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CONVERGENCE RATES IN THE STRONG LAWS OF ASYMPTOTICALLY NEGATIVELY ASSOCIATED RANDOM FIELDS

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Abstract. In this paper, a notion of negative side ρ-mixing (ρ^--mixing) which can be regarded as asymptotic negative association is defined, and some Rosenthal type inequalities for ρ^--mixing random fields are established. The complete convergence and almost sure summability on the convergence rates with respect to the strong law of large numbers are also discussed for ρ^--mixing random fields. The results obtained extend those for negatively associated sequences and ρ^*-mixing random fields.

§1 Introduction

Let d be a positive integer, $\mathbf{N}^d$ be the d-dimensional lattice equipped with the coordinatewise partial order, $\leqslant$. For any $A\subset\mathbf{N}^d$, set $S_A=\sum_{n\in A}X_n$, $|A|=$ the cardinal number of A. For any $n\in\mathbf{N}^d$, let $(n)=\{m\in\mathbf{N}^d, m\leqslant n\}$, $S_n=S_{(n)}$, $|n|=|(n)|=n_1n_2\dots n_d$ and $\|n\|$ denote the Euclidean norm. Occasionally, n,k, etc. will also denote positive integers, the reader should not be confused from their context. A d-dimensional discrete field of real random variables $\{X_k;k\in\mathbf{N}^d\}$ will be called "centered" if $\mathrm{E}X_k=0$. For a random variable X, define $\|X\|_p=(\mathrm{E}|X|^p)^{1/p}$. For two nonempty disjoint sets $S,T\subset\mathbf{N}^d$, we define dist(S,T) to be $\min\{\|j-k\|;j\in S,k\in T\}$. Let $\sigma(S)$ be the σ-field generated by $\{X_k;k\in S\}$, and define $\sigma(T)$ similarly. Let $\mathscr{C}$ be a class of functions which are coordinatewise increasing. It is easy to see that for any $b\leqslant c$, the functions $b\vee(x\wedge c)$ and $x-b\vee(x\wedge c)$ are in $\mathscr{C}$.

A field $\{X_k;k\in\mathbf{N}^d\}$ is called negatively associated (NA) if for every pair of disjoint subsets S,T of $\mathbf{N}^d$,

$$\mathrm{Cov}\{f(X_i;i\in S),g(X_j;j\in T)\}\leqslant 0,$$

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whenever $f,g\in\mathscr{C}$. $\{X_k;k\in\mathbf{N}^d\}$ is called ρ^*-mixing if

$$\rho^*(s)=\sup\{\rho(S,T);S,T\in\mathbf{N}^d,\mathrm{dist}(S,T)\geqslant s\}\to 0(s\to\infty),$$

where

$$\rho(S,T)=\sup\{|\mathrm{E}(f-\mathrm{E}f)(g-\mathrm{E}g)/(\|f-\mathrm{E}f\|_2\|g-\mathrm{E}g\|_2)|;\ f\in L_2(\sigma(S)),g\in L_2(\sigma(T))\}.$$

In the case of $d=1$, many limit results for NA sequences were obtained recently (cf. [1～5] etc). But, it seems that their methods can not be used to the cases of $d\geqslant 2$. By the way, many weak limit results for ρ^*-mixing fields were obtained in the passed several years. (cf. [6～9] etc.). Laterly, [10,11] studied the almost sure convergence ρ^*-mixing fields. The purpose of this paper is to put this two kinds of dependence together and study the almost sure convergence for this two kinds of dependent fields simultaneously. In this section, we define a concept of ρ^--mixing and state some basic properties of it. In § 2, we establish some Rosenthal-type inequalities for block sums of ρ^--mixing fields, which is the main tool in this paper. In § 3, we obtain the results on the complete convergence and the Marcinkiewicz-Zygmund law of large numbers. In § 4, we study the almost sure summability of partial sums. The results obtained extend those for NA sequences and include some of those for ρ^*-mixing fields.

Definition. A field $\{X_k;k\in\mathbf{N}^d\}$ is called ρ^--mixing if

$$\rho^-(s)=\sup\{\rho^-(S,T);S,T\in\mathbf{N}^d,\mathrm{dist}(S,T)\geqslant s\}\to 0\quad(s\to\infty),$$

where

$$\rho^-(S,T)=0\vee\sup\left\{\frac{\mathrm{Cov}\{f(X_i;i\in S),g(X_j;j\in T)\}}{\mathrm{Var}\{f(X_i;i\in S)\}\cdot\mathrm{Var}\{g(X_j;j\in T)\}};f,g\in\mathscr{C}\right\}.$$

Let $x^+=0\vee x$ and $x^-=-(0\wedge x)$. It is easy to see that if $\{X_k;k\in\mathbf{N}^d\}$ is ρ^--mixing, then $\{X_k^+;k\in\mathbf{N}^d\}$ and $\{X_k^-;k\in\mathbf{N}^d\}$ both are also ρ^--mixing with the mixing coefficients not greater than $\rho^-(s)$.

