_{j=1}^{500} \|x_i - \hat{x}_{i|i}^{(j)}\|^2$.

This example is improved from the one in [1] and [10], in which it is assumed that,

$$A_i(\varepsilon_i) = \begin{bmatrix} 0.9802 & 0.0196 \\ 0.0000 & 0.9802 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0 \end{bmatrix} \times \varepsilon \times [0.0 \ \ 5.0],$$

$$B_{1i}(\varepsilon_i) = \begin{bmatrix} 1.0000 & 0.0 \\ 0.0 & 1.0000 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0 \end{bmatrix} \times \varepsilon \times [0.0 \ \ 5.0],$$

$$B_{2i}(\varepsilon_i) = \begin{bmatrix} 1.0000 & 0.0 \\ 0.0 & 1.0000 \end{bmatrix}, \quad C_i(\varepsilon_i) = [1.0000 \ \ -1.0000],$$

$$R_i = 1.0000, \quad Q_i = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9605 \end{bmatrix},$$

$$\Pi_0 = \begin{bmatrix} 1.0000 & 0.0 \\ 0.0 & 1.0000 \end{bmatrix}.$$

In the first set of simulations the modelling error ε is fixed to be -0.8508 and the input signal u_i is also fixed, $u_i = [1.0; 0.1]$. Fig. 1 shows the variations of estimation error variances with respect to time samples and the filter design parameter γ. When the design parameter γ takes the empirical value which is approximately 0.8, the difference between the performances of the Kalman filter with actual parameter values and the robust state estimator derived in this paper is only 1 dB and nearly 10 dB performance improvement is obtained compared with the Kalman filter based on nominal parameter values. The same conclusion can be drawn from Fig. 2.

Fig. 2 shows that at the sampled instants $i = 500$ and $i = 1000$, if γ takes any value between 0.0000 and 1.0000, the performance of the derived robust filter is better than that of the Kalman filter based on nominal parameter values and the optimal γ is approximately 0.8300.

In Fig. 3, the input signal u_i is fixed to be $[1.0; 0.1]$ and the modelling error ε is produced randomly and independently in each simulation according to a normal distribution with truncations. The mean and the standard variance of the normal distribution are set respectively to 0.0000 and 1.0000. In case that a generated ε has a magnitude greater than 1, it will be got rid of and reproduced until an ε with magnitude not greater than 1 is obtained. From Fig. 4, we can see that there exists a large interval of γ which leads to a robust estimator with better performance than the Kalman filter based on nominal parameters at the sampled instants $i = 500$ and $i = 1000$.

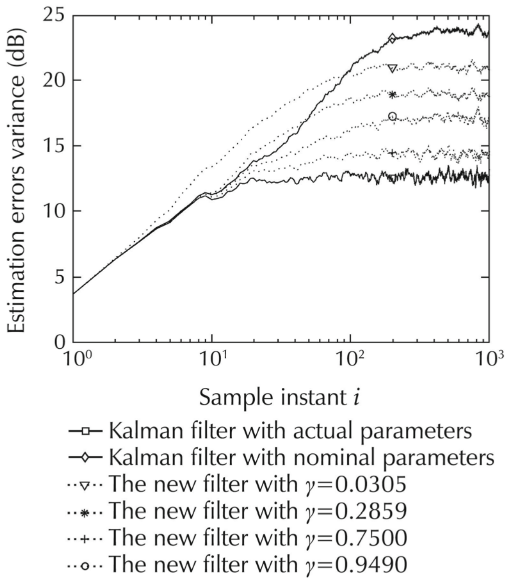

Fig. 1 Estimation error variance with fixed γs.

