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Consensus of high-order dynamic multi-agent systems with switching topology and time-varying delays

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Abstract: This paper studies the consensus problems for a group of agents with switching topology and time-varying communication delays, where the dynamics of agents is modeled as a high-order integrator. A linear distributed consensus protocol is proposed, which only depends on the agent's own information and its neighbors' partial information. By introducing a decomposition of the state vector and performing a state space transformation, the closed-loop dynamics of the multi-agent system is converted into two decoupled subsystems. Based on the decoupled subsystems, some sufficient conditions for the convergence to consensus are established, which provide the upper bounds on the admissible communication delays. Also, the explicit expression of the consensus state is derived. Moreover, the results on the consensus seeking of the group of high-order agents have been extended to a network of agents with dynamics modeled as a completely controllable linear time-invariant system. It is proved that the convergence to consensus of this network is equivalent to that of the group of high-order agents. Finally, some numerical examples are given to demonstrate the effectiveness of the main results.

Keywords: Consensus problems; Distributed control; Multi-agent systems; Switching topology; Time-varying delays; Lyapunov-Krasovskii approach

1 Introduction

Consensus problems for networks of dynamic agents have been extensively studied by researchers from distinct points of view. As to the dynamics of agents, there are discrete-time forms [1∼7], single-integrator dynamics [8∼16], double-integrator dynamics [17, 18] and so on. The topological structures of networks, which are employed to describe the complex interconnections among agents, include bidirectional graphs, unidirectional graphs, random graphs, small-world networks, networks with zero/nonzero communication delays, etc. Some researchers have considered the consensus problems for multi-agent systems based on leader-follower control and model-reference control [13, 14, 18∼20]. Applications of this research pertain to cooperative control of unmanned aircraft, autonomous formation flight, control of communication networks, distributed sensor fusion in sensor networks, swarm-based computing, and rendezvous in space ([21∼25] and the references therein).

In general, the multi-agent systems achieving consensus aim at steering the states of all the agents to a common desired quantity by implementing appropriate consensus protocols. Most designs of consensus protocols are deeply based on the distributed control theory, that is, the control laws of each agent only depend on the local information available to it. In [1], a simple local rule was introduced for a discrete-time multi-agent system, and it was shown that the headings of all the agents converged to a common value. Reference [2] theoretically analyzed and generalized the results of [1] via algebraic graph theory, matrix theory and control theory. Systematically, [8] investigated the consensus problems for networks of single-integrator agents with fixed/switching topologies and zero/nonzero time-delays. Also, further results on the consensus problems for networks of single-integrator agents can be found in [9]. Recently, [17, 18] have proposed some distributed consensus protocols for networks of double-integrator agents. Under the proposed consensus protocol, Xie and Wang [17] solved the average-consensus problem for a group of double-integrator agents with fixed/switching topologies. Reference [18] proved that the states of all the double-integrator agents with jointly connected interactions could converge to the state of a given leader.

In this paper, we mainly investigate the consensus problem for multi-agent systems, where the dynamics of agents is modeled as a high-order integrator. The idea of modeling the dynamics of agents as high-order integrator comes from the following facts. First, it was shown in [26] that any completely controllable continuous-time linear time-invariant (LTI) system could be equivalently broken down into a collection of decoupled and independently controlled chains of integrators under an appropriate nonsingular linear transformation and a suitable state feedback. Second, in practical control systems, almost all the continuous-time LTI systems are completely controllable (see [27] and the references therein). Third, the high-order-integrator model of agents is a generalization of the single-integrator model and the double-integrator model. Finally, based on the consensus protocol of networks of high-order agents, we can propose a consensus protocol for a group of identical agents

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with dynamics modeled as a completely controllable LTI system, such that the convergence to consensus of this group is equivalent to that of a network of high-order agents. Hence, it is of physical interest and of theoretical interest to investigate the consensus problem for networks of high-order agents. Some related works can be found in [20].

For a network of high-order agents, a linear distributed consensus protocol is proposed to solve the consensus problem in the case where the interactions (or communication links) among agents are switching and with time delays. The proposed consensus protocol for each agent depends on the agent's overall information and its neighbors' partial information. Specifically, neighbors' partial information means that the control law only depends on the neighbors' delayed information variables themselves instead of their all-order derivatives. The interactions among agents are described by graphs, which capture the characterization of the topologies of multi-agent systems. On the basis of Lyapnov-Krasovskii theory for the stability of time-delayed systems, some sufficient conditions for the convergence to consensus are established in the form of linear matrix inequalities. Moreover, a method to estimate the maximal upper bound on admissible communication delays is provided. It is shown that the information variables of all the agents achieve a desired common value, and the all-order derivatives of all the information variables converge to zero. To emphasize the physical and theoretical interests of seeking the consensus of the network of high-order agents, a consensus protocol is provided for a group of agents with dynamics modeled as a completely controllable LTI system. It is proved that the convergence to consensus of this group is equivalent to that of the network of high-order agents.

The remainder of this paper is organized as follows. In the next section, we present some mathematical preliminaries on algebraic graph theory. In Section 3, we set up the model of agents and give the definitions of consensus. Section 4 states the main results on the convergence analysis for the network of high-order agents with switching topology and time-varying communication delays. Section 5 presents some numerical examples to illustrate the effectiveness of the theoretical results, and the last section makes some conclusions.

Notation Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of real numbers and the set of nonnegative real numbers, respectively. $\mathbb{R}^N$ is the N-dimensional real vector space. $\mathbb{R}^{N\times N}$ is the set of N-by-N matrices. Let $I_N \in \mathbb{R}^{N\times N}$ be an identity matrix. $\mathbf{0}$ denotes a zero matrix with appropriate order. Let $\mathbf{1}_N = [1 \ \cdots \ 1]^{\mathrm{T}} \in \mathbb{R}^N$ with all the entries being 1, and $e_1 = [1 \ 0 \ \cdots \ 0]^{\mathrm{T}} \in \mathbb{R}^m$. $\underline{N} = \{1, \cdots, N\}$ and $\underline{m-1} = \{1, \cdots, m-1\}$ are two index sets. For symmetric matrices X and Y with the same dimension, we say $X > Y$ if X-Y is positive definite. $\|\cdot\|$ defines the Euclidean norm on $\mathbb{R}^N$. Given a subspace $W \subset \mathbb{R}^N$, $W^{\perp}$ denotes the orthogonal complement space of W. $\otimes$ denotes the Kronecker product.

2 Mathematical preliminaries

A directed graph (or digraph for short) $\mathcal{G}$ consists of a vertex set $\mathcal{V} = \{v_1, \cdots, v_N\}$, an arc set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative entries a_{ij}, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. An arc of $\mathcal{G}$ is denoted by $e_{ij} := (v_i, v_j)$, and $e_{ij} \in \mathcal{E}$ if and only if $a_{ji} > 0$, which means that there exists a link from v_i to v_j; v_i and v_j are called the tail and the head of e_{ij}, respectively. If $e_{ij} = (v_i, v_j)$ is an arc, then we say that v_i is a neighbor of v_j. We assume that $a_{ii} = 0$, namely, the graph has no self-loops. If $a_{ij} = a_{ji}$, then the graph is called undirected graph. It is evident that the adjacency matrix $\mathcal{A}$ is symmetric for an undirected graph. Denote the neighbors of vertex v_i by $\mathcal{N}_i = \{v_j : \ e_{ji} = (v_j, v_i) \in \mathcal{E}\}$. A directed path from v_i to v_j means that there is a sequence of distinct arcs in $\mathcal{E}$, $(v_i, v_1), (v_1, v_2), \cdots, (v_r, v_j)$. A digraph is called strongly connected, if there exists a path between any two distinct vertices of the graph; for undirected graph it is called connected. Denote the in-degree and the out-degree of vertex v_i as $\deg_{\mathrm{in}}(v_i) = \sum\limits_j a_{ij}$ and $\deg_{\mathrm{out}}(v_i) = \sum\limits_j a_{ji}$, respectively. A vertex v_i is said to be balanced if $\deg_{\mathrm{in}}(v_i) = \deg_{\mathrm{out}}(v_i)$. A digraph is said to be balanced if all of its vertices are balanced. An undirected graph is called complete if its adjacency matrix $\mathcal{A}$ satisfies $a_{ij} > 0$ for all $i \neq j$.