It is obvious that $\rho^-(s)\leqslant\rho^*(s)$. It is easy to see that $\{X_k;k\in\mathbf{N}^d\}$ is negatively associated if and only if $\rho^-(s)=0$ for $s\geqslant 1$. So ρ^--mixing is weaker than ρ^*-mixing and can be regarded as the asymptotically negative association or negative side ρ^*-mixing. The following gives an example of a ρ^--mixing sequence which is neither NA nor ρ^*-mixing.

Example 1. Let $\{\xi_n;n\geqslant 1\}$, $\{\eta_n;n\geqslant 1\}$ and $\{\tau_n;n\geqslant 1\}$ be three independent sequences of i. i. d. standard normal random variables. Let

$$X_n=\begin{cases}\xi_m, & \text{if } n=2m-1\\ -\xi_m, & \text{if } n=2m\end{cases},\quad Y_n=\begin{cases}\eta_m, & \text{if } n=2^{2m-1}\\ -\eta_m, & \text{if } n=2^{2m}\\ \tau_n, & \text{otherwise},\end{cases}$$

and $Z_n=X_n^2+Y_n$. Then $\{X_n;n\geqslant 1\}$ and $\{Y_n;n\geqslant 1\}$ are two independent sequences of NA identically distributed normal variables. Also, $\{X_n;n\geqslant 1\}$ is a 2-dependent sequence, so $\{X_n;n\geqslant 1\}$ is a ρ^*-mixing sequence with $\rho^*(2)=0$. From Property P3 below it follows that $\{Z_n;n\geqslant 1\}$ is ρ^--mixing with $\rho^-(2)=0$. But, $\{Z_n;n\geqslant 1\}$ is neither NA nor ρ^*-mixing, since

$$\mathrm{Cov}\{Z_{2m-1}, Z_{2m}\} = \mathrm{Cov}\{X_{2m-1}^2, X_{2m}^2\} = \mathrm{E}\xi_m^4 - (\mathrm{E}\xi_m^2)^2 = 2 > 0$$

and

$$\frac{\mathrm{Cov}\{Z_{2^{2m-1}}, Z_{2^{2m}}\}}{\mathrm{Var}\{Z_{2^{2m-1}}\}\mathrm{Var}\{Z_{2^{2m}}\}} = -\frac{1}{3} \not\to 0 \quad \text{as } \mathrm{dist}(2^{2m-1}, 2^{2m}) = 2^{2m-1} \to \infty$$

The following two properties of ρ^--mixing are obvious from the definitions.

Property P1. A subset of a ρ^--mixing field $\{X_k; k\in\mathbf{N}^d\}$ with mixing coefficients $\rho^-(s)$ is also ρ^--mixing with mixing coefficients not greater than $\rho^-(s)$.

Property P2. Increasing functions defined on disjoint subsets of a ρ^--mixing field $\{X_k; k\in\mathbf{N}^d\}$ with mixing coefficients $\rho^-(s)$ are also ρ^--mixing with mixing coefficients not greater than $\rho^-(s)$.

Property P3. Suppose that $\{X_k; k\in\mathbf{N}^d\}$ and $\{Y_k; k\in\mathbf{N}^d\}$ are two independent ρ^--mixing fields with mixing coefficients $\rho_1^-(s)$ and $\rho_2^-(s)$, respectively. Then $\{(X_k, Y_k); k\in\mathbf{N}^d\}$ is also a ρ^--mixing field with mixing coefficients not greater than $\rho_1^-(s)+\rho_2^-(s)$.

Proof. It is enough to show that

$$\begin{aligned}&\mathrm{E}\{f(X_n,Y_n;n\in A)g(X_n,Y_n;n\in B)\}\leqslant\\&\mathrm{E}\{f(X_n,Y_n;n\in A)\}\mathrm{E}\{g(X_n,Y_n,n\in B)\}+\\&(\rho_1^-(r)+\rho_2^-(r))\{\mathrm{E}f^2(X_n,Y_n;n\in A)\}^{1/2}\{\mathrm{E}g^2(X_n,Y_n;n\in B)\}^{1/2},\end{aligned}$$

where $f, g\in\mathscr{C}$ and $r=\mathrm{dist}(A,B)$. Let $f_1(Y_n; n\in A)=\mathrm{E}_X f$ and $g_1(Y_n; n\in A)=\mathrm{E}_X g$, where E_X is the expectation taken over $\{X_n; n\geqslant 1\}$ only, i.e. $\mathrm{E}_X(\cdot)=\mathrm{E}(\cdot\,|Y_n; n\geqslant 1)$, and E_Y is defined similarly. Then f_1 and g_1 are also in $\mathscr{C}$. From the Fubini theorem, it follows that

$$\begin{aligned}\mathrm{E}fg &= \mathrm{E}_Y\{\mathrm{E}_X fg\} \leqslant \mathrm{E}_Y\{(\mathrm{E}_X f)(\mathrm{E}_X g)\} + \rho_1^-(r)\mathrm{E}_Y\{(\mathrm{E}_X f^2)^{1/2}(\mathrm{E}_X g^2)^{1/2}\} \leqslant\\&\mathrm{E}fg + \rho_2^-(r)\{\mathrm{E}_Y(\mathrm{E}_X f)^2\}^{1/2}\{\mathrm{E}_Y(\mathrm{E}_X g)^2\}^{1/2} + \rho_1^-(r)(\mathrm{E}f^2)^{1/2}(\mathrm{E}g^2)^{1/2} \leqslant\\&\mathrm{E}fg + (\rho_1^-(r)+\rho_2^-(r))(\mathrm{E}f^2)^{1/2}(\mathrm{E}g^2)^{1/2},\end{aligned}$$

which completes the proof.