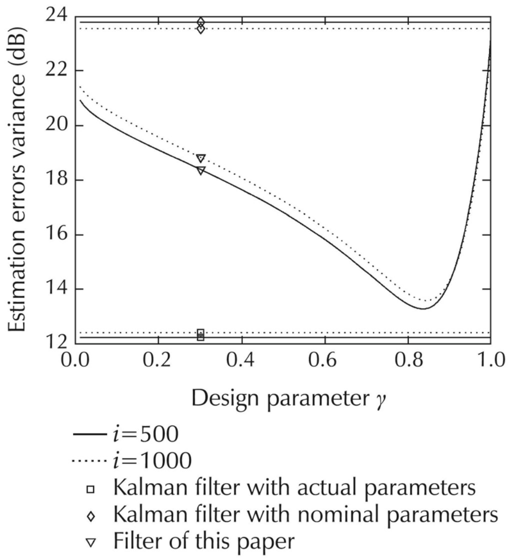

Fig. 2 Estimation error variance at fixed instants.

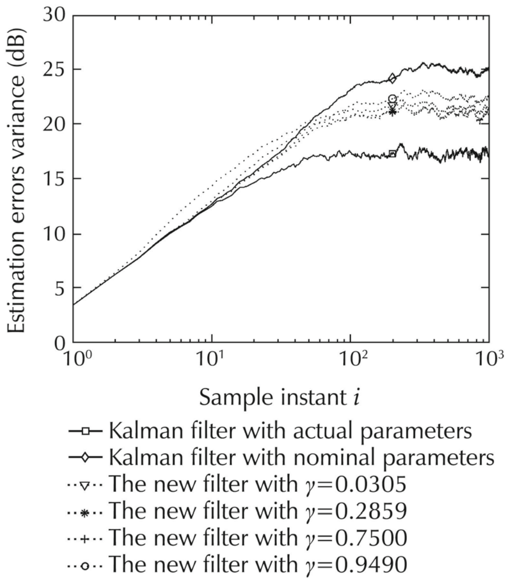

Fig. 3 Estimation error variance with fixed γs.

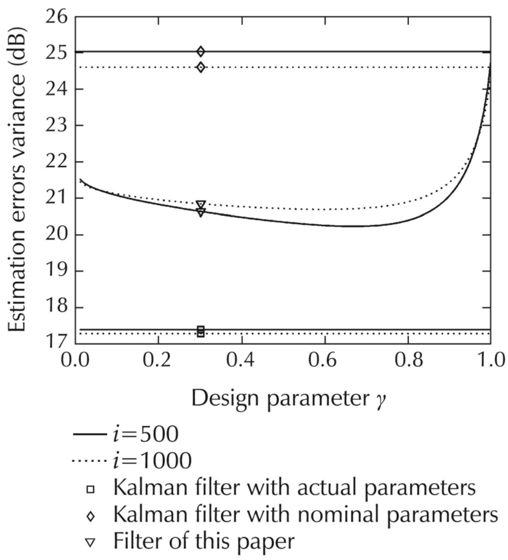

Fig. 4 Estimation error variance at fixed instants.

The deterministic input signal u_i in Fig. 5 is produced randomly and independently according to a normal distribution with truncations and the modelling error ε is produced randomly and independently in each simulation according to a normal distribution with truncations. The mean and the standard variance of the normal distribution ε are set respectively to 0.0000 and 1.0000. The known input vector signal u_i whose entries are independent respectively is generated similarly with mean $\begin{bmatrix} 1.0 \\ 0.1 \end{bmatrix}$ and covariance $\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$ according to normal distributions with truncations. The magnitude difference is not greater than 1. It is observed from Fig. 6 that there also exists a large interval of γ which leads to a robust estimator with better performance than the Kalman filter based on nominal parameter at the sampled instants $i = 500$ and $i = 1000$.

