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Input-output finite-time stability of time-varying linear singular systems

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Abstract: This paper studies the input-output finite-time stabilization problem for time-varying linear singular systems. The output and the input refer to the controlled output and the disturbance input, respectively. Two classes of disturbance inputs are considered, which belong to L-two and L-infinity. Sufficient conditions are firstly provided which guarantee the input-output finite-time stability. Based on this, state feedback controllers are designed such that the resultant closed-loop systems are input-output finite-time stable. The conditions are presented in terms of differential linear matrix inequalities. Finally, an example is presented to show the validity of the proposed results.

Keywords: Differential linear matrix inequality; Finite-time stability; Input-output; Linear singular system

1 Introduction

Singular systems, also known as generalized state systems, have been widely studied in the past several decades. They have broad applications and can be found in many practical systems, such as power systems, electronic circuit systems, socio-economic systems, constrained control systems, chemical processes, network analysis and other engineering fields, see [1–2], just to mention a few. Due to their extensive applications, singular systems attract much attention and great progress has been made in both theory and applications, such as controllability, observability and stability analysis [1–4], robust control and filtering [5].

Stability is an important performance for a system. Many results about stability of singular systems have been proposed, especially Lyapunov stability and asymptotic stability. However, neither of them can reflect the transient performance of systems. A system could be stable but its transient response is undesirable (e.g., large overshoot), which causes an adverse performance or even is impossible for application. In practice, for the short time working systems (such as missile systems, communication network systems, robot control systems), system trajectories required certain performance (e.g., system trajectories within a certain specified bound) are more concerned rather than their stability (mostly in the sense of Lyapunov stability) [6]. Thus, it is reasonable to study the transient behavior of dynamical systems over a finite-time interval. To study the transient performance, short-time stability was proposed by Peter Dorato in 1961 [7]. It, latter known as finite-time stability, has been more extensively studied.

Many results about finite-time stability have been reported, such as [8–10] for linear systems, and [11–13] for singular systems. It should be pointed out that the current literature is about state finite-time stability. But sometimes only the output, not the state, is required to be restrained within a bound. In this case, it is just needed to consider the input-output finite-time stability of a system. The concept of input-output finite-time stability of linear systems has been proposed in [14]. In this paper, the concept is generalized to linear singular systems and the corresponding control problem via state feedback is also discussed.

Notation Throughout this paper, $\mathbb{R}^n$ and $\deg(\cdot)$ denote the n-dimensional Euclidean space and the degree of the polynomial, respectively. The superscript 'T' means the transpose of the matrix. The symbol L_2 and L_∞ refer to the space of square integrable signals and the space of essentially bounded signals, respectively. For a given set $\Omega \subseteq \mathbb{R}$, a positive definite matrix R and a signal $\sigma(\cdot) : \Omega \to \mathbb{R}^l$, $\|\sigma(\cdot)\|_{\Omega,R}$ represents the weighted norm $(\int_\Omega \sigma(\tau)^{\mathrm{T}} R\sigma(\tau)\mathrm{d}\tau)^{1/2}$.

2 Input-output finite-time stability

Consider a time-varying singular system described by

$$\begin{cases} E\dot{x}(t) = A(t)x(t) + B(t)w(t), \ \ x(0) = 0, \\ y(t) = C(t)x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $w(t) \in \mathbb{R}^l$ is the disturbance input; $y(t) \in \mathbb{R}^q$ is the output. $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are piecewise continuous matrix-valued functions. The constant matrix E may be singular; we assume that $\mathrm{rank}(E) = r \leqslant n$. In this paper we consider the following two classes of disturbance inputs, belonging to L_2 and L_∞, respectively:

i) The set of norm bounded square integrable signals over $[0, T]$

$$\mathcal{W}_2(T, R, d) := \{w(\cdot) \in L_{2,[0,T]} : \|w\|_{[0,T],R} \leqslant d\};$$

ii) The set of the uniformly bounded signals over $[0, T]$

$$\mathcal{W}_\infty(T, R, d) := \{w(\cdot) \in L_{\infty,[0,T]} : w(t)^{\mathrm{T}} Rw(t) \leqslant d, \ t \in [0, T]\},$$

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where R and d denote a positive definite symmetric matrix and a positive scalar, respectively. In this section we study the input-output finite-time stability of system (1). First, we give the following definitions.