Define the degree matrix of a graph with N vertices by a diagonal matrix $\mathcal{D} = \mathrm{diag}\{\deg_{\mathrm{in}}(v_1), \cdots, \deg_{\mathrm{in}}(v_N)\}$. Then the Laplacian matrix of a graph is defined as:

$$\mathcal{L} = \mathcal{D} - \mathcal{A}.$$

Some basic and fundamental properties of the Laplacian matrix $\mathcal{L}$ are presented as follows, which will be helpful to develop the main results [8, 28]:

1) $\mathcal{L}$ has a zero eigenvalue and a corresponding right eigenvector $\mathbf{1}_N$, i.e., $\mathcal{L}\mathbf{1}_N = 0$;

2) if a digraph $\mathcal{G}$ is strongly connected, then the associated Laplacian matrix $\mathcal{L}$ has only one zero eigenvalue and all other nonzero eigenvalues have positive real parts; for a connected undirected graph the associated Laplacian matrix is positive semi-definite and $\mathrm{rank}(\mathcal{L}) = N - 1$;

(3) a digraph is balanced if and only if $\mathbf{1}_N^{\mathrm{T}}\mathcal{L} = 0$.

3 Agent model and consensus problem

Consider a multi-agent system of N autonomous agents, which are labeled from 1 to N. The dynamics of each agent is described by the following mth order integrator

$$\begin{cases} \dot{x}_i^{(1)} = x_i^{(2)}, \\ \vdots \\ \dot{x}_i^{(m-1)} = x_i^{(m)}, \\ \dot{x}_i^{(m)} = u_i, \ \ i \in \underline{N}, \end{cases} \tag{1}$$

where m is a positive integer and denotes the dimension of the agents' state space; $x_i = [x_i^{(1)} \ \cdots \ x_i^{(m)}]^{\mathrm{T}}$ is the stacked state of agent i; $x_i^{(1)} \in \mathbb{R}$ is called the information variable of agent i for the convenience of description; $x_i^{(l+1)}, l \in \underline{m-1}$ is the lth order derivative of $x_i^{(1)}$; $u_i \in \mathbb{R}$ is the control input to be designed in a distributed form. Herein, the control input u_i is also called a consensus protocol. We make use of graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ to describe the interactions or communication relations among agents. Each vertex in $\mathcal{V}$ represents an agent of the multi-agent system, each arc e_{ji} in $\mathcal{E}$ means that there is an interaction or a communication link from agent j to agent i, and a_{ij} is the weight of the communication link e_{ij}.

It is known that the disturbance of communication delays is unavoidable in a real network due to the limited communication capacity of sensing/transmitting equipment. The consensus problems for networks of single-integrator/double-integrator agents with communication delays had been studied in [8], [15] and [19]. There are no results on the consensus problem for networks of high-order agents with communication delays in the literature. By taking the communication delays into account, we give the following time-delayed consensus protocol for the high-order multi-agent system (1)

$$u_i(t) = -\sum_{l=1}^{m-1} c_l x_i^{(l+1)}(t) - \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\big(x_i^{(1)}(t-\tau_{ij}(t)) - x_j^{(1)}(t-\tau_{ij}(t))\big), \quad (2)$$

where the parameters $c_l > 0, l \in \underline{m-1}$ are feedback gains and $\tau_{ij}(t)$ are the time-varying communication delays existing in the communication link from agent j to agent i. The above time-delayed consensus protocol involves both the real-time state information of agent i and the relative information between the time-delayed information variables of agent i and those of its neighbors. In practical applications, we assume that agent i can record and transmit not only the values of its state variables but also the associated time stamps. In this paper, we study the simple case where the communication delays $\tau_{ij}(t) = \tau(t), i, j \in \underline{N}$ with $\tau(t)$ being piecewise continuous and satisfying

A1) $0 \leqslant \tau(t) \leqslant \tau_0$ and $0 \leqslant \dot{\tau}(t) \leqslant d < 1$ for $t \geqslant 0$, or

A2) $0 \leqslant \tau(t) \leqslant \tau_0$ for $t \geqslant 0$,

where $\tau_0 > 0$ and $d \geqslant 0$ are constants. Notice that A2) includes the case when there is no prior knowledge about the derivative of $\tau(t)$.

Define the state vector of the multi-agent system (1) as $x(t) = [x_1^{\mathrm{T}}(t), \cdots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Assume the initial state of the system is $\phi(t) = x(0), t \in [-\tau_0, 0]$. Let $\chi:\ \mathbb{R}^{mN} \to \mathbb{R}$ be a continuous function of the agents' initial states. In this paper, we study the following consensus problem for system (1).

Definition 1 For a given protocol u_i, we say the protocol globally asymptotically solves the consensus problem for the multi-agent system (1), if for any initial state $\phi(t) = x(0), t \in [-\tau_0, 0]$ the states of all the agents satisfy that

$$\lim_{t\to\infty}\big(x_i^{(1)}(t) - x_j^{(1)}(t)\big) = 0,$$
$$\lim_{t\to\infty} x_i^{(l+1)}(t) = 0,\ \ l \in \underline{m-1}$$

for all $i, j \in \underline{N}$.

Definition 2 For a given protocol u_i, we say the protocol globally asymptotically solves the χ-consensus problem for the multi-agent system (1), if there exists a function $\chi:\ \mathbb{R}^{mN} \to \mathbb{R}$ such that for any initial state $\phi(t) = x(0), t \in [-\tau_0, 0]$, the states of all the agents satisfy

$$\lim_{t\to\infty}\big(x_i^{(1)}(t) - \chi(x(0))\big) = 0,$$
$$\lim_{t\to\infty} x_i^{(l+1)}(t) = 0,\ \ l \in \underline{m-1}$$

for all $i, j \in \underline{N}$. The function $\chi(\cdot): \mathbb{R}^{mN} \to \mathbb{R}$ is called the consensus function of system (1), $\chi(x(0))e_1$ with $e_1 \in \mathbb{R}^m$ is called the consensus state of the system (1).

