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New versions of Barbalat's lemma with applications

Mingzhe HOU [1], Guangren DUAN [1], Mengshu GUO [2]
(1.Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin Heilongjiang 150001, China;
2.Department of Mathematics, Harbin Institute of Technology, Harbin Heilongjiang 150001, China)

Abstract: This note presents a set of new versions of Barbalat's lemma combining with positive (negative) definite functions. Based on these results, a set of new formulations of Lyapunov-like lemma are established. A simple example shows the usefulness of our results.
Keywords: Barbalat's lemma; Lyapunov-like lemma

1 Introduction

Barbalat's lemma (see [1] Lemma 8.2), which states that a uniformly continuous function converges to zero if its integral has a (finite) limit, is a powerful tool to conclude that signals converge to zero in nonlinear systems, especially in time-varying nonlinear systems. However, this formulation of Barbalat's lemma is not easy to combine with Lyapunov stability theory, which somewhat restricts its application. To cope with this problem, an effective way is to combine Barbalat's lemma with (semi)positive or (semi)negative definite functions which play a key role in Lyapunov stability theory. A typical result is the well-known Lyapunov-like lemma (see [2] Lemma 4.3). However, this lemma is still not convenient for use because it is necessary to check the uniform continuity of a seminegative definite function and it only shows the asymptotical convergence of this seminegative definite function. Another result that has been widely used in the adaptive control literature states that if $x(t) : \mathbb{R}^n$ such that $x \in \mathcal{L}_\infty^n$, $\dot{x} \in \mathcal{L}_\infty^n$ and $x \in \mathcal{L}_p^n$, then $x(t) \to 0$ as $t \to \infty$ (see [3] Lemma A.5 and [4] Lemma B.2.1 for the case $p = 2$). Actually, the condition $x \in \mathcal{L}_\infty^n$ is redundant, which was pointed out by Tao [5]. In this note, it will be further revealed that this is caused by the positive definite and radially unbounded properties of the continuous function $\|x\|^p$. Besides the above results, another result, which has been employed successfully to prove stability results for model predictive control of nonlinear systems (see [6] Lemma 7 and [7] Lemma 4), states that if an absolutely continuous function $x(t)$ is such that $x(t) \in \mathcal{L}_\infty$, $\dot{x}(t) \in \mathcal{L}_\infty^n$ and $M(x(t)) \in \mathcal{L}_1$ where $M : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous positive definite function, then $x(t) \to 0$ as $t \to \infty$. For this result, a natural problem is whether the conclusion holds true or not if the condition $\dot{x}(t) \in \mathcal{L}_\infty^n$ is replaced by a weaker condition that $x(t)$ is a uniformly continuous function. Furthermore, it is significant to find out when the condition $x(t) \in \mathcal{L}_\infty^n$ can be omitted, since in many cases, whether the state is bounded or not is unknown in advance.

Motivated by these two problems, we will propose two new formulations of Barbalat's lemma in conjunction with positive (negative) definite functions in this note. The first one is a natural generalization of Lemma 7 in [6] and Lemma 4 in [7], but it is still required that the signals must be bounded; and the second one together with its corollary not only overcome the disadvantage of the first one by adding some trivial restrictions on the selected positive (negative) definite functions, but also are more general than the other related results, such as [3] Lemma A.5, [4] Lemma B.2.1 and the result in [5], etc. Furthermore, based on these results, a set of new versions of Lyapunov-like lemma are established, which are more convenient for application than the existing one.

For convenience, some notations and concepts used throughout this paper are presented as follows. For each $p \in [1, \infty)$ and $n \in \mathbb{N}$, $\mathcal{L}_p^n$ denotes the set of all measurable functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^n$ with $\int_0^\infty \|f(t)\|^p \,\mathrm{d}t < \infty$. $\mathcal{L}_\infty^n$ denotes the set of all measurable functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^n$ with $\sup\limits_{t \geqslant 0} \|f(t)\| < \infty$. A measurable function f is said to be uniformly, locally integrable if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \geqslant 0$, $\int_t^{t+\delta} \|f(s)\| \,\mathrm{d}s < \varepsilon$. And the other necessary concepts can be consulted in [1].

2 Main results

Before giving the main results, we first introduce some necessary concepts and their properties as follows.