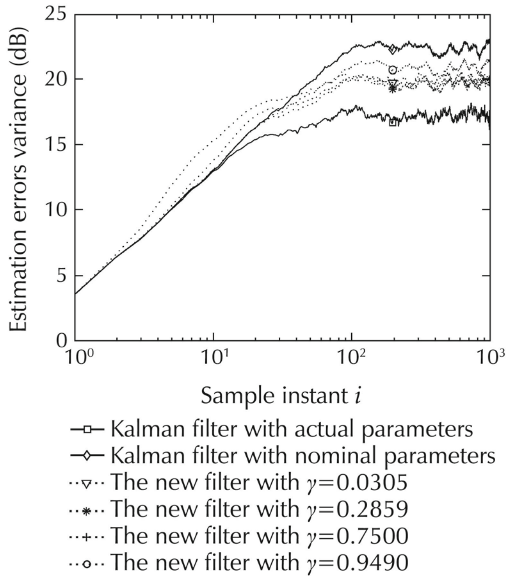

Fig. 5 Estimation error variance with fixed γs.

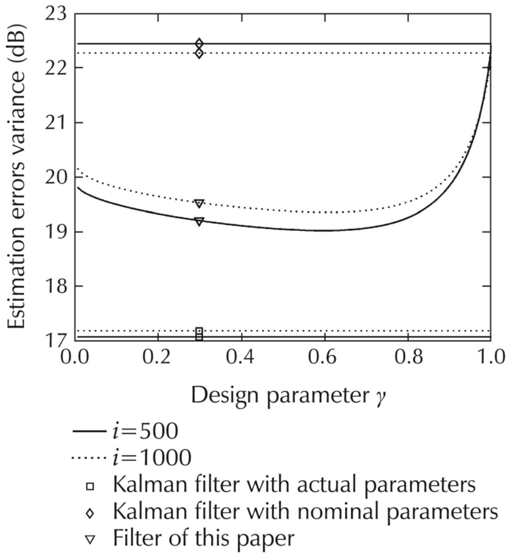

Fig. 6 Estimation error variance at fixed instants.

From Figs. 1–6, it is obvious that the robust state estimator based on the sensitivity penalization of estimation errors to modelling uncertainties can bring about significant robustness improvements in plant state estimator designs. The optimal design parameter γ may lead to the robust estimator with performances close to those of the Kalman filter based on actual plant parameter values. Moreover, there are quite a lot of selections for the parameter γ for that the performances of the robust state estimator are continuous functions of the design parameter. These properties are attractive in actual filter designs and next we aim to find the optimal filter design parameter.

5 Conclusions

This paper investigates a robust state estimator based on modelling errors sensitivity penalization for uncertain linear systems subject to deterministic input signals and norm-bounded parametric uncertainties. The derived state estimator is biased owing to the existence of modelling errors in the input matrix, but the covariance matrix of estimation errors is proved to be bounded. The simulation examples show that this approach significantly improved the estimator's robustness to model uncertainties compared with the designs only based on nominal systems.

It also remains challenging to give an estimate for the interval of desirable penalizing factor γ, as well as an estimate for the size of tolerable modelling errors.

References

[1] A. H. Sayed. A framework for state-space estimation with uncertain models. *IEEE Transactions on Automatic Control*, 2001, 46(7): 998 – 1013.

[2] T. Zhou. Sensitivity penalization based robust state estimation for uncertain linear systems. *IEEE Transactions on Automatic Control*, 2010, 55(4): 1018 – 1024.

[3] T. Kailath, A. H. Sayed, B. Hassibi. *Linear Estimation*. Upper Saddle River: Prentice Hall, 2000.

[4] C. J. Martin, M. Mintz. Robust filtering and prediction for linear systems with uncertain dynamics: a game-theoretic approach. *IEEE Transactions on Automatic Control*, 1983, 28(6): 888 – 896.

[5] A. Garulli, A. Vicino, G. Zappa. Conditional central algorithms for worst case set-membership identificaion and filtering. *IEEE Transactions on Automatic Control*, 2000, 45(1): 14 – 23.

[6] C. E. de Souza, U. Shaked. Robust H_2 filtering for uncertain systems with measurable inputs. *IEEE Transactions on Signal Processing*, 1999, 47(8): 2286 – 2292.

[7] L. Xie, C.E. de Souza, M. Fu. H_∞ estimation for discrete-time linear uncertain systems. *International Journal of Robust Nonlinear Control*, 1991, 1(2): 111 – 123.