Under the protocol (2), we can rewrite the dynamics (1) of agent i as the following concise form

$$\dot{x}_i(t) = E_m x_i(t) - \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t) F_m \times\big(x_i(t-\tau(t)) - x_j(t-\tau(t))\big), \quad (3)$$

where

$$E_m = \begin{bmatrix} \mathbf{0} & I_{m-1} \\ 0 & \theta^{\mathrm{T}} \end{bmatrix},\ \ F_m = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} \end{bmatrix},$$
$$\theta = [-c_1\ -c_2\ \cdots\ -c_{m-1}]^{\mathrm{T}}.$$

4 Main results

In this section, we consider the consensus problem for the multi-agent system (1) with switching topology and time-varying communication delays. It is known that there might exist link failures or creations in a communication network of mobile agents, due to the finite communication/sensing region of sensing devices or the effect of environment (such as in the case of nearest neighbor information exchange [2]). Therefore, the topologies of such a network might be switching or time-varying. For simplicity, we assume that the topological structures of the network are time-variant but the weights of the communication links are time-invariant, that is, at any time t when agent j is a neighbor of agent i, $a_{ij}(t) = a_{ij}, i, j \in \underline{N}$ and a_{ij} are given constants. Let $\mathscr{G}^{\mathrm{b}}$ be a collection of balanced graphs with N vertices, and $\mathscr{G}^{\mathrm{scb}} \subset \mathscr{G}^{\mathrm{b}}$ be a subset composed of strongly connected graphs. It is evident that $\mathscr{G}^{\mathrm{b}}$ is a finite set if the weights of communication links are chosen from a finite set, and all the undirected graphs belong to $\mathscr{G}^{\mathrm{b}}$. Refer to $\mathcal{I}_{\mathscr{G}^{\mathrm{b}}}$ and $\mathcal{I}_{\mathscr{G}^{\mathrm{scb}}}$ as the index sets of $\mathscr{G}^{\mathrm{b}}$ and $\mathscr{G}^{\mathrm{scb}}$, respectively. Next, introduce a switching signal $\sigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\mathrm{b}}}$ and a switching time sequence $t_0 = 0, t_1, \cdots, t_s, \cdots$, at which the network topology changes. Then the graphs remain time-invariant over the time intervals $[t_s, t_{s+1}), s = 0, 1, \cdots$. Under the protocol (2), the multi-agent system (1) with switching topology becomes the following hybrid system

$$\begin{cases} \dot{x}(t) = (I_N \otimes E_m)x(t) - (\mathcal{L}_{\sigma(t)} \otimes F_m)x(t-\tau(t)), \\ \phi(t) = x(0), t \in [-\tau_0, 0], \end{cases} \quad (4)$$

where E_m and F_m are given in (3), $\mathcal{L}_{\sigma(t)}$ (or $\mathcal{L}_\sigma$ for short) is the associated Laplacian matrix of the graph $\mathcal{G}_{\sigma(t)}$.

According to the aforementioned properties of the Laplacian matrix, we can obtain that for any switching signal σ, the function $\chi(x(t)) := \dfrac{1}{Nc_1}\mathbf{1}_N^{\mathrm{T}} \otimes [c_1\ \cdots\ c_{m-1}\ \ 1]x(t)$ is time-invariant along system (4). This is because $\mathbf{1}_N^{\mathrm{T}} \otimes [c_1\ \cdots\ c_{m-1}\ \ 1]$ is a left eigenvector of $I_N \otimes E_m - \mathcal{L}_\sigma \otimes F_m$ associated with the zero eigenvalue for any σ. Consequently, the state $x(t)$ of system (4) can be written as

$$x(t) = \chi(x(0))\mathbf{1}_N \otimes [1\ \underbrace{0\ \cdots\ 0}_{m-1}]^{\mathrm{T}} + \delta(t). \quad (5)$$

From $(I_N \otimes E_m - \mathcal{L}_\sigma \otimes F_m)\mathbf{1}_N \otimes [1\ \underbrace{0\ \cdots\ 0}_{m-1}]^{\mathrm{T}} = \mathbf{0}$, we obtain that $\mathbf{1}_N^{\mathrm{T}} \otimes [c_1\ \cdots\ c_{m-1}\ \ 1]\delta(t) = 0$ and $\delta(t)$ satisfies the following dynamics

$$\dot{\delta}(t) = (I_N \otimes E_m)\delta(t) - (\mathcal{L}_\sigma \otimes F_m)\delta(t-\tau(t)). \quad (6)$$

Denote $W_1 := \mathrm{span}\{\mathbf{1}_N \otimes [c_1 \ \cdots \ c_{m-1} \ 1]^{\mathrm{T}}\}^{\perp}$ and $W_2 := \mathrm{span}\{[c_1 \ \cdots \ c_{m-1} \ 1 \ \underbrace{0 \cdots 0}_{m(N-1)}]^{\mathrm{T}}\}^{\perp}$. The following lemma gives the relationship between them.

Lemma 1 For a given connected graph $\mathcal{G}$,

$$W_1 = (U_{\mathcal{G}} \otimes I_m)W_2, \quad (7)$$

where $U_{\mathcal{G}}$ is an orthogonal matrix such that $U_{\mathcal{G}}^{\mathrm{T}}\mathcal{L}_{\mathcal{G}}U_{\mathcal{G}} = \mathrm{diag}\{0,\mu_2,\cdots,\mu_N\}$ with $\mu_2, \cdots, \mu_N$ being the nonzero eigenvalues of $\mathcal{L}_{\mathcal{G}}$.

Proof Define $\varLambda = \mathrm{diag}\{0,\mu_2,\cdots,\mu_N\}$. Then $\mathcal{L}_{\mathcal{G}}\mathbf{1}_N = 0$ implies that $\varLambda U_{\mathcal{G}}^{\mathrm{T}}\mathbf{1}_N = 0$. As a result,

$$U_{\mathcal{G}}^{\mathrm{T}}\mathbf{1}_N = k[1 \ \underbrace{0 \cdots 0}_{N-1}]^{\mathrm{T}}$$

for $k \in \mathbb{R}$. Hence

$$\begin{aligned}&(U_{\mathcal{G}}^{\mathrm{T}} \otimes I_m)(\mathbf{1}_N \otimes [c_1 \ \cdots \ c_{m-1} \ 1]^{\mathrm{T}})\\ &= k[c_1 \ \cdots \ c_{m-1} \ 1 \underbrace{0 \cdots 0}_{m(N-1)}]^{\mathrm{T}},\end{aligned}$$

and therefore $(U_{\mathcal{G}}^{\mathrm{T}} \otimes I_m)\,\mathrm{span}\{\mathbf{1}_N \otimes [c_1 \ \cdots \ c_{m-1} \ 1]^{\mathrm{T}}\} = \mathrm{span}\{[c_1 \ \cdots \ c_{m-1} \ 1 \ \underbrace{0 \cdots 0}_{m(N-1)}]^{\mathrm{T}}\}$. Since $U_{\mathcal{G}}$ is orthogonal, we can derive that the conclusion holds.

Let $\mathcal{G}_c \in \mathscr{G}^{\mathrm{b}}$ be a complete graph, all the weights of which are 1. Denote the associated Laplacian matrix as $\mathcal{L}_c$. Suppose U_c is an orthogonal matrix with the first column being $\frac{1}{\sqrt{N}}\mathbf{1}_N$, such that $U_c^{\mathrm{T}}\mathcal{L}_cU_c = \mathrm{diag}\{0,\underbrace{N,\cdots,N}_{N-1}\}$. Based on the orthogonal matrix U_c we have the following result.

Lemma 2 For any digraph $\mathcal{G} \in \mathscr{G}^{\mathrm{scb}}$, let $\bar{L}_{\mathcal{G}} = [\mathbf{0} \ I_{N-1}]\,U_c^{\mathrm{T}}\mathcal{L}_{\mathcal{G}}U_c\begin{bmatrix}\mathbf{0}\\ I_{N-1}\end{bmatrix}$. Then the matrix $\bar{L}_{\mathcal{G}} + \bar{L}_{\mathcal{G}}^{\mathrm{T}}$ is positive definite.

Proof Since the graph $\mathcal{G}$ is strongly connected and balanced, we have $U_c^{\mathrm{T}}\mathcal{L}_{\mathcal{G}}U_c = \begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & \bar{L}_{\mathcal{G}}\end{bmatrix}$. According to the result of Theorem 7 in [8], $\mathcal{L}_{\mathrm{sym}} = \frac{\mathcal{L}_{\mathcal{G}} + \mathcal{L}_{\mathcal{G}}^{\mathrm{T}}}{2}$ is a valid Laplacian matrix of a graph. In addition, the strongly connected graph $\mathcal{G}$ indicates that the graph of $\mathcal{L}_{\mathrm{sym}}$ is connected. Thus, $\mathcal{L}_{\mathrm{sym}}$ is positive semi-definite and $\mathrm{rank}(\mathcal{L}_{\mathrm{sym}}) = N-1$. This implies that $\bar{L}_{\mathcal{G}} + \bar{L}_{\mathcal{G}}^{\mathrm{T}}$ is positive definite.