Definition 1 [15] The linear singular system $E\dot{x}(t) = A(t)x(t)+B(t)w(t)$ is said to be impulse-free in time interval $[0,T]$, if $\deg(\det(sE - A(t))) = \mathrm{rank}(E), \forall t \in [0,T]$.

Definition 2 Given two positive scalars T and c, a class of disturbances $\mathcal{W}$ defined over $[0,T]$, a positive definite matrix-valued function $Q(\cdot)$, system (1) is said to be input-output finite-time stable with respect to $(\mathcal{W}, Q(\cdot), T, c)$, if

$$w(\cdot) \in \mathcal{W} \Rightarrow y(t)^{\mathrm{T}}Q(t)y(t) < c,\ \ t \in (0,T].$$

By the assumption of $\mathrm{rank}(E) = r$, for system $E\dot{x}(t) = A(t)x(t) + B(t)w(t)$, we can choose two nonsingular matrices M and N such that

$$MEN = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},\ \ MA(t)N = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}.$$

According to Definition 1, we have the following lemma.

Lemma 1 [5] The linear singular system $E\dot{x}(t) = A(t)x(t)+B(t)w(t)$ is impulse-free in time interval $[0,T]$, if and only if $A_4(t)$ is nonsingular for all $t \in [0,T]$.

For the two classes of disturbance inputs $\mathcal{W}_2(T,R,d)$ and $\mathcal{W}_\infty(T,R,d)$, we will consider the input-output finite-time stability with respect to $(\mathcal{W}_2(T,R,d), Q(\cdot), T, c)$ and $(\mathcal{W}_\infty(T,R,d), Q(\cdot), T, c)$. First, we have the following lemma.

Lemma 2 For the two classes of disturbance inputs, the two following statements hold:

1) Given the class of disturbance inputs $\mathcal{W}_2(T,R,d)$, there exists a positive definite matrix $\tilde{R}$ such that $\mathcal{W}_2(T,R,d) = \mathcal{W}_2(T,\tilde{R},1)$. Similar result holds for the class of disturbance inputs $\mathcal{W}_\infty(T,R,d)$.

2) Given $(\mathcal{W}, Q(\cdot), T, c)$, there exists a positive definite matrix-valued function $\tilde{Q}(\cdot)$ such that a system is input-output finite-time stable with respect to $(\mathcal{W}, Q(\cdot), T, c)$ is equivalent to the system is input-output finite-time stable with respect to $(\mathcal{W}, \tilde{Q}(\cdot), T, 1)$.

Proof Taking $\tilde{R} = R/d^2$ and $\tilde{Q}(t) = Q(t)/c$, the results can be derived easily.

Thus, in the following part, we just need to consider the input-output finite-time stability with respect to $(\mathcal{W}_2(T,R,1), Q(\cdot), T, 1)$ and $(\mathcal{W}_\infty(T,R,1), Q(\cdot), T, 1)$, for simplicity. In this section, we study the input-output finite-time stability of singular systems. For system (1), we will provide sufficient conditions for input-output finite-time stability respectively with respect to $(\mathcal{W}_2(T,R,1), Q(\cdot), T, 1)$ and $(\mathcal{W}_\infty(T,R,1), Q(\cdot), T, 1)$.