Definition 1 The set $\mathcal{M}_r^{\mathcal{L}}$ consists of all continuous positive definite functions defined on $D \supseteq \mathcal{B}_r = \{z |\, z \in \mathbb{R}^n \text{ and } \|z\| \leqslant r\}$ for some $r \in \mathbb{R}^+$. The set $\mathcal{M}^{\mathcal{G}}$ consists of all continuous positive definite functions M which are defined on $\mathbb{R}^n$ and satisfies that there exist two positive scalars c_M and r_M such that $M(z) \geqslant c_M$ when $\|z\| > r_M$.

Obviously, $\mathcal{M}^{\mathcal{G}} \subseteq \mathcal{M}_r^{\mathcal{L}}$, and the following proposition holds.

Proposition 1 If a function M is continuous, positive definite, and radially unbounded, then $M \in \mathcal{M}^{\mathcal{G}}$.

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A set of new versions of Barbalat's lemma combining with positive (negative) definite functions are presented as follows:

Theorem 1 If $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is a uniformly continuous function such that $x(t) \in \mathcal{L}_\infty^n$ (exactly, $\|x(t)\| \leqslant r$ for some $r \in \mathbb{R}^+$) and $M(x(t))$ (or $-M(x(t))$) $\in \mathcal{L}_1$, where M (respectively, $-M$) $\in \mathcal{M}_r^{\mathcal{L}}$, then $x(t) \to 0$ as $t \to \infty$.

Theorem 2 Let M (or $-M$) $\in \mathcal{M}^{\mathcal{G}}$. If $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is a uniformly continuous function such that $M(x(t))$ (respectively, $-M(x(t))$) $\in \mathcal{L}_1$, then $x(t) \to 0$ as $t \to \infty$.

Proof of Theorem 1 We only prove the case of M, for the case of $-M$, the proof is similar. Assume that the assertion that $x(t) \to 0$ as $t \to \infty$ is false, then there exists a positive scalar ε_0 such that for any $T > 0$, there exists $t > T$ satisfying $\|x(t)\| \geqslant \varepsilon_0$. Hence, there exists a positive scalar η_0 and an increasing sequence of positive times $\{t_i\}_{i\in\mathbb{N}}$, with $t_i \to \infty$ as $i \to \infty$, such that $\|x(t_i)\| \geqslant \varepsilon_0$ and $|t_{i+1} - t_i| > \eta_0$ for all i. And the conclusion $|t_{i+1} - t_i| > \eta_0$ for all i implies that the intervals $\left[t_i - \frac{\eta_0}{2}, t_i + \frac{\eta_0}{2}\right]$ are nonoverlapping. Since $x(t)$ is assumed to be uniformly continuous, there exists $\eta > 0$ such that the following inequality

$$\|x(t') - x(t'')\| < \frac{\varepsilon_0}{2}$$

holds for any t' and t'' satisfying $|t' - t''| < \eta$ (without loss of generality, we can assume $\eta < \frac{\eta_0}{2}$, hence the intervals $[t_i - \eta, t_i + \eta]$ are nonoverlapping). This implies that for any t within the η-neighborhood of t_i (i.e., such that $|t - t_i| < \eta$)

$$\begin{aligned}\|x(t)\| &= \|x(t_i) + x(t) - x(t_i)\| \\ &\geqslant \|x(t_i)\| - \|x(t) - x(t_i)\| > \frac{\varepsilon_0}{2}.\end{aligned}$$

Define

$$d = \min\left\{M(z) : z \in \mathbb{R}^n, \frac{\varepsilon_0}{2} \leqslant \|z\| \leqslant r\right\}.$$

Under the assumptions, we have $d > 0$. Meanwhile, $M(x(t)) \geqslant d$ for any t within the η-neighborhood of t_i. Hence, we can conclude that

$$\begin{aligned}\infty > \int_0^\infty M(x(t))\mathrm{d}t &\geqslant \lim_{N\to\infty} \sum_{i=1}^N \int_{t_i-\eta}^{t_i+\eta} M(x(t))\mathrm{d}t \\ &\geqslant \lim_{N\to\infty} \sum_{i=1}^N \int_{t_i-\eta}^{t_i+\eta} d\,\mathrm{d}t = \lim_{N\to\infty} 2N\eta d = \infty.\end{aligned}$$

From this contradiction we have that $x(t) \to 0$ as $t \to \infty$. This completes the proof.

The proof of Theorem 2 resembles that of Theorem 1, so the proof is omitted here. What should be noted is that in this case d should be defined as $d = \min\{\min\{M(z) : z \in \mathbb{R}^n, \frac{\varepsilon_0}{2} \leqslant \|z\| \leqslant r_M\}, c_M\}$.

According to Theorem 2 and Proposition 1, the following result is immediate.