Lemma 3 [15] For any differentiable vector function $y(t) \in \mathbb{R}^N$ and any $N \times N$ positive definite matrix P, the following inequality

$$\begin{aligned}&\tau_0^{-1}[y(t) - y(t-\tau(t))]^{\mathrm{T}}P[y(t) - y(t-\tau(t))]\\ &\leqslant \int_{t-\tau(t)}^{t} \dot{y}^{\mathrm{T}}(s)P\dot{y}(s)\mathrm{d}s, \ \ t \geqslant 0\end{aligned}$$

holds, where $\tau(t)$ satisfies A1) or A2).

Lemma 4 (Schur complement theorem) [29] Let X, Y, Z be some given matrices with appropriate dimensions such that $Z < 0$. Then $\begin{bmatrix}X & Y\\ Y^{\mathrm{T}} & Z\end{bmatrix} < 0$ if and only if $X - YZ^{-1}Y^{\mathrm{T}} < 0$.

We start the main results with investigating the consensus problem for system (1) in the case where the switching topology keeps strongly connected and balanced across each successive interval $[t_s, t_{s+1}), s = 0,1,\cdots$. In this case, we denote the switching signal as $\varsigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\mathrm{scb}}}$ for clarity.

Theorem 1 Consider system (4) with $\sigma(t)$ replaced by $\varsigma(t)$. Suppose that the assumption A1) holds and the parameters $c_1,\cdots,c_{m-1}$ make the polynomial $s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2s + c_1$ Hurwitz stable. For arbitrary switching signal $\varsigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\mathrm{scb}}}$, if there exist symmetric matrices $P > 0, Q > 0, R > 0$ with proper dimensions, such that the following linear matrix inequalities

$$\varPhi_\varsigma = \begin{bmatrix}\varPhi_{11} & \varPhi_{12} & G^{\mathrm{T}}R\\ * & \varPhi_{22} & (G-H_\varsigma)^{\mathrm{T}}R\\ * & * & -\tau_0^{-1}R\end{bmatrix} < 0 \quad (8)$$

hold for a suitable $\tau_0 > 0$, then the protocol (2) globally asymptotically solves the χ-consensus problem for system (1) with $\chi(x(0)) = \frac{1}{Nc_1}\sum_{i=1}^{N}\tilde{c}^{\mathrm{T}}x_i(0)$ and $\tilde{c} = [c_1 \ \cdots \ c_{m-1} \ 1]^{\mathrm{T}} \in \mathbb{R}^m$. Here, "$*$" represents the elements below the main diagonal of a symmetric matrix and

$$\begin{aligned}&G = E_m \otimes I_{N-1}, \ \ H_\varsigma = -F_m \otimes \bar{L}_\varsigma,\\ &\varPhi_{11} = (G+H_\varsigma)^{\mathrm{T}}P + P(G+H_\varsigma) + dQ,\\ &\varPhi_{12} = -PH_\varsigma + (1-d)Q,\\ &\varPhi_{22} = -\tau_0^{-1}R + (d-1)Q\end{aligned}$$

with $\bar{L}_\varsigma$ being associated to the graph $\mathcal{G}_\varsigma$ and defined as in Lemma 2.

Proof Following Lemma 1, we have $W_1 = (U_c \otimes I_m)W_2$ with U_c given in Lemma 2. Then for any $\delta \in W_1$, there exists $\eta = [\eta_1^{\mathrm{T}}, \eta_2^{\mathrm{T}}]^{\mathrm{T}} \in W_2$ with

$$\begin{aligned}&\eta_1 = [\eta_1^{(1)} \ \cdots \ \eta_1^{(m)}]^{\mathrm{T}} \in \mathbb{R}^m,\\ &\eta_2 = [\eta_2^{(1)} \ \cdots \ \eta_2^{(m)} \ \cdots \ \eta_N^{(1)} \ \cdots \ \eta_N^{(m)}]^{\mathrm{T}} \in \mathbb{R}^{m(N-1)},\end{aligned}$$

such that $\delta = (U_c \otimes I_m)\eta$. According to the dynamics (6) with σ replaced by ς, we have

$$\dot{\eta}_1(t) = E_m\eta_1(t), \quad (9a)$$

$$\dot{\eta}_2(t) = (I_{N-1} \otimes E_m)\eta_2(t) - (\bar{L}_\varsigma \otimes F_m)\eta_2(t-\tau(t)). \quad (9b)$$

From $\eta \in W_2$, it follows that $c_1\eta_1^{(1)} + \cdots + c_{m-1}\eta_1^{(m-1)} + \eta_1^{(m)} = 0$. From the assumption that $c_1,\cdots,c_{m-1}$ make the polynomial $s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2s + c_1$ Hurwitz stable, we have that subsystem (9a) is globally asymptotically stable. As to subsystem (9b), we redefine the state vector η_2 as $\bar{\eta} = [\eta_2^{(1)} \ \cdots \ \eta_N^{(1)} \ \cdots \ \eta_2^{(m)} \ \cdots \ \eta_N^{(m)}]^{\mathrm{T}}$. Consequently, subsystem (9b) can be rewritten as

$$\dot{\bar{\eta}}(t) = G\bar{\eta}(t) + H_\varsigma\bar{\eta}(t-\tau(t)), \quad (10)$$

where G and H_ς are defined as in the theorem. In order to investigate the stability of the zero solution for system (6), the remaining work is to prove the stability of the zero solution for subsystem (10).

Take the candidate Lyapunov functional as

$$\begin{aligned}V_1(t) = &\bar{\eta}^{\mathrm{T}}(t)P\bar{\eta}(t) + \int_{t-\tau(t)}^{t}\bar{\eta}^{\mathrm{T}}(s)Q\bar{\eta}(s)\mathrm{d}s\\ &+ \int_{t-\tau_0}^{t}(s-t+\tau_0)\dot{\bar{\eta}}^{\mathrm{T}}(s)R\dot{\bar{\eta}}(s)\mathrm{d}s.\end{aligned}$$

Then the time derivative of $V_1(t)$ along subsystem (10) satisfies

$$\begin{aligned}\dot{V}_1(t) \leqslant\ & \bar{\eta}^{\mathrm{T}}(t)(G^{\mathrm{T}}P + PG)\bar{\eta}(t) + 2\bar{\eta}^{\mathrm{T}}(t)PH_{\varsigma}\bar{\eta}(t-\tau(t))\\ &+\bar{\eta}^{\mathrm{T}}(t)Q\bar{\eta}(t) + (d-1)\bar{\eta}^{\mathrm{T}}(t-\tau(t))Q\bar{\eta}(t-\tau(t))\\ &+\tau_0\dot{\bar{\eta}}^{\mathrm{T}}(t)R\dot{\bar{\eta}}(t) - \int_{t-\tau_0}^{t}\dot{\bar{\eta}}^{\mathrm{T}}(s)R\dot{\bar{\eta}}(s)\mathrm{d}s.\end{aligned}$$