Theorem 1 Linear singular system (1) is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_2(T,R,1), Q(\cdot), T, 1)$, if for all $t \in [0,T]$, there exists a nonsingular and piecewise continuously differential matrix-valued function $P(\cdot)$ such that

$$\begin{pmatrix} E^{\mathrm{T}}\dot{P}(t) + A(t)^{\mathrm{T}}P(t) + P(t)^{\mathrm{T}}A(t) & P(t)^{\mathrm{T}}B(t) \\ B(t)^{\mathrm{T}}P(t) & -R \end{pmatrix} < 0, \tag{2a}$$

$$P(t)^{\mathrm{T}}E = E^{\mathrm{T}}P(t) \geqslant C(t)^{\mathrm{T}}Q(t)C(t) \geqslant 0. \tag{2b}$$

Proof By Schur complement, it is easy to check that inequality (2a) holds if and only if for all $t \in [0,T]$, the following inequality holds

$$\begin{aligned} &E^{\mathrm{T}}\dot{P}(t) + A(t)^{\mathrm{T}}P(t) + P(t)^{\mathrm{T}}A(t) \\ &+P(t)^{\mathrm{T}}B(t)R^{-1}B(t)^{\mathrm{T}}P(t) < 0. \end{aligned} \tag{3}$$

First we show that linear singular system (1) is impulse-free, and later derive the input-output finite-time stability of the system. Since $\mathrm{rank}(E) = r$, we can choose two nonsingular matrices M and N such that

$$MEN = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},\ \ MA(t)N = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}. \tag{4}$$

Write

$$M^{-\mathrm{T}}P(t)N = \begin{pmatrix} P_1(t) & P_2(t) \\ P_3(t) & P_4(t) \end{pmatrix}, \tag{5}$$

where the partition is compatible with that of E in (4). Then, by (2b), it can be shown that $P_2(t) = 0$, and $P_1(t) = P_1(t)^{\mathrm{T}}$. Noting (3) and $R > 0$, we have

$$E^{\mathrm{T}}\dot{P}(t) + A(t)^{\mathrm{T}}P(t) + P(t)^{\mathrm{T}}A(t) < 0. \tag{6}$$

Now, pre- and post-multiplying (6) by N^{T} and N, respectively, and then using the expressions in (4) and (5), we have

$$\begin{pmatrix} \dot{P}_1(t) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} U_1 & U_2 \\ U_2^{\mathrm{T}} & U_3 \end{pmatrix} < 0, \tag{7}$$

where

$$\begin{aligned} U_1 &= A_1^{\mathrm{T}}(t)P_1(t) + P_1(t)A_1(t) + A_3^{\mathrm{T}}(t)P_3(t) \\ &\quad +P_3^{\mathrm{T}}(t)A_3(t), \\ U_2 &= P_1(t)A_2(t) + P_3^{\mathrm{T}}(t)A_4(t) + A_3^{\mathrm{T}}(t)P_4(t), \\ U_3 &= A_4(t)^{\mathrm{T}}P_4(t) + P_4(t)^{\mathrm{T}}A_4(t). \end{aligned}$$

Then, the 2-2 block in (7) gives

$$A_4(t)^{\mathrm{T}}P_4(t) + P_4(t)^{\mathrm{T}}A_4(t) < 0,$$

which implies $A_4(t)$ is nonsingular. By Lemma 1, system (1) is impulse-free.

Next, we show that system (1) is input-output finite-time stable. Consider the generalized Lyapunov function

$$V(\tau, x) = x(\tau)^{\mathrm{T}}E^{\mathrm{T}}P(\tau)x(\tau).$$

Then, differentiating $V(\tau, x)$ with respect to time τ along with the solution to (1), we obtain (time argument is omitted for brevity)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(x^{\mathrm{T}}E^{\mathrm{T}}Px) &= x^{\mathrm{T}}E^{\mathrm{T}}\dot{P}x + \dot{x}^{\mathrm{T}}E^{\mathrm{T}}Px + x^{\mathrm{T}}P^{\mathrm{T}}E\dot{x} \\ &= x^{\mathrm{T}}(E^{\mathrm{T}}\dot{P} + A^{\mathrm{T}}P + P^{\mathrm{T}}A)x \\ &\quad +w^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}P^{\mathrm{T}}Bw. \end{aligned}$$