§ 2 Rosenthal-type Inequalities

The following Rosenthal-type inequality for ρ^--mixing random fields is the main tool for studying the limit results in this paper.

Theorem 2.1. Suppose $\{X_k; k\in\mathbf{N}^d\}$ is a ρ^--mixing random field with $\mathrm{E}X_k=0$ and $\|X_k\|_p<\infty$ for some $p\geqslant 2$ and all k. Then there exists a positive constant B_p depending only on p and $\rho^-(\cdot)$ such that for any finite set $S\subset\mathbf{N}^d$,

$$\mathrm{E}\Big|\sum_{k\in S}X_k\Big|^p \leqslant B_p\Big(\sum_{k\in S}\mathrm{E}|X_k|^p + \Big(\sum_{k\in S}\mathrm{E}|X_k|^2\Big)^{p/2}\Big). \tag{2.1}$$

When $d=1$, the Rosenthal-type inequality remains ture for the maximal partial sums, if some conditions on $\rho^-(\cdot)$ are added.

Theorem 2.2. Suppose $d=1$, and $\{X_k; k\in\mathbf{N}\}$ is a sequence of ρ^--mixing random variables with $EX_k=0$ with $\|X_k\|_p<\infty$ for some $p\geqslant 2$ and all k. For any $k\geqslant 0, n\geqslant 1$, set $S_n(k) = \sum_{i=1}^{n} X_{k+n}$. Assume that $\rho^-(s)\leqslant c(\log s)^{-1-\delta}$ for some $c>0$ and $\delta>0$. Then there exists a positive constant B_p depending only on p and $\rho^-(\cdot)$ such that for any $k\geqslant 0, n\geqslant 1$,

$$E\max_{1\leqslant m\leqslant n}|S_m(k)|^p \leqslant B_p((n\max_{1\leqslant m\leqslant n} EX_{k+m}^2)^{p/2} + n\max_{1\leqslant m\leqslant n} E|X_{k+m}|^p). \tag{2.2}$$

Although we don't konw whether (2.2) holds true or not when $d\geqslant 2$, we establish the following inequalities for the maximal partial sums.

Theorem 2.3. Let $\{X_n; n\in \mathbf{N}^d\}$ be a centered ρ^--mixing field. Set $S_n(k)=\sum_{1\leqslant m\leqslant n} X_{k+m}$. Then for each $p\geqslant 2$, there exists a positive constant $c=c(p,\rho^-)$ depending only on p and $\rho^-(\cdot)$ such that for any $k, n\in \mathbf{N}^d$,

$$E\max_{1\leqslant m\leqslant n}|S_m(k)|^p \leqslant c\{(|n|\max_{1\leqslant m\leqslant n} E|X_{k+m}|^2)^{p/2} + |n|(\log|n|)^{dp}\max_{1\leqslant m\leqslant n} E|X_{k+m}|^p\}. \tag{2.3}$$

Futhermore, assume that $\rho^-(s)\leqslant c(\log s)^{-1-\delta}$ for some $c>0$ and $\delta>0$. Then

$$E\max_{1\leqslant m\leqslant n}|S_m(k)|^p \leqslant c\{(|n|\max_{1\leqslant m\leqslant n} E|X_{k+m}|^2)^{p/2} + |n|(\log|n|)^{(d-1)p}\max_{1\leqslant m\leqslant n} E|X_{k+m}|^p\}. \tag{2.4}$$

In order to prove Theorem 2.1, we need some lammas.

Lamma 2.1. Let $p,q>1$ with $1/p+1/q=1$. Suppose $X=f(X_i; i\in S)$ and $Y=g(X_j; j\in T)$, where $f,g\in \mathscr{C}$ and $S,T\subset \mathbf{N}^d$ with $S\cap T=\varnothing$. Then we have

$$EXY - EXEY \leqslant 6(\rho^-(S,T))^{\frac{2}{p}\wedge\frac{2}{q}}\|X\|_p\|Y\|_q.$$

The following is the Marcinkiewicz-Zygmund-type inequality.

Lemma 2.2. Suppose that $\{X_k; k\in \mathbf{N}^d\}$ is a centered ρ^--mixing random field with $\|X_k\|_p<\infty$ for some $p>1$ and all k. Then there exists a positive constant $D_p=D(p,\rho^-(\cdot))$ such that

$$E\Big|\sum_{j\in S} X_j\Big|^p \leqslant D_p E\Big(\sum_{j\in S} X_j^2\Big)^{p/2} \quad \text{for any } S\in \mathbf{N}^d. \tag{2.5}$$

The proof of the above two lemmas can be found in [12], we omit them here.