Denote $\xi(t) = \bar{\eta}(t) - \bar{\eta}(t-\tau(t))$. Then

$$\begin{aligned}\dot{V}_1(t) \leqslant\ & \bar{\eta}^{\mathrm{T}}(t)[(G+H_{\varsigma})^{\mathrm{T}}P + P(G+H_{\varsigma})]\bar{\eta}(t)\\ &-2\bar{\eta}^{\mathrm{T}}(t)PH_{\varsigma}\xi(t) + \bar{\eta}^{\mathrm{T}}(t)Q\bar{\eta}(t)\\ &+(d-1)\bar{\eta}^{\mathrm{T}}(t-\tau(t))Q\bar{\eta}(t-\tau(t))\\ &+\tau_0\dot{\bar{\eta}}^{\mathrm{T}}(t)R\dot{\bar{\eta}}(t) - \int_{t-\tau(t)}^{t}\dot{\bar{\eta}}^{\mathrm{T}}(s)R\dot{\bar{\eta}}(s)\mathrm{d}s\\ \leqslant\ & \bar{\eta}^{\mathrm{T}}(t)[(G+H_{\varsigma})^{\mathrm{T}}P + P(G+H_{\varsigma}) + dQ]\bar{\eta}(t)\\ &-2\bar{\eta}^{\mathrm{T}}(t)[(d-1)Q + PH_{\varsigma}]\xi(t)\\ &+\xi^{\mathrm{T}}(t)[-\tau_0^{-1}R + (d-1)Q]\xi(t) + \tau_0[G\bar{\eta}(t)\\ &+H_{\varsigma}\bar{\eta}(t-\tau(t))]^{\mathrm{T}}R[G\bar{\eta}(t) + H_{\varsigma}\bar{\eta}(t-\tau(t))]\\ =\ & \left[\bar{\eta}^{\mathrm{T}}(t)\ \xi^{\mathrm{T}}(t)\right]\left\{\begin{bmatrix}\Phi_{11} & \Phi_{12}\\ * & \Phi_{22}\end{bmatrix}\right.\\ &\left.+\begin{bmatrix}G^{\mathrm{T}}\\ (G-H_{\varsigma})^{\mathrm{T}}\end{bmatrix}\tau_0 R\left[G\ (G-H_{\varsigma})\right]\right\}\begin{bmatrix}\bar{\eta}(t)\\ \xi(t)\end{bmatrix}\\ \triangleq\ & \left[\bar{\eta}^{\mathrm{T}}(t)\ \xi^{\mathrm{T}}(t)\right]\Sigma_{\varsigma}\begin{bmatrix}\bar{\eta}(t)\\ \xi(t)\end{bmatrix}.\end{aligned}$$

By adding the terms $\pm(d-1)\xi^{\mathrm{T}}(t)Q\xi(t)$ and $\pm(d-1)\bar{\eta}^{\mathrm{T}}(t)Q\bar{\eta}(t)$ to the right hand of the first inequality, Lemma 3 results in the second inequality. Then following the Schur complement theorem, the inequality (8) implies that the matrix Σ_{ς} is negative definite. Accordingly, there exists some scalar $\gamma > 0$ such that

$$\dot{V}_1(t) \leqslant -\gamma[\bar{\eta}^{\mathrm{T}}(t)\bar{\eta}(t) + \xi^{\mathrm{T}}(t)\xi(t)] \leqslant -\gamma\bar{\eta}^{\mathrm{T}}(t)\bar{\eta}(t),$$

which proves that the zero solution of subsystem (10) is globally asymptotically stable. And so is the zero solution of system (6). Hence, for any $0 \leqslant d < 1$ and a suitable $\tau_0 > 0$ satisfying (8), the protocol (2) globally asymptotically solves the χ-consensus problem for system (1). The proof is completed.

By making use of the linear matrix inequality (LMI for short) toolbox of the MATLAB software, we can find the feasible solution of the inequalities (8) and maximize the upper bound on the admissible communication delays. Specifically, for any given $0 \leqslant d < 1$ and the parameters $c_1, \cdots, c_{m-1}$ which make the polynomial $s^{m-1} - c_{m-1}s^{m-2} - \cdots - c_2 s - c_1$ Hurwitz stable, the GEVP (general eigenvalue problem) solver of the LMI toolbox can work out this problem, that is,

$$\begin{aligned}&\min \quad \tau_0^{-1}\\ &\text{subject to } P > 0, Q > 0, R > 0 \text{ and (8)}.\end{aligned}$$

For the case when the communication delay $\tau(t)$ satisfies A2), we have the following result.

Theorem 2 Consider system (4) with $\sigma(t)$ replaced by $\varsigma(t)$. Suppose that the assumption A2) holds and the parameters $c_1, \cdots, c_{m-1}$ make the polynomial $s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1$ Hurwitz stable. For arbitrary switching signal $\varsigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\mathrm{scb}}}$, if there exist symmetric matrices $P > 0, Q > 0$ with proper dimensions, such that the following linear matrix inequalities

$$\Psi_{\varsigma} = \begin{bmatrix}\Psi_{11} & -PH_{\varsigma} & G^{\mathrm{T}}Q\\ * & -\tau_0^{-1}Q & (G-H_{\varsigma})^{\mathrm{T}}Q\\ * & * & -\tau_0^{-1}Q\end{bmatrix} < 0 \tag{11}$$

hold for a suitable $\tau_0 > 0$, where

$$\Psi_{11} = (G+H_{\varsigma})^{\mathrm{T}}P + P(G+H_{\varsigma}),$$

G and H_{ς} are defined as in Theorem 1, then the protocol (2) globally asymptotically solves the χ-consensus problem for system (1) with $\chi(x(0))$ defined as in Theorem 1.

Proof According to the proof of Theorem 1, system (6) with σ replaced by ς can be transformed into the decoupled systems (9a) and (9b). In order to analyze the stability of the zero solution for subsystem (9b), we take the following candidate Lyapunov functional

$$V_2(t) = \bar{\eta}^{\mathrm{T}}(t)P\bar{\eta}(t) + \int_{t-\tau_0}^{t}(s-t+\tau_0)\dot{\bar{\eta}}^{\mathrm{T}}(s)Q\dot{\bar{\eta}}(s)\mathrm{d}s.$$

Then the time derivative of $V_2(t)$ along subsystem (9b) satisfies

$$\begin{aligned}\dot{V}_2(t) =\ & \bar{\eta}^{\mathrm{T}}(t)(G^{\mathrm{T}}P + PG)\bar{\eta}(t) + 2\bar{\eta}^{\mathrm{T}}(t)PH_{\varsigma}\bar{\eta}(t-\tau(t))\\ &+\tau_0\dot{\bar{\eta}}^{\mathrm{T}}(t)Q\dot{\bar{\eta}}(t) - \int_{t-\tau_0}^{t}\dot{\bar{\eta}}^{\mathrm{T}}(s)Q\dot{\bar{\eta}}(s)\mathrm{d}s\\ \leqslant\ & \bar{\eta}^{\mathrm{T}}(t)\Psi_{11}\bar{\eta}(t) - 2\bar{\eta}^{\mathrm{T}}(t)PH_{\varsigma}\xi(t) + \tau_0\dot{\bar{\eta}}^{\mathrm{T}}(t)Q\dot{\bar{\eta}}(t)\\ &-\tau_0^{-1}\xi^{\mathrm{T}}(t)Q\xi(t)\\ =\ & \left[\bar{\eta}^{\mathrm{T}}(t)\ \xi^{\mathrm{T}}(t)\right]\left\{\begin{bmatrix}\Psi_{11} & -PH_{\varsigma}\\ * & -\tau_0^{-1}Q\end{bmatrix}\right.\\ &\left.+\begin{bmatrix}G^{\mathrm{T}}\\ (G-H_{\varsigma})^{\mathrm{T}}\end{bmatrix}\tau_0 Q\left[G\ G-H_{\varsigma}\right]\right\}\begin{bmatrix}\bar{\eta}(t)\\ \xi(t)\end{bmatrix}\\ \triangleq\ & \left[\bar{\eta}^{\mathrm{T}}(t)\ \xi^{\mathrm{T}}(t)\right]\Xi_{\varsigma}\begin{bmatrix}\bar{\eta}(t)\\ \xi(t)\end{bmatrix},\end{aligned}$$

where $\xi(t) = \bar{\eta}(t) - \bar{\eta}(t-\tau(t))$. Following the proof of Theorem 1, we can derive that for a suitable $\tau_0 > 0$ satisfying (11), the protocol (2) can solve the χ-consensus problem for system (1) globally asymptotically. The proof is completed.