By (3), it is easy to see that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}(x^{\mathrm{T}}E^{\mathrm{T}}Px) \\ &< w^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}P^{\mathrm{T}}Bw - x^{\mathrm{T}}P^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}Px. \end{aligned}$$

Let $z = (R^{1/2}w - R^{-1/2}B^{\mathrm{T}}Px)$, then

$$\begin{aligned} z^{\mathrm{T}}z &= w^{\mathrm{T}}Rw + x^{\mathrm{T}}P^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}Px - w^{\mathrm{T}}B^{\mathrm{T}}Px \\ &\quad -x^{\mathrm{T}}P^{\mathrm{T}}Bw. \end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(x^{\mathrm{T}}E^{\mathrm{T}}Px) < w^{\mathrm{T}}Rw - z^{\mathrm{T}}z < w^{\mathrm{T}}Rw. \tag{8}$$

Integrating both sides of (8) from 0 to $t \leqslant T$ and noting that

$x(0) = 0$ and $w(\cdot)$ belongs to $\mathcal{W}_2(T, R, 1)$, we obtain

$$\begin{aligned} & x(t)^{\mathrm T}E^{\mathrm T}P(t)x(t) \\ & < \int_0^t w(s)^{\mathrm T}Rw(s)\mathrm{d}s < \|w\|^2_{[0,t],R} < \|w\|^2_{[0,T],R} \leqslant 1. \end{aligned}$$

Considering the condition (2b), we get

$$\begin{aligned} & y(t)^{\mathrm T}Q(t)y(t) \\ & = x(t)^{\mathrm T}C(t)^{\mathrm T}Q(t)C(t)x(t) \\ & \leqslant x(t)^{\mathrm T}E^{\mathrm T}P(t)x(t) < 1. \end{aligned}$$

The proof is completed.

When system (1) is time-invariant, it is reduced to

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bw(t), \ \ x(0) = 0, \\ y(t) = Cx(t). \end{cases} \tag{9}$$

Given a positive scalar T, a positive definite matrix Q, and two classes of disturbance inputs, $\mathcal{W}_2(T, R, 1)$ and $\mathcal{W}_\infty(T, R, 1)$, we study the input-output finite-time stability with respect to $(\mathcal{W}_2(T, R, 1), Q, T, 1)$ and $(\mathcal{W}_\infty(T, R, 1), Q, T, 1)$ for system (9). For the class of disturbance inputs $\mathcal{W}_2(T, R, 1)$, we have the following corollary.

Corollary 1 Time-invariant system (9) is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_2(T, R, 1), Q, T, 1)$, if there exists a nonsingular matrix P such that

$$\begin{pmatrix} A^{\mathrm T}P + P^{\mathrm T}A & P^{\mathrm T}B \\ B^{\mathrm T}P & -R \end{pmatrix} < 0,$$

$$P^{\mathrm T}E = E^{\mathrm T}P \geqslant C^{\mathrm T}QC \geqslant 0.$$

Sufficient conditions for input-output finite-time stability with respect to $(\mathcal{W}_2(T, R, 1), Q(\cdot), T, 1)$ have been provided. In a similar way, we can develop the sufficient conditions for the case of $\mathcal{W}_\infty(T, R, 1)$.

Theorem 2 Linear singular system (1) is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_\infty(T, R, 1), Q(\cdot), T, 1)$, if there exists a nonsingular and piecewise continuously differential matrix-valued function $P(\cdot)$ such that

$$\begin{pmatrix} E^{\mathrm T}\dot{P}(t) + A(t)^{\mathrm T}P(t) + P(t)^{\mathrm T}A(t) & P(t)^{\mathrm T}B(t) \\ B(t)^{\mathrm T}P(t) & -R \end{pmatrix} < 0,$$

$$\forall t \in [0, T], \tag{10a}$$

$$P(t)^{\mathrm T}E = E^{\mathrm T}P(t) \geqslant C(t)^{\mathrm T}\tilde{Q}(t)C(t) \geqslant 0,$$

$$\forall t \in [0, T], \tag{10b}$$

where $\tilde{Q}(t) = tQ(t)$.