[8] P. Bolzern, P. Colaneri, G. De Nicolao. Guaranteed-cost prediction of discrete-time systems: the finite and infinite-horizon case. *IFAC Symposium on Robust Control Design*. Budapest: Pergamon, 1997: 471 – 476.

[9] P. Neveux, E. Blanco, G. Thomas. Robust filtering for linear time invariant continuous systems. *IEEE Transactions on Signal Processing*, 2007, 55(10): 4752 – 4757.

[10] T. Zhou. Robust state estimation using error sensitivity penalizing. *Proceedings of the 47th IEEE Conference on Decision and Control*. Cancun: IEEE, 2008: 2563 – 2568.

[11] Y. S. Hung, F. Yang. Robust H_∞ filtering for discrete time-varying uncertain systems with a known deterministic input. *International Journal of Control*, 2002, 75(15): 1159 – 1169.

[12] C. E. de Souza, U. Shaked, M. Fu. Robust H_∞ filtering for continuous time varying uncertain systems with deterministic input signals. *IEEE Transactions on Signal Processing*, 1995, 43(3): 709 – 719.

[13] J. Qiu, G. Feng, H. Gao. Nonsynchronized state estimation of uncertain discrete-time piecewise affine systems. *Journal of Control Theory and Applications*, 2010, 8(3): 286 – 292.

[14] T. Zhou, H. Liang. On asymptotic behaviors of a sensitivity penalization based robust state estimator. *Systems & Control Letters*, 2011, 60(3): 174 – 180.

Appendix A Derivation of the estimation procedure

To estimate the initial state x_0, the cost function $J(\alpha_0)$ is set as follows, in which $e_0(\varepsilon_0) = y_0 - C_0(\varepsilon_0)x_0$.

$$J(\alpha_0) = \gamma_0[\|x_0\|^2_{\Pi_0^{-1}} + \|y_0 - C_0(0)x_0\|^2_{R_0^{-1}}] + (1-\gamma_0)\sum_{k=1}^{L}(\|\frac{\partial e_0(\varepsilon_0)}{\partial \varepsilon_{0,k}}\|^2)|_{\varepsilon_0=0}.$$

We obtain the following estimate of the initial state

$$\hat{x}_{0|0} = (\hat{\Pi}_0^{-1} + C_0^{\mathrm{T}}(0)R_0^{-1}C_0(0))^{-1}C_0^{\mathrm{T}}(0)R_0^{-1}y_0,$$

where $\hat{\Pi}_0^{-1} = (\Pi_0^{-1} + (1-\gamma_0)/\gamma_0\sum_{k=1}^{L}(\frac{\partial C_0^{\mathrm{T}}(\varepsilon_0)}{\partial \varepsilon_{0,k}})(\frac{\partial C_0(\varepsilon_0)}{\partial \varepsilon_{0,k}})|_{\varepsilon_0=0})^{-1}$. Moreover, we know $\mathrm{Cov}(x_0 - \hat{x}_{0|0}) = (\Pi_0^{-1} + C_0^{\mathrm{T}}(0)R_0^{-1}C_0(0))^{-1}$ when no modelling errors exist.

Denote $\alpha_{i\mathrm{opt}}, C_{i+1}(0)[A_i(0)\ \ \hat{B}_{2i}(0)], T_{2i} - \lambda_i S_i\hat{P}_{i|i}S_i^{\mathrm{T}}T_{2i}$ and $\hat{x}_{i|i+1} + \lambda_i\hat{P}_{i|i}S_i^{\mathrm{T}}T_{2i}\hat{w}_{i|i+1}$ as $\mathrm{col}(\hat{x}_{i|i+1} - \hat{x}_{i|i}, \hat{w}_{i|i+1})$, $\hat{H}_i$, $\hat{T}_{2i}$ and $\tilde{x}_{i|i+1}$. From the following algebraic relation:

$$\begin{aligned}&\begin{bmatrix} P_{i|i}^{-1} & 0 \\ 0 & Q_i^{-1}\end{bmatrix} + \lambda_i[\,S_i\ \ T_{2i}\,]^{\mathrm{T}}[\,S_i\ \ T_{2i}\,] \\ &= \begin{bmatrix} I & 0 \\ \lambda_i T_{2i}^{\mathrm{T}}S_i\hat{P}_{i|i} & I\end{bmatrix}\begin{bmatrix}\hat{P}_{i|i}^{-1} & 0 \\ 0 & \hat{Q}_i^{-1}\end{bmatrix}\begin{bmatrix} I & \lambda_i\hat{P}_{i|i}S_i^{\mathrm{T}}T_{2i} \\ 0 & I\end{bmatrix},\end{aligned} \tag{a1}$$

instituting (a1) to (4), and multiplying $[I\ 0; \lambda_i T_{2i}^{\mathrm{T}}S_i\hat{P}_{i|i}\ \ I]^{-1}$ from the left sides of (4), we can obtain that

$$\begin{aligned}&(\begin{bmatrix}\hat{P}_{i|i}^{-1} & 0 \\ 0 & \hat{Q}_i^{-1}\end{bmatrix} + \hat{H}_i^{\mathrm{T}}\Psi_i\hat{H}_i)\begin{bmatrix}\tilde{x}_{i|i+1} - \hat{x}_{i|i} \\ \hat{w}_{i|i+1}\end{bmatrix} \\ &= \hat{H}_i^{\mathrm{T}}\Psi_i(y_{i+1} - C_{i+1}(0)(A_i(0)\hat{x}_{i|i} + B_{1i}u_i)) \\ &\quad - \lambda_i\begin{bmatrix} S_i^{\mathrm{T}} \\ T_{2i}^{\mathrm{T}}\end{bmatrix}(S_i\hat{x}_{i|i} + T_{1i}u_i).\end{aligned}$$

Define variable $\tilde{x}_{i+1|i+1} = A_i(0)\tilde{x}_{i|i+1} + \hat{B}_{2i}(0)\hat{w}_{i|i+1} + B_{1i}(0)u_i$, we can obtain the following two expressions, for which the direct computation of matrix inverse is avoided.

$$\begin{cases}\tilde{x}_{i|i+1} = \hat{x}_{i|i} + \hat{P}_{i|i}A_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0) \\ \qquad \times(B_{1i}(0)u_i + A_i(0)\tilde{x}_{i|i+1} + \hat{B}_{2i}(0)\hat{w}_{i|i+1})) \\ \qquad -\lambda_i\hat{P}_{i|i}S_i^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i), \\ \hat{w}_{i|i+1} = \hat{Q}_i\hat{B}_{2i}^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0) \\ \qquad \times(B_{1i}(0)u_i + A_i(0)\tilde{x}_{i|i+1} + \hat{B}_{2i}(0)\hat{w}_{i|i+1})) \\ \qquad -\lambda_i\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i).\end{cases} \tag{a2}$$

Therefore,

$$\begin{aligned}&\tilde{x}_{i+1|i+1} \\ &= B_{1i}(0)u_i + A_i(0)(\hat{x}_{i|i} + \hat{P}_{i|i}A_i^{\mathrm{T}}(0) \\ &\quad \times C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1})) \\ &\quad - \lambda_i A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i) \\ &\quad + \hat{B}_{2i}(0)(\hat{Q}_i\hat{B}_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1})) \\ &\quad - \lambda_i\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i) \\ &= B_{1i}(0)u_i + A_i(0)\hat{x}_{i|i} + A_i(0)\hat{P}_{i|i}A_i^{\mathrm{T}}(0) \\ &\quad \times C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1}) \\ &\quad - \lambda_i A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i) \\ &\quad + \hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1}) \\ &\quad - \lambda_i\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}}(S_i\hat{x}_{i|i} + T_{1i}u_i) \\ &= B_{1i}(0)u_i + A_i(0)\hat{x}_{i|i} + (A_i(0)\hat{P}_{i|i}A_i^{\mathrm{T}}(0) \\ &\quad + \hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^{\mathrm{T}}(0))C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1}) \\ &\quad - \lambda_i(A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}} + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}})(S_i\hat{x}_{i|i} + T_{1i}u_i).\end{aligned}$$