Proof of Theorem 2.1. For any $p\geqslant 2$, there exists $k\in \mathbf{N}$ and $1\leqslant q<2$ such that $p=2^k q$, so (2.1) is equivalent to

$$E\Big|\sum_{j\in S} X_j\Big|^{2^k q} \leqslant C_{k,q}\Big\{\sum_{j\in S} E|X_j|^{2^k q} + \Big(\sum_{j\in S} E|X_j|^2\Big)^{2^{k-1}q}\Big\}. \tag{2.6}$$

From Lemma 2.2, it follows that

$$E\Big|\sum_{j\in S} X_j\Big|^{2^k q} \leqslant D_{2^k q} E\Big(\sum_{j\in S} X_j^2\Big)^{2^{k-1}q} \leqslant 2^{2^{k-1}q-1} D_{2^k q}\Big\{E\Big(\sum_{j\in S} X_j^{+2}\Big)^{2^{k-1}q} + E\Big(\sum_{j\in S} X_j^{-2}\Big)^{2^{k-1}q}\Big\}. \tag{2.7}$$

By noting that $\{X_j^{+2}\}$ and $\{X_j^{-2}\}$ are ρ^--mixing fields and by the induction method, the proof of (2.6) is similar to that of (6) in [11] with Lemma 2.3 taking the place of its inequality (5).

To prove Theorem 2. 2, we need one more lemma.

Lemma 2. 4. Suppose $d=1$, and $\{X_k; k\in \mathbf{N}\}$ is a centered ρ^--mixing random sequence with $2\sqrt{N}\rho^-(1)<1$, where $N\geqslant 1$. Set $S_n(k)=\sum_{m=1}^n X_{k+m}$. Then for any $k\geqslant 0$ and $1\leqslant n\leqslant N$, we have

$$\mathrm{E}\max_{1\leqslant m\leqslant n}|S_m(k)|^2\leqslant 6n\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2. \tag{2.8}$$

Proof. Let $\rho=\rho^-(1)$. It is enough to show that for any $k\geqslant 0$ and $1\leqslant n\leqslant N$,

$$\mathrm{E}(0\vee\max_{1\leqslant m\leqslant n}S_m(k))^2\leqslant(n+2n^{3/2}\rho)\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2. \tag{2.9}$$

It is obvious that (2. 9) holds for $n=1$. Now we assume that (2. 9) holds for $n-1$, we will prove that it holds also for n. Note that

$$\max_{1\leqslant m\leqslant n}S_m(k)=X_k+0\vee\max_{2\leqslant m\leqslant n}(S_m(k)-X_k),$$

and we have

$$(0\vee\max_{1\leqslant m\leqslant n}S_m(k))^2\leqslant X_k^2+2X_k(0\vee\max_{2\leqslant m\leqslant n}(S_m(k)-X_k))+(0\vee\max_{2\leqslant m\leqslant n}(S_m(k)-X_k))^2.$$

Hence by the hypothesis of induction, it follows that

$$\begin{aligned}
&\mathrm{E}((0\vee\max_{1\leqslant m\leqslant n}S_m(k)))^2\leqslant\\
&\mathrm{E}X_k^2+2\rho(\mathrm{E}X_k^2)^{1/2}(\mathrm{E}(0\vee\max_{2\leqslant m\leqslant n}(S_m(k)-X_k))^2)^{1/2}+\mathrm{E}(0\vee\max_{2\leqslant m\leqslant n}(S_m(k)-X_k))^2\leqslant\\
&\mathrm{E}X_k^2+2\rho(\mathrm{E}X_k^2)^{1/2}(n-1+2(n-1)^{3/2}\rho)^{1/2}(\max_{2\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2))^{1/2}+\\
&(n-1+2(n-1)^{3/2}\rho)\max_{2\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2\leqslant\\
&\{n+2\rho(((n-1)+2\rho(n-1)^{3/2})^{1/2}+(n-1)^{3/2})\}\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2.
\end{aligned}$$

By noting that $2\rho(n-1)^{1/2}\leqslant 1$, it follows that

$$\begin{aligned}
&\mathrm{E}((0\vee\max_{1\leqslant m\leqslant n}S_m(k))^2\leqslant\\
&\{n+2\rho\{(2(n-1))^{1/2}+(n-1)^{3/2}\}\}\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2\leqslant\\
&\{n+2\rho n^{3/2}\}\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2.
\end{aligned}$$

The lemma is thus proved.

Proof of Theorem 2. 2. First, we consider the special case that $p=2$. The following Theorem is the result.

Theorem 2. 4. Suppose $d=1$, and $\{X_k; k\in\mathbf{N}\}$ is a centered ρ^--mixing random field with $\rho^-(s)\leqslant c(\log s)^{-1-\delta}$ for some $c>0$ and $\delta>0$. Set $S_n(k)=\sum_{m=1}^n X_{k+m}$. Then there exists a positive constant $c=c(\rho^-)$ such that for any $k,n\in\mathbf{N}$,

$$\mathrm{E}\max_{1\leqslant m\leqslant n}|S_m(k)|^2\leqslant cn\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2. \tag{2.10}$$