Proof Taking the same line as Theorem 1, it turns out that inequality (8) holds. As $u(\cdot) \in \mathcal{W}_\infty$, it is easy to see that

$$\frac{\mathrm d}{\mathrm d\tau}(x^{\mathrm T}E^{\mathrm T}Px) < 1. \tag{11}$$

Integrating (11) from 0 to $t \leqslant T$, with $x(0) = 0$, we obtain $x(t)^{\mathrm T}E^{\mathrm T}P(t)x(t) < t$. Considering the condition (10b) and $\bar{Q}(t) = tQ(t)$, it follows that $y(t)^{\mathrm T}Q(t)y(t) < 1$. The proof is completed.

Corollary 2 Time-invariant system (9) is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_\infty(T, R, 1), Q, T, 1)$, if there exists a nonsingular matrix P such that

$$\begin{pmatrix} A^{\mathrm T}P + P^{\mathrm T}A & P^{\mathrm T}B \\ B^{\mathrm T}P & -R \end{pmatrix} < 0,$$

$$P^{\mathrm T}E = E^{\mathrm T}P \geqslant TC^{\mathrm T}QC \geqslant 0.$$

3 Input-output finite-time stabilization via state feedback

In this section, we will study the following finite-time control problem, that is, investigating the design of state feedback controllers such that the closed-loop systems are impulse-free and input-output finite-time stable.

Problem 1 (Stabilization via state feedback) Consider the following linear singular system

$$\begin{cases} E\dot{x}(t) = A(t)x(t) + F(t)u(t) + B(t)w(t), \ x(0) = 0, \\ y(t) = C(t)x(t), \end{cases} \tag{12}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(\cdot) \in \mathbb{R}^m$ is the control input; $w(t) \in \mathbb{R}^l$ is the disturbance input. Given a positive scalar T, a class of disturbances $\mathcal{W}_2(T, R, 1)$ (or $\mathcal{W}_\infty(T, R, 1)$) defined over $[0, T]$, and a positive definite matrix-valued function $Q(\cdot)$, find a state feedback control law

$$u(t) = K(t)x(t), \tag{13}$$

such that the closed-loop system

$$\begin{cases} E\dot{x}(t) = A_c(t)x(t) + B(t)w(t), \\ y(t) = C(t)x(t), \end{cases} \tag{14}$$

with $A_c(t) = (A(t) + F(t)K(t))$, is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_2(T, R, 1), Q(\cdot), T, 1)$ (or $(\mathcal{W}_\infty(T, R, 1), Q(\cdot), T, 1)$).

When system (12) is time-invariant, try to find a state feedback control law $u(t) = Kx(t)$ such the closed-loop system is impulse-free and input-output finite-time stable with respect to $(\mathcal{W}_2(T, R, 1), Q, T, 1)$ (or $(\mathcal{W}_\infty(T, R, 1), Q, T, 1)$).

Our aim is to design the state feedback controller (13) such that the closed-loop system (14) is input-output finite-time stable. On the basis of Theorems 1–2, we have the following results.

Theorem 3 For the class of disturbances $\mathcal{W}_2(T, R, 1)$, Problem 1 is solvable if there exist a nonsingular and piecewise continuously differential matrix-valued function $\bar{P}(\cdot)$, and a matrix-valued function $L(\cdot)$ such that

$$\begin{pmatrix} \Gamma_1(t) & B(t) \\ B(t)^{\mathrm T} & -R \end{pmatrix} < 0, \ \ \forall t \in [0, T], \tag{15a}$$

$$\begin{pmatrix} E\bar{P}(t) & \bar{P}(t)^{\mathrm T}C(t)^{\mathrm T} \\ C(t)\bar{P}(t) & Q(t)^{-1} \end{pmatrix} \geqslant 0, \ \ \forall t \in [0, T]. \tag{15b}$$