Moreover,

$$\begin{aligned}&\tilde{x}_{i+1|i+1} \\ &= B_{1i}(0)u_i + A_i(0)\hat{x}_{i|i} + P_{i+1|i}C_{i+1}^{\mathrm{T}}(0) \\ &\quad \times R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)\tilde{x}_{i+1|i+1}) - \lambda_i(A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}} \\ &\quad + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}})(S_i\hat{x}_{i|i} + T_{1i}u_i).\end{aligned}$$

Therefore,

$$\begin{aligned}&(I + P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0))\tilde{x}_{i+1|i+1} \\ &= B_{1i}(0)u_i + A_i(0)\hat{x}_{i|i} + P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}y_{i+1} \\ &\quad - \lambda(A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}} + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}})(S_i\hat{x}_{i|i} + T_{1i}u_i) \\ &= (B_{1i}(0) - \lambda(A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}} + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}})T_{1i})u_i \\ &\quad + (A_i(0) - \lambda(A_i(0)\hat{P}_{i|i}S_i^{\mathrm{T}} + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^{\mathrm{T}})S_i)\hat{x}_{i|i} \\ &\quad + P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}y_{i+1}.\end{aligned}$$

Moreover, we can obtain that $[P_{i+1|i}^{-1} + C_{i+1}(0)^{\mathrm{T}}R_{i+1}^{-1}C_{i+1}(0)]^{-1} = [I + P_{i+1|i}C_{i+1}(0)^{\mathrm{T}}R_{i+1}^{-1}C_{i+1}(0)]^{-1}P_{i+1|i} = P_{i+1|i+1}$ based on the matrix inversion lemma, then $\tilde{x}_{i+1|i+1} = P_{i+1|i+1}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}(y_{i+1} - C_{i+1}(0)(\hat{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{x}_{i|i})) + (\hat{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{x}_{i|i})$. Note that equation (a2) have the same forms as those of [1–3], which implies that we can reasonably designate $\hat{x}_{i+1|i+1}$ as $\tilde{x}_{i+1|i+1}$.

Appendix B Proof of Theorem 2

From the derivation aforementioned, we know that there exists a finite constant N_2 for every sampled instant such that $\|\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i+1|i+1} \\ \hat{\tilde{x}}_{i+1|i+1}\end{bmatrix}\}\| \leqslant N_2$.

Define matrix $M_i = \mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}^{\mathrm{T}}\}$, then from equation (6) and the whiteness of w_i and v_i, as well as the assumption that w_i and v_i are unrelated, we have that

$$\begin{aligned}M_{i+1} =&\tilde{A}_i(\varepsilon_i,\varepsilon_{i+1})M_i\tilde{A}_i^{\mathrm{T}}(\varepsilon_i,\varepsilon_{i+1})\\&+\tilde{B}_{2i}(\varepsilon_i,\varepsilon_{i+1})\mathrm{diag}\{Q_i,R_{i+1}\}\tilde{B}_{2i}^{\mathrm{T}}(\varepsilon_i,\varepsilon_{i+1})\\&+\tilde{A}_i(\varepsilon_i,\varepsilon_{i+1})\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}\}u_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i,\varepsilon_{i+1})\\&+\tilde{B}_{1i}(\varepsilon_i,\varepsilon_{i+1})u_i\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}^{\mathrm{T}}\}\tilde{A}_i^{\mathrm{T}}(\varepsilon_i,\varepsilon_{i+1})\\&+\tilde{B}_{1i}(\varepsilon_i,\varepsilon_{i+1})u_iu_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i,\varepsilon_{i+1}).\end{aligned}$$