Proof. For the sake of convenience of statement, we assume that $\{X, X_n; n\geqslant 1\}$ is identically distributed and $k=0$, otherwise, the following proof will be the same with $\max_{1\leqslant m\leqslant n}\|X_{k+m}\|_2$ taking the place of $\|X\|_2$ and $S_i(k)$ taking the place of S_i, etc. Let $q>3$ such that $q/(q-2)<1+\delta$. Define

$$T=(\log(2n))^{q/(q-2)}, p=T^2, r=[n/p], X_i^{(n)}=(-c)\vee(X_i\wedge c),$$

where $c=n^{1/2}\|X\|_2/T$,

$$X_i^{(n1)} = X_i^{(n)} - \mathrm{E}X_i^{(n)}, X_i^{(n2)} = X_i - X_i^{(n1)}, S_k^{(n1)} = \sum_{i=1}^{k} X_i^{(n1)}, S_k^{(n2)} = \sum_{i=1}^{k} X_i^{(n2)}.$$

Then

$$\mathrm{E}\max_{i\leqslant n}|S_i|^2 \leqslant 2\mathrm{E}\max_{i\leqslant n}|S_i^{(n1)}|^2 + 2\mathrm{E}\max_{i\leqslant n}|S_i^{(n2)}|^2. \tag{2.11}$$

From Theorem 2.1, it follows that for any $1\leqslant l<m\leqslant n$,

$$\mathrm{E}|S_m^{(n1)} - S_l^{(n1)}|^q \leqslant D_q\{(m-l)^{q/2}\|X\|_2^q + (m-l)\|(-c)\vee(X\wedge c)\|_q^q\}.$$

Hence, by Corollary 3 of [13],

$$\mathrm{E}\max_{i\leqslant n}|S_i^{(n1)}|^q \leqslant D_q\{n^{q/2}\|X\|_2^q + n\log^q(2n)\|(-c)\vee(X\wedge c)\|_q^q\} \leqslant$$

$$D_q\{n^{q/2} + n^{q/2}\log^q(2n)/T^{q-2}\}\|X\|_2^q = 2D_q n^{q/2}\|X\|_2^q.$$

It follows that

$$\mathrm{E}\max_{i\leqslant n}|S_i^{(n1)}|^2 \leqslant (\mathrm{E}\max_{i\leqslant n}|S_i^{(n1)}|^q)^{2/q} \leqslant Kn\|X\|_2^2. \tag{2.12}$$

Now, we consider $S_i^{(n2)}$. Let $p_1=[p/2]$,

$$Y_{i1} = \sum_{j=1+(2i-1)r}^{2ir} X_j^{(n2)}, Y_{i2} = \sum_{j=1+2(i-1)r}^{(2i-1)r} X_j^{(n2)}, T_1(i) = \sum_{j=1}^{i} Y_{j1}, T_2(i) = \sum_{j=1}^{i} Y_{j2}.$$

Then

$$\max_{i\leqslant n}|S_i^{(n2)}| \leqslant \max_{i\leqslant p_1}|T_1(i)| + \max_{i\leqslant p_1}|T_2(i)| + \max_{i\leqslant p+2}\max_{(i-1)r<j\leqslant ir}|S_j^{(n2)} - S_{(i-1)r}^{(n2)}|.$$

It follows that

$$\mathrm{E}\max_{i\leqslant n}|S_i^{(n2)}|^2 \leqslant 3\mathrm{E}\max_{i\leqslant p_1}|T_1(i)|^2 + 3\mathrm{E}\max_{i\leqslant p_1}|T_2(i)|^2 +$$

$$3\mathrm{E}\max_{i\leqslant p+2}\max_{(i-1)r<j\leqslant ir}|S_j^{(n2)} - S_{(i-1)r}^{(n2)}|^2 \leqslant$$

$$3\mathrm{E}\max_{i\leqslant p_1}|T_1(i)|^2 + 3\mathrm{E}\max_{i\leqslant p_1}|T_2(i)|^2 + 3(p+2)\max_{i\leqslant p+2}\mathrm{E}\max_{j\leqslant r}|S_{j+(i-1)r}^{(n2)}|^2 \leqslant$$

$$3\mathrm{E}\max_{i\leqslant p_1}|T_1(i)|^2 + 3\mathrm{E}\max_{i\leqslant p_1}|T_2(i)|^2 + 3(p+2)\max_{i\leqslant p+2}\mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r}|X_j^{(n2)}|\Big)^2 \leqslant$$

$$3\mathrm{E}\max_{i\leqslant p_1}|T_1(i)|^2 + 3\mathrm{E}\max_{i\leqslant p_1}|T_2(i)|^2 +$$

$$6(p+2)\max_{i\leqslant p+2}\Big\{\mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r} X_j^{(n2)^+}\Big)^2 + \mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r} X_j^{(n2)^-}\Big)^2\Big\} = I_1 + I_2 + I_3. \tag{2.13}$$