As to the linear matrix inequalities (11), we can follow the same method as (8) to find a feasible solution and maximize the upper bound on the admissible communication delays.

Theorem 1 and Theorem 2 indicate that the consensus problem for system (1) can be solved when the switching topology keeps strongly connected and balanced across each successive interval $[t_s, t_{s+1}), s = 0, 1, \cdots$. We next extend the results above to the case where the switching topology does not always keep strongly connected. Suppose there exists a sequence of time intervals, $[T_k, T_{k+1}), k = 0, 1, \cdots$ with $T_0 = 0$, $T_{k+1} = T_k + T$ and $T > 0$ being a sufficiently large constant. Each interval $[T_k, T_{k+1})$ contains a number of small intervals $[t_s, t_{s+1})$, across which the topology is time-invariant. Here we assume that $t_{s+1} - t_s \geqslant d_0, s = 0, 1, \cdots$, where $d_0 > 0$ is called the dwell time. Refer to T_k^{scb} as the total length of the intervals, across which the graphs are in $\mathscr{G}^{\mathrm{scb}}$, over $[T_k, T_{k+1})$; T_k^{b} as the total length of the intervals, across which the graphs are in $\mathscr{G}^{\mathrm{b}}/\mathscr{G}^{\mathrm{scb}}$, over $[T_k, T_{k+1})$. Suppose that there is a positive constant T^{b} such that $T_k^{\mathrm{b}} \leqslant T^{\mathrm{b}}, k = 0, 1, \cdots$. It is evident that $T_k^{\mathrm{scb}} \geqslant T - T^{\mathrm{b}}$, $k = 0, 1, \cdots$. Let $T^{\mathrm{scb}} = T - T^{\mathrm{b}}$.

Theorem 3 Consider system (4). Suppose that the assumption A1) holds and the parameters $c_1, \cdots, c_{m-1}$ make the polynomial $s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1$ Hurwitz stable. For a given constant $\alpha > 0$ and a switching signal $\sigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\rm b}}$, if $T_k^{\rm scb}, k = 0, 1, \cdots$ are sufficiently large and there exist symmetric matrices $P > 0, Q > 0, R > 0$ such that the following linear matrix inequalities

$$\Omega_s = \begin{bmatrix} \Omega_{11} & \Omega_{12} & G^{\rm T}R \\ * & \Omega_{22} & (G-H_s)^{\rm T}R \\ * & * & -\tau_0^{-1}R \end{bmatrix} < 0 \qquad (12)$$

hold for a suitable $\tau_0 > 0$, where the index $s \in \mathcal{I}_{\mathscr{G}^{\rm scb}} \bigcap \{\sigma(t) : t \geqslant 0\}$,

$$\begin{aligned} \Omega_{11} &= (G+H_s)^{\rm T}P + P(G+H_s) + \alpha P \\ &\quad + Q + (d-1){\rm e}^{-\alpha\tau_0}Q, \\ \Omega_{12} &= -PH_s + (1-d){\rm e}^{-\alpha\tau_0}Q, \\ \Omega_{22} &= -{\rm e}^{-\alpha\tau_0}\tau_0^{-1}R + (d-1){\rm e}^{-\alpha\tau_0}Q, \end{aligned}$$

G and H_s are given in Theorem 1, then the protocol (2) globally asymptotically solves the χ-consensus problem for system (1) with $\chi(x(0))$ defined as in Theorem 1.

Proof Analogously, we only need to analyze the stability of the zero solution for subsystem (10) with ς replaced by σ. Take the candidate Lyapunov functional as

$$\begin{aligned} V_3(t) &= \bar\eta^{\rm T}(t)P\bar\eta(t) + \int_{t-\tau(t)}^t {\rm e}^{\alpha(s-t)}\bar\eta^{\rm T}(s)Q\bar\eta(s){\rm d}s \\ &\quad + \int_{t-\tau_0}^t {\rm e}^{\alpha(s-t)}(s-t+\tau_0)\dot{\bar\eta}^{\rm T}(s)R\dot{\bar\eta}(s){\rm d}s. \end{aligned}$$

Then the time derivative of $V_3(t)$ along subsystem (10) is

$$\begin{aligned} &\dot V_3(t) \\ =\ & -\alpha V_3(t) + \alpha\bar\eta^{\rm T}(t)P\bar\eta(t) + 2\bar\eta^{\rm T}(t)P\dot{\bar\eta}(t) + \bar\eta^{\rm T}(t)Q\bar\eta(t) \\ & + (\dot\tau(t)-1){\rm e}^{-\alpha\tau(t)}\bar\eta^{\rm T}(t-\tau(t))Q\bar\eta(t-\tau(t)) \\ & + \tau_0\dot{\bar\eta}^{\rm T}(t)R\dot{\bar\eta}(t) - \int_{t-\tau_0}^t {\rm e}^{\alpha(s-t)}\dot{\bar\eta}^{\rm T}(s)R\dot{\bar\eta}(s){\rm d}s. \end{aligned}$$

Denote $\xi(t) = \bar\eta(t) - \bar\eta(t-\tau(t))$, we have

$$\begin{aligned} &\dot V_3(t) + \alpha V_3(t) \\ &\leqslant \left[\bar\eta^{\rm T}(t)\ \xi^{\rm T}(t)\right]\left\{\begin{bmatrix}\Omega_{11} & \Omega_{12} \\ * & \Omega_{22}\end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} G^{\rm T} \\ (G-H_\sigma)^{\rm T}\end{bmatrix}\tau_0 R\left[G\ (G-H_\sigma)\right]\right\}\begin{bmatrix}\bar\eta(t) \\ \xi(t)\end{bmatrix} \\ &\triangleq \left[\bar\eta^{\rm T}(t)\ \xi^{\rm T}(t)\right]\widehat\Sigma_\sigma\begin{bmatrix}\bar\eta(t) \\ \xi(t)\end{bmatrix}. \end{aligned} \qquad (13)$$

For the sake of notational convenience, we use Ω_{11} and Ω_{12} in (13) to denote the matrices $(G+H_\sigma)^{\rm T}P + P(G+H_\sigma) + \alpha P + Q + (d-1){\rm e}^{-\alpha\tau_0}Q$ and $-PH_\sigma + (1-d){\rm e}^{-\alpha\tau_0}Q$, respectively. First, for $t \in [t_s, t_{s+1})$, over which the corresponding graph belongs to $\mathscr{G}^{\rm scb}$, the Schur complement theorem and (12) imply that $\widehat\Sigma_\sigma$ is negative definite. Hence $\dot V_3(t) + \alpha V_3(t) \leqslant 0, t \in [t_s, t_{s+1})$. This proves that

$$V_3(t) \leqslant {\rm e}^{-\alpha(t-t_s)}V_3(t_s),\ \ t \in [t_s, t_{s+1}). \qquad (14)$$