Based on this relation, a direct application of mathematical inductions shows that

$$\begin{aligned}M_{i+1} =&\sum_{k=0}^{i}\{\prod_{j=k+1}^{i}\tilde{A}_j(\varepsilon_i,\varepsilon_{i+1})(\tilde{B}_{2i}(\varepsilon_i)\mathrm{diag}\{Q_i,R_{i+1}\}\tilde{B}_{2i}^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{A}_i(\varepsilon_i)\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}\}u_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i)+\tilde{B}_{1i}(\varepsilon_i)u_i\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}^{\mathrm{T}}\}\tilde{A}_i^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{B}_{1i}(\varepsilon_i)u_iu_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i))(\prod_{j=k+1}^{i}\tilde{A}_j(\varepsilon_i,\varepsilon_{i+1}))^{\mathrm{T}}\}.\end{aligned}$$

Define N_3 as

$$\begin{aligned}N_3 =&\sup_{i\geqslant 0}\sup_{\varepsilon_i,\varepsilon_{i+1}\in\mathcal{E}}\bar{\sigma}(\tilde{B}_{2i}(\varepsilon_i)\mathrm{diag}\{Q_i,R_{i+1}\}\tilde{B}_{2i}^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{A}_i(\varepsilon_i)\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}\}u_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i)+\tilde{B}_{1i}(\varepsilon_i)u_i\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}^{\mathrm{T}}\}\tilde{A}_i^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{B}_{1i}(\varepsilon_i)u_iu_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i)).\end{aligned}$$

Then, from the boundedness of $B_{1i}(\varepsilon_i)$, $B_{2i}(\varepsilon_i)$, $C_i(\varepsilon_i)$, Q_i and R_i, as well as the boundedness of estimation errors for the derived estimator, we have that N_3 is a finite positive number and,

$$\begin{aligned}\bar{\sigma}(M_{i+1}) \leqslant&\sum_{k=0}^{i}\{\bar{\sigma}(\prod_{j=k+1}^{i}\tilde{A}_j(\varepsilon_j,\varepsilon_{j+1}))\bar{\sigma}(\tilde{B}_{2i}(\varepsilon_i)\mathrm{diag}\{Q_i,R_{i+1}\}\tilde{B}_{2i}^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{A}_i(\varepsilon_i)\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}\}u_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i)+\tilde{B}_{1i}(\varepsilon_i)u_i\mathrm{E}\{\begin{bmatrix}\tilde{\tilde{x}}_{i|i}\\ \hat{\tilde{x}}_{i|i}\end{bmatrix}^{\mathrm{T}}\}\tilde{A}_i^{\mathrm{T}}(\varepsilon_i)\\&+\tilde{B}_{1i}(\varepsilon_i)u_iu_i^{\mathrm{T}}\tilde{B}_{1i}^{\mathrm{T}}(\varepsilon_i))\bar{\sigma}((\prod_{j=k+1}^{i}\tilde{A}_j(\varepsilon_j,\varepsilon_{j+1}))^{\mathrm{T}})\}\\ \leqslant&\,\alpha\sum_{k=0}^{i}\{\bar{\sigma}(\prod_{j=k+1}^{i}\tilde{A}_j(\varepsilon_j,\varepsilon_{j+1}))\}^2\\ \leqslant&\,N_3\sum_{k=0}^{i}\{\frac{3+\sqrt{5}}{2}\sqrt{M_1^2+M_2^2+(i-k)^2M_3^2}\rho_3^{i-k}\}^2\\ =&\,\frac{7+3\sqrt{5}}{2}N_3\{(M_1^2+M_2^2)\frac{1-\rho_3^{2(i+1)}}{1-\rho_3^2}\\&+M_3^2\frac{\rho_3^2(1+\rho_3^2)-\rho_3^{2(i+1)}[(i+1)^2-(2i^2+2i-1)\rho_3^{2i}+i^2\rho_3^{4i}]}{(1-\rho_3^2)^3}\}\\ <&\,+\infty.\end{aligned}$$

That is, the covariance matrix of estimation errors is always upper bounded.

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