By noting that

$$\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r}\mathrm{E}X_j^{(n2)^+}\Big)^2 = (r\mathrm{E}X^{(n2)^+})^2 \leqslant (r2\mathrm{E}|X|I\{|X|\geqslant c\})^2 \leqslant$$

$$4(rT/(n^{1/2}\|X\|_2)\mathrm{E}X^2)^2 = 4r^2T^2\mathrm{E}X^2/n \leqslant 4\frac{n^2}{p^2}T^2\mathrm{E}X^2/n = 4n\mathrm{E}X^2/p,$$

from Theorem 2.1 it follows that

$$\mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r} X_j^{(n2)^+}\Big)^2 \leqslant$$

$$2\Big\{\mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r}[X_j^{(n2)^+} - \mathrm{E}X_j^{(n2)^+}]\Big)^2 + \Big(\sum_{j=1+(i-1)r}^{r+(i-1)r}\mathrm{E}X_j^{(n2)^+}\Big)^2\Big\} \leqslant$$

$$8n\mathrm{E}X^2/p + Kr\mathrm{E}X^2. \tag{2.14}$$

Similarly,

$$\mathrm{E}\Big(\sum_{j=1+(i-1)r}^{r+(i-1)r} X_j^{(n2)^-}\Big)^2 \leqslant 8n\mathrm{E}X^2/p + Kr\mathrm{E}X^2. \tag{2.15}$$

From (2.14) and (2.15) it follows that

$$I_3 \leqslant Kn\mathrm{E}X^2. \tag{2.16}$$

Now, from the condition $\rho^-(s) \leqslant c(\log s)^{-1-\delta}$ it follows that for n large enough,

$$2\rho^-(r)\sqrt{p_1} \leqslant C(\log n)^{q/(q-2)}(\log n)^{-1-\delta} < 1.$$

So, from Lemma 2.4 and Theorem 2.1, it follows that for n_0 large enough and $n \geqslant n_0$,

$$I_1 \leqslant 6p_1 \max_{i\leqslant p_1} \mathrm{E}Y_{i1}^2 \leqslant Kp_1 r\mathrm{E}X^2 \leqslant Kn\mathrm{E}X^2. \tag{2.17}$$

The above inequality holds also for $n \leqslant n_0$. Similarly,

$$I_2 \leqslant Kn\mathrm{E}X^2. \tag{2.18}$$

Putting (2.16), (2.17) and (2.18) into (2.13) yields $\mathrm{E}\max_{i\leqslant n}|S_n|^2 \leqslant Kn\mathrm{E}X^2$. This completes the proof.

Now, we prove (2.2) by induction on p. When $p=2$, (2.2) follows from Theorem 2.4 immediately.

When $p(>2)$ is not an integer, assume that (2.2) holds for $[p]$, that is, there exists $K_1 \geqslant 2$ such that for every $k \geqslant 0, n \geqslant 1$,

$$\mathrm{E}\max_{1\leqslant m\leqslant n}|S_m(k)|^{[p]} \leqslant K_1\big((n\max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2)^{[p]/2} + n\max_{1\leqslant m\leqslant n}\mathrm{E}|X_{k+m}|^{[p]}\big) \tag{2.19}$$

and

$$\mathrm{E}\max_{1\leqslant m\leqslant n}|S_m(k)|^2 \leqslant K_1 n \max_{1\leqslant m\leqslant n}\mathrm{E}X_{k+m}^2. \tag{2.20}$$

We will show that (2.2) remains valid for p. Without a loss of generality, we can assume that $k=0$. For the sake of convenience of statement, we assume that $\{X, X_k; k\geqslant 1\}$ is identically distributed. Also, we can assume that $(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}} < \frac{1}{24p}$, where $q=\frac{p}{p-1}$. Otherwise, there exists a positive integer J such that $(\rho^-(J))^{\frac{2}{p}\wedge\frac{2}{q}} < \frac{1}{24p}$. It is easy to see that

$$\max_{1\leqslant m\leqslant n}|S_m| \leqslant \sum_{k=0}^{J}\max_{1\leqslant j\leqslant n/J}\Big|\sum_{i=1}^{j}X_{iJ+k}\Big|.$$

Then we can consider $\{X_{iJ+k}; i\geqslant 1\}$ for each k separately, instead.

Now, let $\varepsilon(j)$'s be i.i.d. r.v.s with $\mathrm{P}(\varepsilon(j)=1)=\mathrm{P}(\varepsilon(j)=-1)=1/2$, which are also independent of $\{X_k\}$. Define $Y_j = \sum_{i=1,\varepsilon(i)=1}^{j} X_i$ and $Z_j = \sum_{i=1,\varepsilon(i)=-1}^{j} X_i$. Then $Y_j + Z_j = S_j$ and $Y_j - Z_j = \sum_{i=1}^{j}\varepsilon(i)X_i$. From the fact that $|y-z|^p - |y|^p \geqslant -zD_p(y)$, where $D_p(y) = p|y|^{p-1}\mathrm{sgn}\,y$, it follows that $|Y_j|^p \leqslant |Y_j - Z_j|^p + Z_jD_p(Y_j) \leqslant |Y_j - Z_j|^p + Z_j^+D_p(Y_j)^+ + Z_j^-D_p(Y_j)^-$. So

$$\mathrm{E}\max_{j\leqslant n}|Y_j|^p \leqslant \mathrm{E}\max_{j\leqslant n}|Y_j - Z_j|^p +$$

$$\mathrm{E}(\max_{j\leqslant n} Z_j^+)(\max_{j\leqslant n} D_p(Y_j)^+) + \mathrm{E}(\max_{j\leqslant n} Z_j^-)(\max_{j\leqslant n} D_p(Y_j)^-). \tag{2.21}$$