Next, for $t \in [t_b, t_{b+1})$, over which the corresponding graph belongs to $\mathscr{G}^{\rm b}/\mathscr{G}^{\rm scb}$, we obtain from (13) that

$$\dot V_3(t) + \alpha V_3(t) \leqslant \lambda_{\max}(\widehat\Sigma_\sigma)(\|\bar\eta(t)\|^2 + \|\xi(t)\|^2),$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix. If $\lambda_{\max}(\widehat\Sigma_\sigma) \leqslant 0$, then $\dot V_3(t) + \alpha V_3(t) \leqslant 0, t \in [t_b, t_{b+1})$, which enhances the stability of the zero solution of subsystem (10). If $\lambda_{\max}(\widehat\Sigma_\sigma) > 0$, then from $\|\xi(t)\|^2 \leqslant 2(\|\bar\eta(t)\|^2 + \|\bar\eta(t-\tau(t))\|^2)$ we have

$$\dot V_3(t) + \alpha V_3(t) \leqslant \lambda_{\max}(\widehat\Sigma_\sigma)\Big(\frac{3}{\lambda_{\min}(P)} + \frac{2{\rm e}^{\alpha\tau_0}}{\lambda_{\min}(Q)}\Big)V_3(t),$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix. Let $\beta = \lambda_{\max}(\widehat\Sigma_\sigma)\Big(\dfrac{3}{\lambda_{\min}(P)} + \dfrac{2{\rm e}^{\alpha\tau_0}}{\lambda_{\min}(Q)}\Big)$. Then $\beta > 0$. Consequently,

$$V_3(t) \leqslant {\rm e}^{(\beta-\alpha)(t-t_b)}V_3(t_b),\ \ t \in [t_b, t_{b+1}). \qquad (15)$$

According to (14) and (15), we obtain

$$V_3(T_{k+1}) \leqslant {\rm e}^{-\alpha T + \beta(T - T^{\rm scb})}V_3(T_k).$$

Denote $\gamma = \beta T^{\rm scb} - (\beta-\alpha)T$. If $\beta - \alpha \leqslant 0$, it is evident that $\gamma > 0$; if $\beta - \alpha > 0$, then we take $T^{\rm scb} > \dfrac{(\beta-\alpha)T}{\beta}$ which guarantees $\gamma > 0$. Thus,

$$V_3(T_{k+1}) \leqslant {\rm e}^{-(k+1)\gamma}V_3(0).$$

Till now, for any $t > 0$, there exists an appropriate integer k satisfying $T_k \leqslant t < T_{k+1}$ such that

$$V_3(t) \leqslant {\rm e}^{\beta T^{\rm b}}V_3(T_k) \leqslant {\rm e}^{\beta T^{\rm b}}{\rm e}^{-k\gamma}V_3(0) \to 0,\ \ t \to \infty.$$

This implies that the zero solution of subsystem (10) is globally asymptotically stable. The proof is completed.

If the assumption A2) holds, we can take the following candidate Lyapunov functional for subsystem (10) with ς replaced by σ

$$\begin{aligned} V_4(t) &= \bar\eta^{\rm T}(t)P\bar\eta(t) \\ &\quad + \int_{t-\tau_0}^t {\rm e}^{\alpha(s-t)}(s-t+\tau_0)\dot{\bar\eta}^{\rm T}(s)Q\dot{\bar\eta}(s){\rm d}s. \end{aligned}$$

The related result is stated in the following theorem.

Theorem 4 Consider system (4). Suppose that the assumption A2) holds and the parameters $c_1, \cdots, c_{m-1}$ make the polynomial $s^{m-1} + c_{m-1}s^{m-2} + \cdots + c_2 s + c_1$ Hurwitz stable. For a given constant $\alpha > 0$ and a switching signal $\sigma(t) : \mathbb{R}_+ \to \mathcal{I}_{\mathscr{G}^{\rm b}}$, if $T_k^{\rm scb}, k = 0, 1, \cdots$ are sufficiently large, and there exist symmetric matrices $P > 0, Q > 0$ such that the following linear matrix inequalities

$$\Gamma_s = \begin{bmatrix} \Gamma_{11} & -PH_s & G^{\rm T}Q \\ * & -{\rm e}^{-\alpha\tau_0}\tau_0^{-1}Q & (G-H_s)^{\rm T}Q \\ * & * & -\tau_0^{-1}Q \end{bmatrix} < 0 \qquad (16)$$

hold for a suitable $\tau_0 > 0$, where $\Gamma_{11} = (G+H_s)^{\rm T}P + P(G+H_s) + \alpha P$, the index s, G and H_s are defined as in Theorem 3, then the protocol (2) globally asymptotically solves the χ-consensus problem for system (1) with $\chi(x(0))$ defined as in Theorem 1.

The proof is similar to that of Theorem 3, and hence is omitted.

Remark 1 As aforementioned, a consensus protocol for a group of identical agents with dynamics modeled by a completely controllable LTI system, can be derived from the consensus protocol (2) of the network of high-order agents. For simplicity of expression, we take the completely controllable single-input LTI system for example. Consider a group of N agents with dynamics modeled by

$$\dot\xi_i = A\xi_i + bv_i,\ \ i \in \underline{N}, \qquad (17)$$

where $\xi_i \in \mathbb{R}^m$ is the state of agent i, $v_i \in \mathbb{R}$ is the control input and (A, b) is completely controllable. Suppose the characteristic polynomial of A is $s^m - a_m s^{m-1} - \cdots - a_2 s - a_1$. Based on the consensus protocol (2), we can provide a distributed consensus protocol for the multi-agent system (17) as

$$\begin{aligned} v_i = \; & f^{\mathrm{T}} T^{-1} \xi_i(t) - \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) e_1^{\mathrm{T}} T^{-1} \\ & \times \big(\xi_i(t - \tau_{ij}(t)) - \xi_j(t - \tau_{ij}(t))\big), \end{aligned} \qquad (18)$$

where $f = [-a_1 \ -(c_1 + a_2) \ \cdots \ -(c_{m-1} + a_m)]^{\mathrm{T}} \in \mathbb{R}^m$ with $c_1, \cdots, c_{m-1}$ given in (2); $e_1 \in \mathbb{R}^m$; $T \in \mathbb{R}^{m \times m}$ is a nonsingular matrix such that $T^{-1}AT = A_c, T^{-1}b = [\underbrace{0 \cdots 0}_{m-1} \ 1]^{\mathrm{T}} \triangleq b_c$ with (A_c, b_c) being the associated controllable canonical form. Then there exists a nonsingular linear transformation between the closed-loop dynamics (17)(18) and system (3). As a matter of fact, if we let $\xi_i = T x_i, i \in \underline{N}$, then

$$\begin{aligned} \dot{x}_i(t) = \; & A_c x_i(t) + b_c f^{\mathrm{T}} x_i(t) - \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) b_c e_1^{\mathrm{T}} \\ & \times \big(x_i(t - \tau_{ij}(t)) - x_j(t - \tau_{ij}(t))\big) \\ = \; & E_m x_i(t) - \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) F_m \\ & \times \big(x_i(t - \tau_{ij}(t)) - x_j(t - \tau_{ij}(t))\big), \end{aligned}$$

where E_m and F_m are defined as in (3). This indicates that the protocol (18) solves a consensus problem for system (17) if and only if the protocol (2) solves a consensus problem for the high-order multi-agent system (1).

5 Numerical examples

In this section, we present some numerical examples for the multi-agent system (1) with $m = 3$ to demonstrate the effectiveness of the theoretical results.

These numerical simulations are performed with six agents. Fig.1 shows four possible connected graphs $\mathcal{G}_1 \sim \mathcal{G}_4$ with the six agents.