Then a desired controller (13) is obtained with $K(t) = L(t)\bar{P}(t)^{-1}$, where

$$\begin{aligned} \Gamma_1(t) = & -E\dot{\bar{P}}(t) + \bar{P}(t)^{\mathrm T}A(t)^{\mathrm T} + A(t)\bar{P}(t) \\ & + L(t)^{\mathrm T}F(t)^{\mathrm T} + F(t)L(t). \end{aligned}$$

Proof Applying the state feedback controller $K(t) = L(t)\bar{P}(t)^{-1}$ to (12), we can get the following closed-loop system

$$\begin{cases} E\dot{x}(t) = A_c(t)x(t) + B(t)w(t), \\ y(t) = C(t)x(t), \end{cases} \tag{16}$$

where $A_c(t) = A(t)+F(t)L(t)\bar{P}(t)^{-1}$. On the other hand, it is easy to see that (15) can be rewritten as

$$\begin{pmatrix} \Gamma_2(t) & B(t) \\ B(t)^{\mathrm{T}} & -R \end{pmatrix} < 0,$$

$$E\bar{P}(t) = \bar{P}(t)^{\mathrm{T}}E^{\mathrm{T}} \geqslant \bar{P}(t)^{\mathrm{T}}C(t)^{\mathrm{T}}Q(t)C(t)\bar{P}(t) \geqslant 0,$$

where

$$\begin{aligned}\Gamma_2(t) = & -E\dot{\bar{P}}(t) + \bar{P}(t)^{\mathrm{T}}(A(t) + F(t)L(t)\bar{P}(t)^{-1})^{\mathrm{T}} \\ & +(A(t) + F(t)L(t)\bar{P}(t)^{-1})\bar{P}(t).\end{aligned}$$

Noting $A_c(t) = A(t) + F(t)L(t)\bar{P}(t)^{-1}$, we have

$$\begin{pmatrix} -E\dot{\bar{P}}(t) + \bar{P}(t)^{\mathrm{T}}A_c(t)^{\mathrm{T}} + A_c(t)\bar{P}(t) & B(t) \\ B(t)^{\mathrm{T}} & -R \end{pmatrix} < 0, \tag{17a}$$

$$E\bar{P}(t) = \bar{P}(t)^{\mathrm{T}}E^{\mathrm{T}} \geqslant \bar{P}(t)^{\mathrm{T}}C(t)^{\mathrm{T}}Q(t)C(t)\bar{P}(t) \geqslant 0. \tag{17b}$$

Taking $P(t) = \bar{P}(t)^{-1}$, and pre- and post-multiplying (17a) by diag$\{P(t)^{\mathrm{T}}, I\}$ and its transpose, (17b) by $P(t)^{\mathrm{T}}$ and $P(t)$, we have

$$\begin{pmatrix} \Gamma_3(t) & P(t)^{\mathrm{T}}B(t) \\ B(t)^{\mathrm{T}}P(t) & -R \end{pmatrix} < 0,$$

$$P(t)^{\mathrm{T}}E = E^{\mathrm{T}}P(t) \geqslant C(t)^{\mathrm{T}}Q(t)C(t) \geqslant 0.$$

where

$$\Gamma_3(t) = -P(t)^{\mathrm{T}}E\dot{\bar{P}}(t)P(t) + A_c(t)^{\mathrm{T}}P(t) + P(t)^{\mathrm{T}}A_c(t).$$

Since $P(t) = \bar{P}(t)^{-1}$, we get $I = \bar{P}(t)P(t)$. Considering the derivation, we have $0 = \dot{\bar{P}}(t)P(t) + \bar{P}(t)\dot{P}(t)$. It is easy to see that $\dot{P}(t) = -P(t)\dot{\bar{P}}(t)P(t)$. Noting that $P(t)^{\mathrm{T}}E = E^{\mathrm{T}}P(t)$, it derives $-P(t)^{\mathrm{T}}E\dot{\bar{P}}(t)P(t) = -E^{\mathrm{T}}P(t)\dot{\bar{P}}(t)P(t) = E^{\mathrm{T}}\dot{P}(t)$. It turns out that