Corollary 1 Let M (or $-M$) be a continuous positive definite and radially unbounded scalar function. If $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is an uniformly continuous function such that $M(x(t))$ (respectively, $-M(x(t))$) $\in \mathcal{L}_1$, then $x(t) \to 0$ as $t \to \infty$.

Remark 1 One of the most common ways of guaranteeing that a function $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is uniformly continuous on $\mathbb{R}^+$ is to assume that its derivative $\dot{x}(t) \in \mathcal{L}_\infty^n$. A weaker condition to guarantee the uniform continuity of $x(t)$ on $\mathbb{R}^+$ is that if $x(t)$ is absolutely continuous and $\dot{x}(t)$ is uniformly, locally integrable (e.g., $\dot{x} \in \mathcal{L}_1^n$), then $x(t)$ is uniformly continuous. This condition can be checked easily if one can recall the fact that an absolutely continuous function is the integral function of its own derivative.

Remark 2 It is easy to see that the related results in [6] and [7] are a special case of Theorem 1. Furthermore, let $M = \|x\|^p$, then M is a continuous positive definite and radially unbounded scalar function, hence it is easy from Corrolary 1 to conclude the related results in [3], [4] and [5]. Therefore, it is reasonable to say that the results proposed in this note are more general than the ones cited before.

Remark 3 It should be noted that the radially unbounded condition in Corollary 1 is necessary. Otherwise, let $x(t) = t$ and $M(x) = 2|x|\mathrm{e}^{-x^2}$, then it is easy to check that $x(t)$ is uniformly continuous on $\mathbb{R}^+$, $M(x)$ is a continuous positive definite scalar function, and $\lim\limits_{t\to\infty} \int_0^t M(x(t)))\mathrm{d}t = 1$(i.e., $M(x(t)) \in \mathcal{L}_1$). But obviously, $x(t) \to \infty$ as $t \to \infty$, this is caused by the fact that $M(x)$ is not a radially unbounded scalar function.

Based on Theorem 1, Theorem 2 and Corollary 1, a set of new versions of Lyapunov-like lemma can be established as follows.

Theorem 3 If $x(t)$ is a uniformly continuous function such that $\|x(t)\| \leqslant r$ for some $r \in \mathbb{R}^+$, and if there exist a lower bounded scalar function V and a function $M \in \mathcal{M}_r^{\mathcal{L}}$ such that $-\dot{V} \geqslant M(x)$, then $x(t) \to 0$ as $t \to \infty$.

Theorem 4 If $x(t)$ is a uniformly continuous function and if there exist a lower bounded scalar function V and a function $M \in \mathcal{M}^{\mathcal{G}}$ such that $-\dot{V} \geqslant M(x)$, then $x(t) \to 0$ as $t \to \infty$.

Corollary 2 If $x(t)$ is a uniformly continuous function and if there exist a lower bounded scalar function V and a continuous positive definite and radially unbounded scalar function M such that $-\dot{V} \geqslant M(x(t))$, then $x(t) \to 0$ as $t \to \infty$.

Proof of Theorem 3 First, it is necessary to recall a fact that if a lower bounded function $V : \mathbb{R} \to \mathbb{R}$ is nonincreasing, then V converges (see [8] Fact 1). From this fact and the presented assumptions, it is easy to conclude that V_∞ exists and

$$\lim_{t\to\infty} \int_0^t M(x(t)))\mathrm{d}t \leqslant \lim_{t\to\infty} \int_0^t -\dot{V}\mathrm{d}t = V_0 - V_\infty < \infty,$$

i.e., $M(x(t)) \in \mathcal{L}_1$. Combinging this with the uniform continuity and boundedness of $x(t)$, it is obtained from Theorem 1 that $x(t) \to 0$ as $t \to \infty$. This completes the proof.

The proof of Theorem 4 and Corollary 2 is similar to that of Theorem 3, and hence omitted here.

3 A simple example

To illustrate the usefulness of our results, we give an example as follows.

Example 1 Consider the following second-order control system

$$\begin{cases} \dot{e} = -e + \theta w(t), \\ \dot{\theta} = -ew(t), \end{cases}$$

where $w(t)$ is a bounded continuous function. Let us analyze the asymptotic properties of this system.