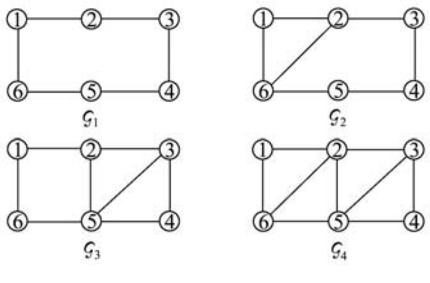

Fig. 1 Four possible connected graphs with six vertices: $\mathcal{G}_1 \sim \mathcal{G}_4$.

Fig.2 displays a switching law of the four graphs, which starts from $\mathcal{G}_1$ and then switches one by one along the arrows.

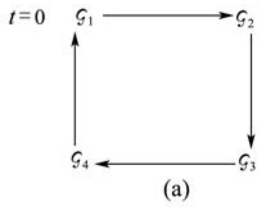

Fig. 2 A switching law of the graphs $\mathcal{G}_1 \sim \mathcal{G}_4$ shown in Fig. 1.

The switching period is one time unit. We first estimate the upper bound on the allowable communication delays via the LMI toolbox of MATLAB software. Take $c_1 = c_2 = 12$. The inequalities (8) of Theorem 1 result in the upper bound $\tau_0 = 2.1939$ with $d = 0.6$, while the inequalities (11) of Theorem 2 lead to the upper bound $\tau_0 = 0.0915$. (Note that when the time-varying delay $\tau(t)$ satisfies both A1) and A2), plenty of numerical simulations indicate that the upper bound on tolerant time delays derived from (8) is always larger than that derived from (11).) Next, by taking $\tau(t) = 0.5\arctan(t), t \geqslant 0$, Fig.3 presents the state trajectories of the six agents with the switching topology shown in Fig.2.

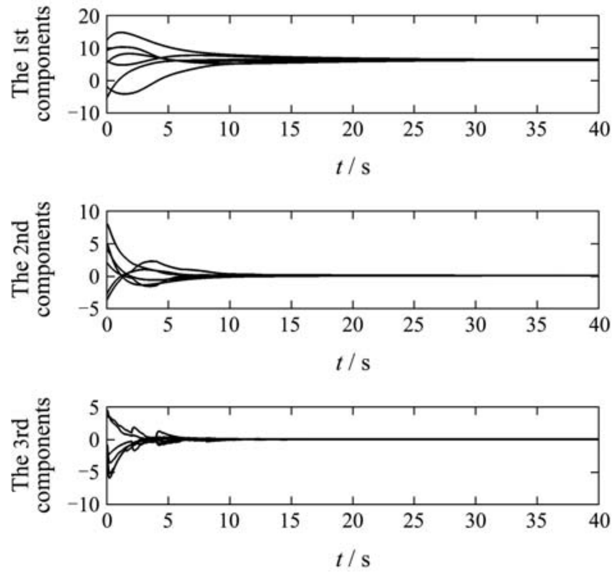

Fig. 3 For $m = 3$, $c_1 = c_2 = 12$ and $\tau(t) = 0.5\arctan(t)$, the state trajectories of six agents with switching topology shown in 2.

By taking $\tau(t) = 0.0518|\sin t|, t \geqslant 0$, Fig.4 shows the state trajectories of the six agents with the switching topology shown in Fig.2.

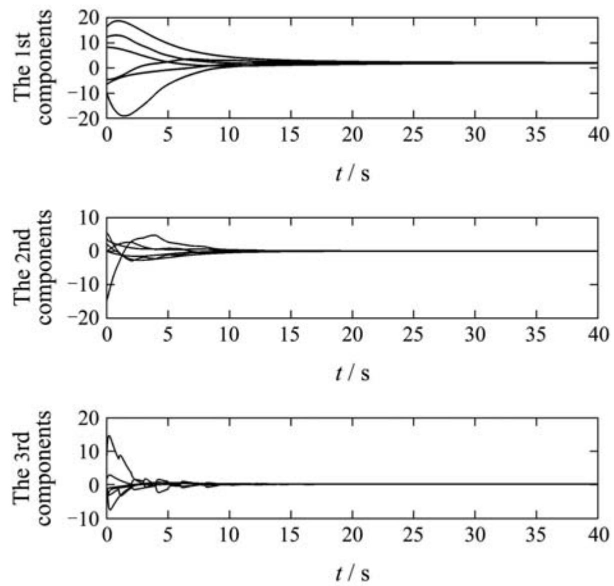

Fig. 4 For $m = 3$, $c_1 = c_2 = 12$ and $\tau(t) = \tau(t) = 0.0518|\sin t|$, the state trajectories of six agents with switching topology shown in Fig.2.

The last numerical simulation reports the consensus problem for the six agents with switching topology shown in Fig.5.

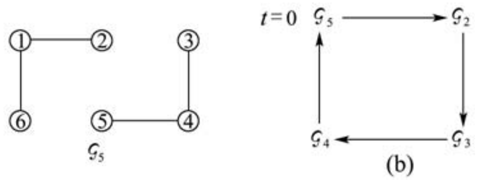

Fig. 5 A possible unconnected graph with six vertices and a switching law of the graphs $\mathcal{G}_2 \sim \mathcal{G}_5$.

The switching law starts from $\mathcal{G}_5$ and then switches one by one along the arrows with switching period being one time unit. This kind of switching topology indicates that there exist balanced but not strongly connected graphs among the switching topology, such as $\mathcal{G}_5$. Following Theorem 4, if we take $\alpha = 0.1$ and $c_1 = c_2 = 12$, the linear matrix inequalities (16) are solvable with $\tau_0 = 0.09$. Fig.6 depicts the state trajectories of the six agents with the switching topology shown in Fig.4 and the time-varying delay $\tau(t) = 0.09|\sin t|, t \geqslant 0$.

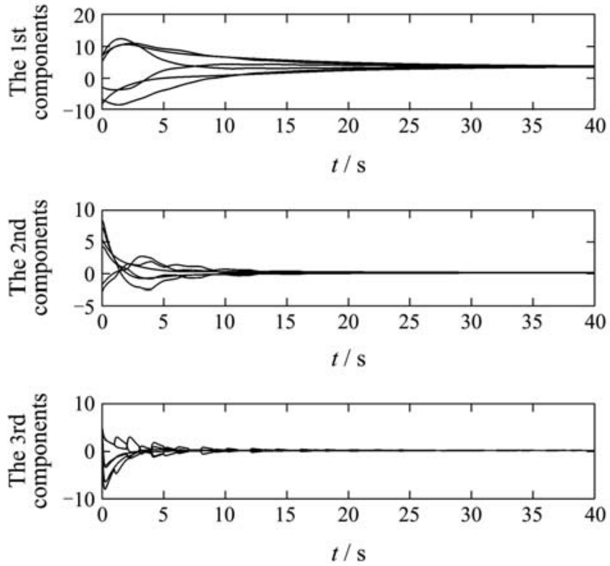

Fig. 6 For $m = 3$, $c_1 = c_2 = 12$ and $\tau(t) = \tau(t) = 0.09|\sin t|$, the state trajectories of six agents with switching topology shown in Fig.5.

6 Conclusions

This paper has considered the consensus problem for a network of high-order dynamic agents with switching topology and time-varying communication delays. A linear distributed consensus protocol has been proposed, which only depends on the agent's overall information and its neighbors' partial information. Based on the Lyapunov-Krasovskii approach, some sufficient conditions for the convergence to consensus have been established in the form of linear matrix inequalities. The results from the consensus seeking of the network of high-order agents have been extended to a group of agents with dynamics modeled as a completely controllable linear time-invariant system. It has been proved that the convergence to consensus of this group is equivalent to that of the network of high-order agents. Another topic is to consider the case where the communication time-delays are nonuniform and time-varying.

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