$$\begin{pmatrix} \Gamma_4(t) & P(t)^{\mathrm{T}}B(t) \\ B(t)^{\mathrm{T}}P(t) & -R \end{pmatrix} < 0,\ \ \forall t \in [0, T],$$

$$P(t)^{\mathrm{T}}E = E^{\mathrm{T}}P(t) \geqslant C(t)^{\mathrm{T}}Q(t)C(t) \geqslant 0,\ \ \forall t \in [0, T],$$

where $\Gamma_4(t) = E^{\mathrm{T}}\dot{P}(t) + A_c(t)^{\mathrm{T}}P(t) + P(t)^{\mathrm{T}}A_c(t)$. By Theorem 1, it is easy to see closed-loop system (7) is impulse-free and input-output finite-time stable. The proof is completed.

Corollary 3 For the time-invariant case, given the class of disturbances $\mathcal{W}_2(T, R, 1)$, Problem 1 is solvable if there exist a nonsingular matrix $\bar{P}$ and a matrix L such that

$$\begin{aligned}&\begin{pmatrix} \bar{P}^{\mathrm{T}}A^{\mathrm{T}} + A\bar{P} + L^{\mathrm{T}}F^{\mathrm{T}} + FL & B \\ B^{\mathrm{T}} & -R \end{pmatrix} < 0, \\ &\begin{pmatrix} E\bar{P} & \bar{P}^{\mathrm{T}}C^{\mathrm{T}} \\ C\bar{P} & Q^{-1} \end{pmatrix} \geqslant 0.\end{aligned} \tag{18}$$

Then a desired controller is obtained with $K = L\bar{P}^{-1}$.

In a similar way, we can develop the results for the case of $\mathcal{W}_\infty(T, R, 1)$.

Theorem 4 Given the class of disturbances $\mathcal{W}_\infty(T, R, 1)$, Problem 1 is solvable if there exist a nonsingular and piecewise continuously differential matrix-valued function $\bar{P}(\cdot)$, and a matrix-valued function $L(\cdot)$ such that

$$\begin{pmatrix} \Gamma_1(t) & B(t) \\ B(t)^{\mathrm{T}} & -R \end{pmatrix} < 0,\ \ \forall t \in [0, T],$$

$$\begin{pmatrix} E\bar{P}(t) & \bar{P}(t)^{\mathrm{T}}C(t)^{\mathrm{T}} \\ C(t)\bar{P}(t) & (tQ(t))^{-1} \end{pmatrix} \geqslant 0,\ \ \forall t \in [0, T].$$

Then a desired controller (13) is obtained with $K(t) = L(t)\bar{P}(t)^{-1}$.

Corollary 4 For the time-invariant case, given the class of disturbances $\mathcal{W}_\infty(T, R, 1)$, Problem 1 is solvable if there exist a nonsingular matrix $\bar{P}$ and a matrix L such that

$$\begin{pmatrix} \bar{P}^{\mathrm{T}}A^{\mathrm{T}} + A\bar{P} + L^{\mathrm{T}}F^{\mathrm{T}} + FL & B \\ B^{\mathrm{T}} & -R \end{pmatrix} < 0,$$

$$\begin{pmatrix} E\bar{P} & \bar{P}^{\mathrm{T}}C^{\mathrm{T}} \\ C\bar{P} & (TQ)^{-1} \end{pmatrix} \geqslant 0.$$

Then a desired controller is obtained with $K = L\bar{P}^{-1}$.

Remark 1 When $E = I$, system (1) is reduced to a state-space system. It is easy to check that Theorems 1-4 coincide with the results in [14]. Therefor, the results can be regarded as an extension of the input-output finite-time stability theory from state-space systems to singular systems.