Fixed $\varepsilon(j)$'s, the distance of the two sets $\{j; \varepsilon(j)=1\}$ and $\{j; \varepsilon(j)=-1\}$ is one. By Lemma 2.1 it follows that

$$\mathrm{E}_X(\max_{j\leqslant n} Z_j^+)(\max_{j\leqslant n} D_p(Y_j)^+) \leqslant (\mathrm{E}_X \max_{j\leqslant n} Z_j^+)(\mathrm{E}_X \max_{j\leqslant n} D_p(Y_j)^+) +$$

$$6(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}}(\mathrm{E}_X \max_{j\leqslant n} Z_j^{+p})^{1/p}(\mathrm{E}_X \max_{j\leqslant n} D_p(Y_j)^{+q})^{1/q} \leqslant$$

$$p(\mathrm{E}_X \max_{j\leqslant n} |Z_j|^2)^{1/2}(\mathrm{E}_X \max_{j\leqslant n} |Y_j|^{[p]})^{\frac{p-1}{[p]}} +$$

$$6p(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}}(\mathrm{E}_X \max_{j\leqslant n} |Z_j|^p + \mathrm{E}_X \max_{j\leqslant n} |Y_j|^p) =: I_1 + I_2. \tag{2.22}$$

Similarly,

$$\mathrm{E}_X(\max_{j\leqslant n} Z_j^-)(\max_{j\leqslant n} D_p(Y_j)^-) \leqslant I_1 + I_2. \tag{2.23}$$

By the hypothesis of induction,

$$\mathrm{E}_X \max_{j\leqslant n} |Z_j|^2 = \mathrm{E}_X \max_{j\leqslant n} |\sum_{i=1}^{j} X_j I\{\varepsilon(i) = -1\}|^2 \leqslant K_1 n \mathrm{E}X^2,$$

$$\mathrm{E}_X \max_{j\leqslant n} |Y_j|^{[p]} = \mathrm{E}_X \max_{j\leqslant n} |\sum_{i=1}^{j} X_j I\{\varepsilon(i) = 1\}|^{[p]} \leqslant K_1\{(n\mathrm{E}X^2)^{[p]/2} + n\mathrm{E}|X|^{[p]}\}.$$

It follows that

$$I_1 \leqslant K_2\{n^{p/2}\|X\|_2^p + n^{\{1/2+(p-1)/[p]\}}\|X\|_2\|X\|_{[p]}^{p-1}\}. \tag{2.24}$$

On the other hand, by Theorem 2.1 it follows that

$$\mathrm{E}\max_{j\leqslant n}|Y_j - Z_j| = \mathrm{E}_X\mathrm{E}_\varepsilon \max_{j\leqslant n}|\sum_{i=1}^{j}\varepsilon(i)X_i|^p \leqslant 2\mathrm{E}_X\mathrm{E}_\varepsilon|\sum_{i=1}^{n}\varepsilon(i)X_i|^p =$$

$$2\mathrm{E}_\varepsilon\mathrm{E}_X|\sum_{i=1}^{n}\varepsilon(i)X_i|^p \leqslant K_3\{n^{p/2}\|X\|_2^p + n\|X\|_p^p\}. \tag{2.25}$$

Putting (2.21)~(2.25) together yields

$$\mathrm{E}\max_{j\leqslant n}|Y_j|^p \leqslant K_4\{n^{p/2}\|X\|_2^p + n\|X\|_p^p + n^{\{1/2+(p-1)/[p]\}}\|X\|_2\|X\|_{[p]}^{p-1}\} +$$

$$12p(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}}(\mathrm{E}\max_{j\leqslant n}|Z_j|^p + \mathrm{E}\max_{j\leqslant n}|Y_j|^p). \tag{2.26}$$

Similarly,

$$\mathrm{E}\max_{j\leqslant n}|Z_j|^p \leqslant K_4\{n^{p/2}\|X\|_2^p + n\|X\|_p^p + n^{\{1/2+(p-1)/[p]\}}\|X\|_2\|X\|_{[p]}^{p-1}\} +$$

$$12p(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}}(\mathrm{E}\max_{j\leqslant n}|Z_j|^p + \mathrm{E}\max_{j\leqslant n}|Y_j|^p). \tag{2.27}$$

From (2.26) and (2.27) it follows that

$$\mathrm{E}\max_{j\leqslant n}|Y_j|^p + \mathrm{E}\max_{j\leqslant n}|Z_j|^p \leqslant$$

$$K_4(1 - 24p(\rho^-(1))^{\frac{2}{p}\wedge\frac{2}{q}})^{-1}\{n^{p/2}\|X\|_2^p + n\|X\|_p^p + n^{\{1/2+(p-1)/[p]\}}\|X\|_2\|X\|_{[p]}^{p-1}\}.$$

By noting that

$$\mathrm{E}\max_{j\leqslant n}|S_j|^p = \mathrm{E}\max_{j\leqslant n}|Y_j + Z_j|^p \leqslant 2^{p-1}\{\mathrm{E}\max_{j\leqslant n}|Y_j|^p + \mathrm{E}\max_{j\leqslant n}|Z_j|^p\},$$

it follows that

$$\mathrm{E}\max_{j\leqslant n}|S_j|^p\leqslant K\{n^{p/2}\|X\|_2^p+n\|X\|_p^p+n^{(1/2+(p-1)/[p])}\|X\|_2\|X\|_{[p]}^{p-1}\}. \tag{2.28}$$

If $2<p<3$, then $[p]=2$. In this case (2.2) follows from (2.28) immediately.