Consider the lower bounded function

$$V = e^2 + \theta^2.$$

Its derivative is

$$\dot{V} = 2e(-e + \theta w(t)) + 2\theta(-ew(t)) = -2e^2 \leqslant 0.$$

This implies that $V_t \leqslant V_0$, and hence, e and θ is bounded. So it is easy to see that $\dot{e}$ is also bounded, and hence e is uniformly continuous. Combining these with the fact that $-\dot{V} = 2e^2 = M(e)$ is continuous, positive definite, and radially unbounded, we can use Corollary 2 to obtain that $e \to 0$ as $t \to \infty$.

Remark 4 The above example was also taken by Slotine to illustrate the use of the Lyapunov-like lemma in [2] (see Example 4.13), where the uniform continuity of the function $\dot{V}$ is checked, and then it is concluded that $\dot{V}$ converges to zero, which further indicates that e converges to zero. But as shown in the above example, to use our results, we just need to check the uniform continuity of the signal e; and furthermore, we can conclude directly that e converges to zero. Therefore, it stands to reason that, compared to the well-known Lyapunov-like lemma, our results make it more convenient to analyze the asymptotic convergence of signals.

4 Conclusions

In this note, a set of new versions of Barbalat's lemma combining with positive (negative) definite functions are proposed. These results allow us to establish a set of new versions of Lyapunov-like lemma. A simple example illustrates that, compared to the existing Lyapunov-like lemma, the results proposed in this note can make it more convenient to analyze the asymptotic stability of a control system.

References

[1] H. K. Khalil. *Nonlinear Systems*[M]. 3rd ed. Englewood Cliffs: Prentice-Hall, 2002.

[2] J. J. E. Slotine, W. P. Li. *Applied Nonlinear Control*[M]. Englewood Cliffs: Prentice-Hall, 1991.

[3] R. Kelly, V. Santibanez, A. Loria. *Control of Manipulators in Joint Space*[M]. London: Springer-Verlag, 2005.

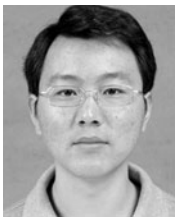
[4] R. Marino, P. Tomei. *Nonlinear Control Design: Geometric, Adaptive and Robust*[M]. Englewood Cliffs: Prenctice-Hall, 1995.

[5] G. Tao. A simple alternative to the Barbalat lemma[J]. *IEEE Transactions on Automatic Control*, 2000, 42(5): 698.

[6] F. A. C. C. Fontes. A general framework to design stabilizing nonlinear model predictive controllers[J]. *Systems & Control Letters*, 2001, 42(2): 127 – 143.

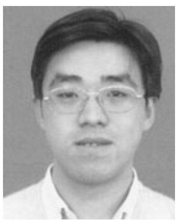
[7] H. Michalska, R. B. Vinter. Nonlinear stabilization using discontinuous moving-horizon control[J]. *IMA Journal of Mathematical Control Information*, 1994, 11(4): 321 – 340.

[8] B. Brogliato, R. Lozano, B. Maschke, et al. *Dissipative Systems Analysis and Control Theory and Applications*[M]. 2rd ed. London: Springer-Verlag, 2007.



Mingzhe HOU received his B.E. degree in Automation in 2005 from Harbin Institute of Technology, China. Currently, he is working toward the Ph.D. degree in the Center for Control Theory and Guidance Technology at Harbin Institute of Technology. His main research interests include nonlinear control theory and guidance and control for aircrafts. E-mail: houlechuan@126.com.

Guangren DUAN received his Ph.D. degree in Control Systems Theory in 1989 from Harbin Institute of Technology, China. From 1989 to 1991, he was a post-doctoral researcher at Harbin Institute of Technology, where he became a professor of control systems theory in 1991. Dr. Duan visited the University of Hull, UK, and the University of Sheffield, UK from December 1996 to October 1998, and worked at the Queen's University of Belfast, UK from October 1998 to October 2002. Since August 2000, he has been elected Specially Employed Professor at Harbin Institute of Technology sponsored by the Cheung Kong Scholars Program of the Chinese government. He is currently the Director of the Center for Control Systems and Guidance Technology at Harbin Institute of Technology. His research interests include robust control, eigenstructure assignment, descriptor systems, missile autopilot design and magnetic bearing control. E-mail: g.r.duan@hit.edu.cn.



Mengshu GUO received his B.S. degree from Harbin Normal University, China, in 1995 and the M.S. and Ph.D. degrees in Mathematics from Harbin Institute of Technology, Harbin, China, in 1997 and 2003, respectively. Since 2006, he has been working as an associate professor at the Department of Mathematics, Harbin Institute of Technology. His research areas include differential inclusions, optimal control and optimization. E-mail: msguo@hit.edu.cn.