4 Example

For simplicity, we take a time-invariant linear singular system as an example. Consider a linear singular system (12) with parameters as follows:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 4.5 & -2 & 0 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} 0.05 & -0.1 & 0 \\ 0 & 0 & 0.1 \\ -0.1 & 0 & 0.1 \end{pmatrix},\ B = \begin{pmatrix} 0.2 \\ 0.7 \\ -1 \end{pmatrix},$$

$$C = (2\ \ -1\ \ 0).$$

Let $R = 1$, $Q = 1$, $T = 2$ and $c = d = 1$. Set $u(t) = 0$ and take $w(t) = \mathrm{e}^{-t} \in L_2$, $t \in [0, 2]$. From Fig. 1, it is easy to see the free system is not input-output finite-time stable with respect to $(\mathcal{W}_2(T, R, 1), Q, T, 1)$.

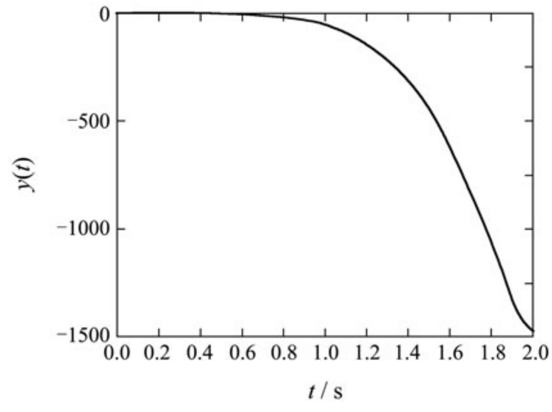

Fig. 1 The open-loop system output.

Then we can design a state feedback controller such that the closed-loop system is input-output finite-time stable with respect to $(\mathcal{W}_2(T, R, 1), Q, T, 1)$. Solving LMIs in

(18), we obtain

$$P=\begin{pmatrix}0.3731 & 0.5562 & 0\\ 0.5562 & 1.2075 & 0\\ -0.6151 & -6.9320 & 5.5865\end{pmatrix},$$

$$L=\begin{pmatrix}2.4333 & -37.9787 & 44.4600\\ 14.4995 & 0.3882 & 9.0837\\ -0.8314 & 12.2168 & -23.5814\end{pmatrix},$$

and the desired state feedback controller with

$$K=\begin{pmatrix}-5.0463 & 16.5603 & 7.9584\\ 86.6657 & -30.2675 & 1.6260\\ 37.8515 & -31.5524 & -4.2211\end{pmatrix}.$$

With $w(t)=\mathrm{e}^{-t}$, the output of the closed-loop system performs as Fig. 2. It is obvious that the closed-loop system is input-output finite-time stable.

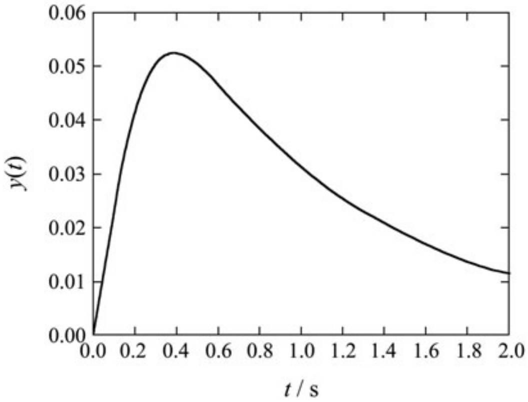

Fig. 2 The closed-loop system output.

5 Conclusions

In this paper, input-output finite-time stability has been analyzed for time-varying linear singular systems. The definition of input-output finite-time stability has been extended to singular systems. Sufficient conditions for input-output finite-time stability with the disturbance belonging to the space L_2 and L_∞ have been derived, respectively. Then stabilization problems via state feedback control have been investigated such that the resultant closed-loop systems are input-output finite-time stable. The conditions are given in terms of differential linear matrix inequalities. Finally, an example was presented to show the solvability and practicality.

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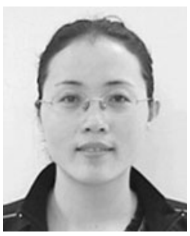

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