If $p>3$. From the Lyapunov inequality

$$\mathrm{E}|X|^{[p]}\leqslant(\mathrm{E}X^2)^{\frac{p-[p]}{p-2}}(\mathrm{E}|X|^p)^{\frac{[p]-2}{p-2}},$$

it follows easily that (for details see [14])

$$n^{(1/2+(p-1)/[p])}\|X\|_2\|X\|_{[p]}^{p-1}\leqslant n^{p/2}\|X\|_2+n\|X\|_p^p. \tag{2.29}$$

This proves (2.2) for p by (2.28) and (2.29).

When $p>3$ is an integer, along the same lines as in the above proof, with $p-1$ instead of $[p]$, we can deduce that (2.2) remains true for p. Now, the proof of Theorem 2.2 is complete.

Proof of Theorem 2.3. If $d=1$, from Theorem 2.1 and Corollary 3 of Maricz[13], it follows that (2.3) holds. Also, from Theorem 2.2 it follows that (2.4) holds for $d=1$ if $\rho^-(s)\leqslant(\log s)^{-1-\delta}$. In the case of $d\geqslant2$, by using the method of [13], one can prove (2.3) and (2.4) by induction on d. For details see [10].

§ 3 Complete Convergence

Applying Thoerem 2.1 and Theorem 2.3, one can obtain the following results on the complete convergence for ρ^--mixing field, which is a kind of convergence rate with respect to the strong law of large numbers[15].

Theorem 3.1. Let $\alpha>1/2, p\alpha>1, p\geqslant1$, and $\{X, X_n; n\in\mathbf{N}^d\}$ be a ρ^--mixing field of identically distributed random variables with $\mathrm{E}X=0$ if $\alpha\leqslant1$, and

$$\mathrm{E}|X|^p\log^{d-1}(|X|)<\infty. \tag{3.1}$$

Then

$$\forall\,\varepsilon>0,\quad\sum_n|n|^{p\alpha-2}\mathrm{P}(\max_{1\leqslant j\leqslant n}|S_j|>\varepsilon|n|^\alpha)<\infty. \tag{3.2}$$

The following theorem deals with the case of $p\alpha=1$.

Theorem 3.2. Let $1\leqslant p<2$, and $\{X, X_n; n\in\mathbf{N}^d\}$ be a ρ^--mixing field of identically distributed random variables with $\mathrm{E}X=0$. Assume that

$$\mathrm{E}|X|^p\log^{\beta\vee(d-1)}(|X|)<\infty,\quad\text{for some }\beta>d(p-1). \tag{3.3}$$

Then

$$\forall\,\varepsilon>0,\sum_n|n|^{-1}\mathrm{P}(\max_{1\leqslant j\leqslant n}|S_j|>\varepsilon|n|^{1/p})<\infty. \tag{3.4}$$

The ideal condition for (3.4) is the condition (3.1). If $p<2-1/d$, then condition (3.3) just is (3.1). When $p\geqslant2-1/d$, (3.4) also holds under condition (3.1), if some conditions on the mixing coefficient are added.

Theorem 3.3. Let $1\leqslant p<2$, and $\{X, X_n; n\in\mathbf{N}^d\}$ be a ρ^--mixing field of identically distributed random variables with $\mathrm{E}X=0$ and (3.1). Assume that

$$\rho^-(s) \leqslant c(\log s)^{-1-\delta}, \quad \text{for some } c > 0, \delta > 0, \tag{3.5}$$

then (3.4) holds.

An immediate consequence of Theorem 3.2 and Theorem 3.3 is the following Marcinkiewicz-Zygmund law of large numbers.

Corollary 3.1. Let $1 \leqslant p < 2$, and $\{X, X_n; n \in \mathbf{N}^d\}$ be a ρ^--mxing field of identically distributed random variables with $EX = 0$ and (3.1). Assume that (3.3) or (3.5) holds, then

$$\lim_{n\to\infty} S_n / |n|^{1/p} = 0 \quad \text{a. s.} \tag{3.6}$$

§ 4 Almost Sure Summability

Based on Thoerem 2.1, we present the following result on the almost sure summability of partial sums, which is the another way to describe the convergence rate in the strong law of large numbers. (cf. [14,16]).

Theorem 4.1. Let $\{X_n, n \in \mathbf{N}^d\}$ be a ρ^--mixing random field with $EX_n = 0$ and

$$\sup_n E|X_n|^2 \log^D(|X_n|) < \infty \text{ for some } D > d.$$

Then for any $p > 0$ and any $\{q(|n|); n \in \mathbf{N}^d\}$ of positive numbers satisfying

$$\sum_n |n|^{p/2} / q(|n|) < \infty, \tag{4.1}$$

we have

$$\sum_n \max_{1 \leqslant i \leqslant n} |S_i|^p / q(|n|) < \infty \quad \text{a. s.} \tag{4.2}$$

The proof of Theorems 3.1, 3.3 and 4.1 is similar to that in [10] with some changes only, and so is omited here.

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